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A note on decomposable maps on operator systems

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Abstract. This article contains a characterization of operator systems $\mathcal{S}$ with the property that every positive map $\phi : \mathcal{S} \to M_n$ is decomposable, as well as an alternate and a more direct proof of a characterization of decomposable maps due to E. Størmer.

Contents

1. Introduction

Let H denote a Hilbert space over $\mathbb{C}$ and $B(H)$ the C*-algebra of bounded operators on H. Let $\mathcal{A}$ be a unital C*-algebra. Without loss of generality, we shall assume $\mathcal{A}$ to be a C*-subalgebra of $B(H)$ for some Hilbert space H. An operator system $\mathcal{S} \subseteq \mathcal{A}$ is a unital self-adjoint subspace of $\mathcal{A}$. Letting $M_n = B(\mathbb{C}^n)$ to denote the C*-algebra of $n \times n$ complex matrices, a linear map $\phi : \mathcal{S} \to M_n$ is **positive** if $\phi(s) \succeq 0$ whenever s is a positive element of $\mathcal{S}$. Given a positive integer k, let $\phi_k = \phi \otimes I_k : \mathcal{S} \otimes M_k \to M_n \otimes M_k$ denote the linear map determined by $\phi_k(s \otimes X) = \phi(s) \otimes X$. The map ϕ is **completely positive**, or **cp** for short, if each ϕ_k is positive; that is, if $S \in \mathcal{S} \otimes M_k$ is positive as an element of the algebra $B(H) \otimes M_k = B(\oplus_1^k H)$ of $k \times k$ matrices with entries from $B(H)$, then $\phi_k(S)$ is positive in $M_n \otimes M_k = B(\mathbb{C}^n \otimes C^k)$.

Let t denote a transpose on M_n. A mapping $\phi : \mathcal{S} \to M_n$ is **co-cp** if $t \circ \phi$ is cp. As is well known, the definition of co-cp is independent of the choice of transpose since any two transposes are unitarily equivalent. The

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linear map ϕ is said to be **decomposable** if it is a sum of a cp map and a co-cp map. Maps that are positive, but not completely so, like the generic decomposable map, are of importance in quantum information theory as entanglement detecting maps.

Let $\mathcal{S}^+$ denote its positive elements of an operator space $\mathcal{S} \subseteq \mathcal{A}$. Given a positive integer n, it is evident that $\mathcal{S}^+ \otimes M_n^+$, which is the cone generated by elementary tensors $s \otimes X$ where both s and X are positive, is a subset of $(\mathcal{S} \otimes M_n)^+$. Operator systems $\mathcal{S}$ with the property that every positive map $\phi : \mathcal{S} \to M_n$ is completely positive are characterized as follows. See for instance Theorem 6.6 in [5].

Proposition 1.1. *Every positive map $\phi : \mathcal{S} \to M_n$ is completely positive if and only if $\mathcal{S}^+ \otimes M_n^+$ is dense in $(\mathcal{S} \otimes M_n)^+$.*

Using techniques from [3] and [12], here we establish the analog of Proposition 1.1 for decomposable maps. Let $k \in \mathbb{N}$, t denote a transpose on M_k and

$$J_k(\mathcal{S}) := \left\{ S = \sum_{j=1}^{k} s_j \otimes x_j \,:\, S \succeq 0, \sum_{j=1}^{k} s_j \otimes t(x_j) \succeq 0 \right\} \subseteq \mathcal{S} \otimes M_k. \quad (1)$$

Theorem 1.2. *Let $n \in \mathbb{N}$ and $\mathcal{S}$ be an operator system in the unital C*-algebra $\mathcal{A}$. Every positive linear map $\psi : \mathcal{S} \to M_n$ is decomposable if and only if $J_n(\mathcal{S}) \subseteq \overline{\mathcal{S}^+ \otimes M_n^+}$.*

It is well known that every positive linear map $\phi : M_p \to M_q$ is decomposable, whenever $p, q \in \mathbb{N}$ and $pq \leq 6$. (please see [7], [11], [3] and [12]). By combining this fact with Theorem 1.2, one can immediately conclude the following.

Corollary 1.3. *If $p, q \in \mathbb{N}$ and $pq \leq 6$, then $J_p(M_q) \subseteq \overline{M_q^+ \otimes M_p^+}$.*

The proof of Theorem 1.2 is based upon a result of E. Størmer. Let $L(\mathcal{S}, M_n)$ denote the vector space of linear maps from $\mathcal{S}$ to M_n. The **dual functional** $s_\phi : \mathcal{S} \otimes M_n \to \mathbb{C}$, associated to the linear map $\phi \in L(\mathcal{S}, M_n)$, is the mapping defined by

$$s_\phi(s \otimes x) = \langle (\phi(s) \otimes x)\, \mathbf{e}, \mathbf{e} \rangle, \quad (2)$$

where $\otimes$ denotes the Kronecker product, $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{C}^n$ and $\mathbf{e} = \sum_{j=1}^n e_j \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n$. It is customary to identify $M_n(\mathcal{S})$ with $\mathcal{S} \otimes M_n$, via the mapping

$$M_n(\mathcal{S}) \ni [x_{i,j}] \mapsto \sum_{i,j=1}^{n} x_{i,j} \otimes E_{i,j} \in \mathcal{S} \otimes M_n, \quad (3)$$

where $E_{i,j} = e_i e_j^*$ are the standard matrix units in M_n. Under this identification, the dual functional $s_\phi : M_n(\mathcal{S}) \to \mathbb{C}$ becomes

$$s_\phi([x_{i,j}]) = \langle [\phi(x_{i,j})]\mathbf{e}, \mathbf{e} \rangle,$$

where $\mathbf{e} = e_1 \oplus e_2 \cdots \oplus e_n \in \mathbb{C}^{n^2}$. It is also to be noted that the definition of $s_\phi : \mathcal{S} \otimes M_n \to \mathbb{C}$ given above, coincides with that given in [10] namely,

$$s_\phi(s \otimes x) = n(Trace(\phi(s)t(x))),$$

where $t(x)$ is the (standard) transpose of x.

Remark 1.4. *Suppose that $f : \mathcal{S} \otimes M_n \to \mathbb{C}$ is linear, then with $\phi : \mathcal{S} \to M_n$ denoting the linear map determined by*

$$\langle \phi(s)e_k, e_j \rangle = f(s \otimes e_j e_k^*), \tag{4}$$

one gets that $s_\phi = f$. It follows from equations (2) *and* (4) *that the mapping*

$$L(\mathcal{S}, M_n) \ni \phi \mapsto s_\phi \in L(\mathcal{S} \otimes M_n, \mathbb{C})$$

is bijective.

Theorem 1.5 ([10])**.** *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, $\phi : \mathcal{S} \to M_n$ be a linear map. The map $\phi : \mathcal{S} \to M_n$ is decomposable if and only if its associated dual functional $s_\phi : M_n(\mathcal{S}) \to \mathbb{C}$ satisfies $s_\phi(S) \geq 0$ whenever $S \in J_n(\mathcal{S})$.*

A second contribution of this article is to give an alternate and a more direct proof of Theorem 1.5 by using the techniques developed in Chapter 6 of [5]. This approach also yields a simpler proof of a characterization of a cp map $\phi : \mathcal{S} \to M_n$ in terms of its associated dual functional. Please see Theorem 3.2 in Section 3.

2. Preliminaries

This section contains some lemmas that will be used in the sequel.

Given an orthonormal basis $\mathcal{E}$ of a Hilbert space E, the linear map $t_\mathcal{E} : B(E) \to B(E)$ uniquely determined by the property

$$\langle t_\mathcal{E}(T)y, x \rangle = \langle Tx, y \rangle$$

for all $x, y \in \mathcal{E}$ is positive and isometric and is the **transpose on E associated to $\mathcal{E}$**.

If $\mathcal{F}$ is an orthonormal basis on a Hilbert space F, then $\mathcal{E} \otimes \mathcal{F} = \{e \otimes f : e \in \mathcal{E},\ f \in \mathcal{F}\}$ is an orthonormal basis for $E \otimes F$ and moreover,

$$t_{\mathcal{E} \otimes \mathcal{F}} = t_\mathcal{E} \otimes t_\mathcal{F}.$$

In particular, $t_\mathcal{E} \otimes t_\mathcal{F}$ is a positive map.

Given a unitary U on E, the set $\mathcal{F} = U\mathcal{E}$ is also an orthonormal basis and an elementary computation shows, for $T \in B(E)$,

$$t_\mathcal{F}(T) = Vt_\mathcal{E}(T)V^*,$$

where $V = Ut_\mathcal{E}(U)^*$ is unitary. Thus, any two transposes on E are unitarily equivalent. As a consequence, the notion of co-cp for a linear map from an operator space into $B(\mathbb{C}^n)$ is independent of the choice of transpose (basis) on $B(\mathbb{C}^n)$ (of $\mathbb{C}^n$).

Recall the identification of $\mathcal{S} \otimes M_n$ with $M_n(\mathcal{S})$ from (3). The following result (Lemma 1 in [9]) & Lemma 6.5 in [5]) explains the significance of the dual functional.

Lemma 2.1. *The linear map $\phi : \mathcal{S} \to M_n$ is positive if and only if the linear functional $s_\phi : M_n(\mathcal{S}) \to \mathbb{C}$ takes positive values on $\mathcal{S}^+ \otimes M_n^+$.*

The notion of cp and co-cp maps easily extends to maps from an operator system into $B(E)$ for a Hilbert space E.

Lemma 2.2. *Suppose E is a Hilbert space. If $\eta : \mathcal{S} \to B(E)$ is co-cp, $m \in \mathbb{N}$ and $S \in J_m(\mathcal{S})$, then $(\eta \otimes I_m)(S) \succeq 0$.*

Proof. Let t denote a transpose on $B(E)$ and t_m the standard transpose on M_m. Suppose $S = \sum_{j=1}^m s_j \otimes x_j \in J_m(\mathcal{S})$. Thus S and also $S' = \sum s_j \otimes t_m(x_j)$ are positive. Let I_m denote the identity operator on M_m. Since η is co-cp, $S' \succeq 0$, and $t \otimes t_m$ is positive, it follows that

$$0 \preceq (t \otimes t_m)(t \circ \eta \otimes I_m)(S') = (\eta \otimes t_m)(S') = (\eta \otimes I_m)(S).$$

□

Recall the standard matrix units $E_{j,k} \in M_m$. The following is a key positivity property that will be utilized to prove our main results.

Lemma 2.3. *Suppose that $m \in \mathbb{N}$, $S = \sum_{j,k=1}^m s_{j,k} \otimes E_{j,k} \in \mathcal{S} \otimes M_m$, $y_1, \dots, y_m \in \mathbb{C}^n$ and $T = \sum_{j,k=1}^m s_{j,k} \otimes y_j y_k^* \in \mathcal{S} \otimes M_n$.*
(i) If $S \succeq 0$, then $T \succeq 0$.
(ii) If $S \in J_m(\mathcal{S})$, then $T \in J_n(\mathcal{S})$.

Proof. (i) Let 1 denote the unit element in $\mathcal{S}$, $\{e_1, \dots, e_m\}$ denote the standard orthonormal basis for $\mathbb{C}^m$ and $Y = 1 \otimes \sum_{\alpha=1}^m e_\alpha y_\alpha^*$. It follows that

$$Y^* S Y = \sum_{\alpha,\beta=1}^m \sum_{j,k=1}^m s_{j,k} \otimes y_\beta \, [e_\beta^* \, E_{j,k} \, e_\alpha] \, y_\alpha^* = \sum_{\alpha,\beta=1}^m s_{\beta,\alpha} \otimes y_\beta y_\alpha^* = T,$$

since $e_\beta^* E_{j,k} e_\alpha = 1$ if $(\alpha, \beta) = (k, j)$ and 0 otherwise. Thus if $S \succeq 0$, then so is T.

(ii) Let $z_j = \overline{y_j}$, the entrywise complex conjugate. Suppose $S \in J_m(\mathcal{S})$. By definition of $J_m(\mathcal{S})$,

$$S' = (I \otimes t_m)(S) = \sum_{j,k=1}^m s_{k,j} \otimes E_{j,k} \succeq 0,$$

where I is the identity operator on $\mathcal{S}$ and t_m is the transpose on M_m. From part (i) it follows that $T \succeq 0$ and

$$T' = (I \otimes t_m)(T) = \sum_{j,k=1}^m s_{k,j} \otimes y_j y_k^* = \sum_{j,k=1}^m s_{j,k} \otimes z_j z_k^* \succeq 0.$$

Hence, $T \in J_n(\mathcal{S})$. □

Recall the dual functional s_ϕ associated to ϕ, from equation (2).

Lemma 2.4. *Suppose $\mathcal{S} \subset \mathcal{A}$ is an operator system and $\phi : \mathcal{S} \to M_n$ is a linear map. If $s \in \mathcal{S}$ and $y, z \in \mathbb{C}^n$, then*

$$\langle \phi(s)\overline{z}, \overline{y}\rangle = s_\phi(s \otimes yz^*).$$

Moreover, if $S = \sum_{j,k=1}^m s_{j,k} \otimes E_{j,k} \in \mathcal{S} \otimes M_m$ and $w = w_1 \oplus w_2 \oplus \cdots \oplus w_m = \sum_{j=1}^m e_j \otimes w_j \in \mathbb{C}^m \otimes \mathbb{C}^n$, then

$$\langle \phi_m(S)\overline{w}, \overline{w}\rangle = s_\phi\left(\sum_{j,k=1}^m s_{j,k} \otimes w_j w_k^*\right).$$

Proof. Compute

$$\begin{aligned}
s_\phi(s \otimes yz^*) &= \left\langle (\phi(s) \otimes yz^*) \sum_{j=1}^n e_j \otimes e_j, \sum_{k=1}^n e_k \otimes e_k \right\rangle \\
&= \sum_{j,k} (e_k^* \phi(s) e_j)\, (e_k^* y)(z^* e_j) \\
&= \sum_{j,k} (e_j^* \phi(s) e_k)\, (z^* e_k)(e_j^* y) \\
&= \langle \phi(s)\overline{z}, \overline{y}\rangle.
\end{aligned}$$

The second part can be obtained from the first part by linearity, as follows.

$$\begin{aligned}
s_\phi\left(\sum_{j,k=1}^m s_{j,k} \otimes w_j w_k^*\right) &= \sum_{j,k=1}^m s_\phi(s_{j,k} \otimes w_j w_k^*) \\
&= \sum_{j,k=1}^m \langle \phi(s_{j,k})\overline{w_k}, \overline{w_j}\rangle = \langle \phi_m(S)\overline{w}, \overline{w}\rangle.
\end{aligned}$$

□

3. The proofs

This section contains our main results. We begin with the following theorem which is a more elaborate version of our main Theorem 1.5 stated in Section 1. The elaboration is in the sense that it also integrates another characterization of decomposable maps on C*-algebras due to E. Størmer ([8]).

Theorem 3.1. *Let $\mathcal{A}$ be a unital C*-algebra, $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and $\phi : \mathcal{S} \to M_n$ be a linear map. The following statements are equivalent.*

(i) ϕ is decomposable.
(ii) $\phi_m(S) \succeq 0$ for all $m \in \mathbb{N}$ and $S \in J_m(\mathcal{S})$.
(iii) $\phi_n(S) \succeq 0$, for all $S \in J_n(\mathcal{S})$.
(iv) The linear functional $s_\phi : M_n(\mathcal{S}) \to \mathbb{C}$ is positive on $J_n(\mathcal{S})$.

To prove (i) $\Rightarrow$ (ii), let $\phi = \psi + \eta$, where ψ is cp and η is co-cp, $m \in \mathbb{N}$ and $S \in J_m(\mathcal{S})$. By Lemma 2.2, $(\eta \otimes I_m)(S) \succeq 0$ and by the complete positivity of ψ, $(\psi \otimes I_m)(S) \succeq 0$. Hence $\phi_m(S) \succeq 0$.

A proof of (ii) $\Rightarrow$ (i) for the case $\mathcal{S} = \mathcal{A}$, can be found in [8]. With minor modifications, the proof can be made to work for any non-trivial operator system $\mathcal{S} \subset \mathcal{A}$. Hence, we omit the proof.

The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are immediate.

Using Lemmas 2.3 and 2.4 and techniques from Chapter 6 of [5], we give a streamlined proof of (iv) implies (ii) below. Let $e_1, \ldots, e_m$ denote the standard orthonormal basis for $\mathbb{C}^m$ and $E_{j,k} = e_j e_k^*$, the resulting matrix units in M_m.

Proof of *(iv)* $\Rightarrow$ *(ii)***.** Let $m \in \mathbb{N}$, $S = \sum_{j,k=1}^m s_{j,k} \otimes E_{j,k} \in J_m(\mathcal{S})$ and $w = w_1 \oplus \cdots \oplus w_m = \sum_{j=1}^m e_j \otimes w_j \in \mathbb{C}^m \otimes \mathbb{C}^n$. From Lemma 2.4,

$$\langle \phi_m(S)\overline{w}, \overline{w} \rangle = s_\phi \left(\sum_{j,k=1}^m s_{j,k} \otimes w_j w_k^* \right) = s_\phi(T). \tag{5}$$

where $T = \sum_{j,k=1}^m s_{j,k} \otimes w_j w_k^*$. Since $S \in J_m(\mathcal{S})$, Lemma 2.3 implies that $T \in J_n(\mathcal{S})$. Thus, by hypothesis, $s_\phi(T) \succeq 0$. Since $w \in \mathbb{C}^m \otimes \mathbb{C}^n$ is arbitrary, it follows that $\phi_m(S)$ is positive and the proof is complete. □

The following theorem can be found in Chapter 6 of [5]. The proof given there uses the fact that every positive matrix in $M_k(\mathcal{A})$ is a finite sum of matrices of the form $[a_i^* a_j]$, where $a_1, \ldots, a_k \in \mathcal{A}$. It is observed that, one can obtain a proof without using this property, by using Lemma 2.3 instead, as indicated below.

Theorem 3.2. *Let $\mathcal{A}$ be a unital C*-algebra, $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and $\phi : \mathcal{S} \to M_n$ be a linear map. The following statements are equivalent.*

(i) ϕ is cp.

(ii) The linear functional $s_\phi : M_n(\mathcal{S}) \to \mathbb{C}$ is positive on $(\mathcal{S} \otimes M_n)^+$.

Proof. That (i) implies (ii) is immediate from the complete positivity of ϕ and the definition of s_ϕ. To prove (ii) implies (i), let $m \in \mathbb{N}$ and $S = \sum_{j,k=1}^m s_{j,k} \otimes E_{j,k} \in (\mathcal{S} \otimes M_m)^+$ be given. Given $w = \sum_{j,k=1}^m e_j \otimes w_j \in \mathbb{C}^m \otimes \mathbb{C}^n$, it follows from part (i) of Lemma 2.3 that $T = \sum s_{j,k} \otimes w_j w_k^* \in (\mathcal{S} \otimes M_n)^+$. Hence, using Lemma 2.4,

$$\langle \phi_m(S)\overline{w}, \overline{w} \rangle = s_\phi(T) \succeq 0.$$

Thus, $\phi_m(S)$ is positive and the result follows. □

Proof of Theorem 1.2. *(i)* $\Rightarrow$ *(ii)*: Suppose not. Choose $p \in J_n(\mathcal{S})$ such that $p \notin \overline{\mathcal{S}^+ \otimes M_n^+}$. Let $A = \{p\}$ and $B = \overline{\mathcal{S}^+ \otimes M_n^+}$. Observe that A

and B satisfy the hypotheses of the Hahn-Banach separation theorem [6, Theorem 3.4]. It follows that there exists a continuous linear functional $\Lambda : \mathcal{S} \otimes M_n \to \mathbb{C}$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$Re(\Lambda(p)) < \gamma_1 < \gamma_2 < Re(\Lambda(x))$$

for all $x \in B$. Since $0 \in B$, it must be the case that $\gamma_2 \leq 0$. Suppose that $Re(\Lambda(x_0)) < 0$ for some $x_0 \in B$. Since B is a cone, $nx_0 \in B$ for all $n \in \mathbb{N}$. The above equation implies that $Re(\Lambda(nx_0)) = nRe(\Lambda(x_0)) > \gamma_2$ for all $n \in \mathbb{N}$. This is impossible, since $\gamma_2 \leq 0$. Thus,

$$Re(\Lambda(p)) < \gamma_1 < \gamma_2 \leq 0 \leq Re(\Lambda(x)),$$

for all $x \in B$. Define $f : M_n(\mathcal{S}) \to \mathbb{C}$ by $f(x) = \frac{1}{2}\left(\Lambda(x) + \overline{\Lambda(x^*)}\right)$. Observe that f is a continuous linear functional which satisfies

$$f(p) < 0 \text{ and } f(x) \geq 0 \tag{6}$$

for all $x \in B$. By equation (4), there exists $\phi : \mathcal{S} \to M_n$ such that $f = s_\phi$. Since f is positive on B, by Lemma 2.1, it follows that $\phi : \mathcal{S} \to M_n$ is positive. Since $p \in J_n(\mathcal{S})$ and $f(p) = s_\phi(p) < 0$, it follows from Theorem 3.1 that $\phi : \mathcal{S} \to M_n$ is not decomposable, a contradiction.

$(ii) \Rightarrow (i)$: Let $\psi : \mathcal{S} \to M_n$ be a positive map. It follows from Lemma 2.1 that, s_ψ takes positive values on $\mathcal{S}^+ \otimes M_n^+$, and hence also on $\overline{\mathcal{S}^+ \otimes M_n^+}$. Since $J_n(\mathcal{S}) \subseteq \overline{\mathcal{S}^+ \otimes M_n^+}$, it follows that s_ψ takes positive values on $J_n(\mathcal{S})$. An application of Theorem 3.1 yields the decomposability of ψ, and the proof is complete. □

Following [8], we end with an application of Theorem 3.1.

Example 3.3. Consider the map $\phi : M_3 \to M_3$ defined by

$$\phi\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} x_{11} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} \end{pmatrix} + \mu \begin{pmatrix} x_{33} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & x_{22} \end{pmatrix}, \tag{7}$$

where $\mu \geq 1$. It was shown by M.D. Choi that the above map is a positive map but not decomposable (See [1] and [2]). Consider the matrix

$$A(a) := \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1/a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right). \tag{8}$$

We note that the matrix $A(a)$ is a minor refinement of the matrix that appears in page 403 of [8] and that $A(a)$ belongs to $J_3(M_3)$, if $a > 0$ (Ex. 5(a) on Page 32 of [4]). Also observe that $s_\phi(A(a)) = (a\mu - 1)$. Since $\mu \geq 1$, if one chooses $0 < a < \frac{1}{\mu}$, then it follows easily from Theorem 3.1 that $\phi : M_3 \to M_3$ is not decomposable. Since ϕ is a positive map, using Lemma 2.1, one can also conclude that the matrix $A(a)$ does not belong to $\overline{M_3^+ \otimes M_3^+}$, whenever $0 < a < \frac{1}{\mu}$.

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