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On simultaneous rational approximation to a real number and its integral powers, II

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Abstract. For a positive integer n and a real number ξ, let $\lambda_n(\xi)$ denote the supremum of the real numbers λ for which there are arbitrarily large positive integers q such that $||q\xi||, ||q\xi^2||, \ldots, ||q\xi^n||$ are all less than $q^{-\lambda}$. Here, $||\cdot||$ denotes the distance to the nearest integer. We establish new results on the Hausdorff dimension of the set of real numbers ξ such that $\lambda_n(\xi)$ is equal (or greater than or equal) to a given value.

Contents

1. Introduction

In 1932, in order to define his classification of real numbers, Mahler [15, 16] introduced the exponents of Diophantine approximation w_n, which measure how small an integer linear form in the first n powers of a given real number can be.

Definition 1.1. *Let $n \geq 1$ be an integer and ξ a real number. We denote by $w_n(\xi)$ the supremum of the real numbers w such that, for arbitrarily large real numbers X, the inequalities*

$$0 < |x_n\xi^n + \ldots + x_1\xi + x_0| \leq X^{-w}, \qquad \max_{0\leq m\leq n} |x_m| \leq X,$$

have a solution in integers $x_0, \ldots, x_n$.

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We refer to [5, 8] for an overview of the known results on the exponents w_n. In particular, it follows from the Schmidt Subspace Theorem that $w_n(\xi) = \min\{n, d-1\}$ for every positive integer n and every real algebraic number ξ of degree d. In the sequel, by spectrum of a function, we mean the set of values taken by this function at transcendental real numbers.

It is easy to apply the theory of continued fractions to show that the spectrum of w_1 is equal to the whole interval $[1, +\infty]$. Moreover, the classical Jarník–Besicovich theorem [14] asserts that, for any $w \geq 1$, we have

$$\dim\{\xi \in \mathbb{R} : w_1(\xi) \geq w\} = \dim\{\xi \in \mathbb{R} : w_1(\xi) = w\} = \frac{2}{1+w}. \tag{1}$$

Here, and throughout this paper, $1/+\infty$ is understood to be 0 and dim stands for the Hausdorff dimension. To be precise, the Jarník–Besicovich theorem concerns the set $\{\xi \in \mathbb{R} : w_1(\xi) \geq w\}$ and not the level set $\{\xi \in \mathbb{R} : w_1(\xi) = w\}$. However, we easily deduce (1) from [14]. In the sequel, we state several metric results on level sets which, sometimes, are not explicitly stated in the original papers, but whose validity is known. For $n \geq 2$, the fact that the spectrum of w_n equals $[n, +\infty]$ is an immediate consequence of the extension of (1) established in 1983 by Bernik [3], which states that

$$\dim\{\xi \in \mathbb{R} : w_n(\xi) \geq w\} = \frac{n+1}{w+1}, \tag{2}$$

for every positive integer n and every real number w with $w \geq n$.

Another exponent of Diophantine approximation, introduced in [10], measures the quality of the simultaneous rational approximation to the first n integral powers of a real number by rational numbers with the same denominator.

Definition 1.2. *Let $n \geq 1$ be an integer and ξ a real number. We denote by $\lambda_n(\xi)$ the supremum of the real numbers λ such that, for arbitrarily large real numbers X, the inequalities*

$$0 < |x_0| \leq X, \qquad \max_{1 \leq m \leq n} |x_0\xi^m - x_m| \leq X^{-\lambda}, \tag{3}$$

have a solution in integers $x_0, \ldots, x_n$.

Observe that λ_1 and w_1 coincide. The Dirichlet theorem implies that $\lambda_n(\xi)$ is at least equal to $1/n$ for every real number ξ which is not algebraic of degree at most n. Furthermore, there is equality for almost all ξ, with respect to the Lebesgue measure; see [6, 8, 17] for further results. The following question reproduces Problems 2.9 and 2.10 of the survey [7].

Problem 1.3. *Let $n \geq 1$ be an integer. Is the spectrum of the function λ_n equal to $[1/n, +\infty]$? For $\lambda \geq 1/n$, what are the Hausdorff dimensions of the set $\{\xi \in \mathbb{R} : \lambda_n(\xi) \geq \lambda\}$ and of the level set $\{\xi \in \mathbb{R} : \lambda_n(\xi) = \lambda\}$?*

The above mentioned Jarník–Besicovich theorem answers the case $n = 1$ of Problem 1.3. For $n \geq 2$, the state-of-the-art is as follows. It has been proved in [6] that, for any positive integer n and any real number λ with $\lambda \geq 1$, we can construct explicitly uncountably many real numbers ξ such that $\lambda_n(\xi) = \lambda$. Since any real number ξ such that $w_1(\xi)$ is infinite satisfies $\lambda_n(\xi) = +\infty$ (see Corollary 3.2 of [6]), we get that the spectrum of λ_n includes the interval $[1, +\infty]$.

Problem 1.3 for $n = 2$ and λ in $[1/2, 1]$ was solved completely by Beresnevich, Dickinson, Vaughan and Velani [2, 20].

Theorem 1.4. *For any real number λ with $1/2 \leq \lambda \leq 1$, we have*

$$\dim\{\xi \in \mathbb{R} : \lambda_2(\xi) \geq \lambda\} = \dim\{\xi \in \mathbb{R} : \lambda_2(\xi) = \lambda\} = \frac{2-\lambda}{1+\lambda}.$$

For $n \geq 2$ the dimension of the level sets $\{\xi \in \mathbb{R} : \lambda_n(\xi) = \lambda\}$ has been determined by Schleischitz [17] for $\lambda > 1$.

Theorem 1.5. *Let $n \geq 2$ be an integer and $\lambda > 1$ a real number. Then, we have*

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \geq \lambda\} = \dim\{\xi \in \mathbb{R} : \lambda_n(\xi) = \lambda\} = \frac{2}{n(1+\lambda)}. \tag{4}$$

Let us briefly explain the easy part of the proof of Theorem 1.5. One way to construct a good rational approximation $(\frac{p_1}{q}, \ldots, \frac{p_n}{q})$ to $(\xi, \ldots, \xi^n)$ is to start with a rational number p/q very close to ξ, that is, such that $|q\xi - p| = q^{-\lambda}$, for some $\lambda > 1$. We then observe that, for $j = 1, \ldots, n$, we have

$$|q^n \xi^j - q^{n-j} p^j| \ll_n q^{n-1} q^{-\lambda} \ll_n (q^n)^{-(\lambda-n+1)/n}, \quad j = 1, \ldots, n.$$

This gives at once the lower bound

$$\lambda_n(\xi) \geq \frac{\lambda_1(\xi) - n + 1}{n}, \tag{5}$$

which is non-trivial if $\lambda_1(\xi)$ exceeds n. In particular, it then follows from (1) that

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \geq \lambda\} \geq \dim\{\xi \in \mathbb{R} : \lambda_1(\xi) \geq n\lambda + n - 1\} = \frac{2}{n(1+\lambda)}. \tag{6}$$

This inequality is valid for all λ with $\lambda \geq 1/n$, but the lower bound is not greater than $2/(n+1)$ for $\lambda = 1/n$, thus it is far from the truth when $n \geq 2$. To establish Theorem 1.5, Schleischitz proved that, for $\lambda > 1$, all but finitely many rational n-tuples which are the best approximations of the real n-tuple $(\xi, \ldots, \xi^n)$ are of the form $(p/q, (p/q)^2, \ldots, (p/q)^n)$, that is, lie on the Veronese curve $x \mapsto (x, \ldots, x^n)$. In Section 5, we give a new, shorter (and, we believe, illuminating) proof of this assertion.

As a first observation towards Problem 1.3 for $\lambda \leq 1$, let us note that the transference inequality (due to Khintchine, see e.g. [8])

$$\lambda_n(\xi) \geq \frac{w_n(\xi)}{(n-1)w_n(\xi) + n},$$

combined with (2), shows that, for $1/n \le \lambda < 1/(n-1)$, we get

$$\begin{aligned} \dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \ge \lambda\} &\ge \dim\left\{\xi \in \mathbb{R} : w_n(\xi) \ge \frac{n\lambda}{1-\lambda(n-1)}\right\} \\ &\ge \frac{(n+1)(1-\lambda(n-1))}{1+\lambda}. \end{aligned} \tag{7}$$

This (easy) lower estimate, which applies to a very small set of values of λ, gives, unlike (6), that the Hausdorff dimension of the set $\{\xi \in \mathbb{R} : \lambda_n(\xi) \ge \lambda\}$ tends to 1 as λ tends to $1/n$. It is superseded by a deep result of Beresnevich [1] dealing with values of λ close to $1/n$.

Theorem 1.6. *Let $n \ge 2$ be an integer. Let λ be a real number with $1/n \le \lambda < 3/(2n-1)$. Then, we have*

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \ge \lambda\} \ge \frac{n+1}{\lambda+1} - (n-1). \tag{8}$$

The lower bound (8) is not surprising since we may often expect that the codimension of the intersection of two fractal sets is the sum of their codimensions. Here, we intersect the Veronese curve, of dimension 1, with the set of real n-tuples $(\xi_1, \ldots, \xi_n)$ for which there exist infinitely many integers q such that $\max_{1\le j\le n} \|q\xi_j\| < q^{-\lambda}$, where $\|\cdot\|$ denotes the distance to the nearest integer. The Hausdorff dimension of the latter set is equal to $(n+1)/(\lambda+1)$, by a result of Dodson [13].

Observe that the lower bounds in (6) and (8) coincide for $\lambda = 2/n$ and are equal to $2/(n+2)$ at this value of λ. Thus, it could be tempting to conjecture that we have equalities in (6) and (8) for $\lambda \ge 2/n$ and for $\lambda \le 2/n$, respectively. This is, however, not the case for $n \ge 3$: namely, we show that the graph of the function $\lambda \mapsto \dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \ge \lambda\}$ is more complicated and presumably composed of about n parts. Among our results, stated in Section 2, we extend the range of validity of (4) and obtain new lower and upper bounds for the Hausdorff dimension of the set of real numbers ξ such that $\lambda_n(\xi) \ge \lambda$, for $\lambda > 1/n$.

Throughout this paper, $\lfloor\cdot\rfloor$ denotes the integer part function and $\lceil\cdot\rceil$ the ceiling function. The notation $a \gg_d b$ means that a exceeds b times a constant depending only on d. When $\gg$ is written without any subscript, it means that the constant is absolute. We write $a \asymp b$ if both $a \gg b$ and $a \ll b$ hold.

2. Main results

Our first result is an extension of the range of validity of (4).

Theorem 2.1. *Let $n \ge 2$ be an integer. The spectrum of λ_n contains the interval $[(n+4)/(3n), +\infty]$. Let $\lambda \ge (n+4)/(3n)$ be a real number. Then, we have*

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) = \lambda\} = \frac{2}{n(1+\lambda)}.$$

In particular, for any real number λ with $\lambda > 1/3$, there exists an integer n_0 such that

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) = \lambda\} = \frac{2}{n(1+\lambda)},$$

for any integer n greater than n_0.

Our next result shows that the assumption '$\lambda > 1/3$' in the last assertion of Theorem 2.1 is sharp.

Theorem 2.2. *For any integer $n \geq 2$, we have*

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \geq 1/3\} \geq \frac{2}{(n-1)(1+1/3)}.$$

Theorems 2.1 and 2.2 above are special cases of the following general statement.

Theorem 2.3. *Let k, n be integers with $1 \leq k \leq n$. Let λ be a real number with $\lambda \geq 1/n$. Then we have*

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \geq \lambda\} \geq \frac{(k+1)(1-(k-1)\lambda)}{(n-k+1)(1+\lambda)}. \tag{9}$$

If $\lambda > 1/\lfloor \frac{n+1}{2} \rfloor$, then, setting $m = 1 + \lfloor 1/\lambda \rfloor$, we have

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \geq \lambda\} \leq \max_{1 \leq h \leq m} \left\{ \frac{(h+1)(1-(h-1)\lambda)}{(n-2h+2)(1+\lambda)} \right\}. \tag{10}$$

Observe that (7) is the special case $k = n$ of (9).

The lower bounds (9) have been independently obtained by Schleischitz (Theorem 4.9 of [18]) with a different proof.

Theorem 2.2 corresponds to (9) applied with $\lambda = 1/3$ and $k = 2$.

We briefly show how Theorem 2.1 follows from Theorem 2.3. For $n = 2$ it follows immediately from Theorem 1.4 and Theorem 1.5, hence we can assume that $n \geq 3$.

For n with $3 \leq n \leq 7$ and $\lambda \geq \frac{n+4}{3n}$, we have $m \leq 2$. Therefore, for $\lambda \geq \frac{n+4}{3n}$, we get by (10) that

$$\begin{aligned} \max_{1 \leq h \leq 2} \left\{ \frac{(h+1)(1-(h-1)\lambda)}{(n-2h+2)(1+\lambda)} \right\} &= \max\left\{ \frac{2}{n(1+\lambda)}, \frac{3(1-\lambda)}{(n-2)(1+\lambda)} \right\} \\ &= \frac{2}{n(1+\lambda)}. \end{aligned}$$

For $n \geq 8$ and $\lambda \geq \frac{n+4}{3n}$, we have $m \leq 3$. Therefore, for $\lambda \geq \frac{n+4}{3n}$, we get by (10) that

$$\max_{1\le h\le 3}\left\{\frac{(h+1)(1-(h-1)\lambda)}{(n-2h+2)(1+\lambda)}\right\}$$
$$=\max\left\{\frac{2}{n(1+\lambda)},\frac{3(1-\lambda)}{(n-2)(1+\lambda)},\frac{4(1-2\lambda)}{(n-4)(1+\lambda)}\right\}=\frac{2}{n(1+\lambda)}.$$

Combined with (6), this gives

$$\dim\{\xi\in\mathbb{R}:\lambda_n(\xi)\ge\lambda\}=\frac{2}{n(1+\lambda)},$$

for $\lambda\ge\frac{n+4}{3n}$. The remaining part of the proof is standard. Since the function $x\mapsto\frac{2}{x(1+\lambda)}$ is strictly decreasing, we get

$$\dim\{\xi\in\mathbb{R}:\lambda_n(\xi)=\lambda\}=\dim\bigcap_{\varepsilon>0}\{\xi\in\mathbb{R}:\lambda\le\lambda_n(\xi)\le\lambda+\varepsilon\}=\frac{2}{n(1+\lambda)}.$$

The special case $k=m$ of (9) asserts that, for any positive integer m with $m\le n$,

$$\dim\Big\{\xi\in\mathbb{R}:\lambda_n(\xi)\ge\frac{1}{m}\Big\}\ge\frac{1}{n-m+1}.$$

We believe that the graph of $\lambda\mapsto\dim\{\xi\in\mathbb{R}:\lambda_n(\xi)\ge\lambda\}$ is composed of about n parts.

Inequality (5) is a special case of Lemma 3.1 of [6], which asserts that, for any positive integers k and n with k dividing n, and, for any transcendental real number ξ, we have

$$\lambda_n(\xi)\ge\frac{k\lambda_k(\xi)-n+k}{n}. \tag{11}$$

Schleischitz [17] conjectured that (11) remains true when k is less than n but does not divide n. Our next theorem confirms this conjecture.

Theorem 2.4. *Let ξ be a real transcendental number. For any positive integer k, we have*

$$(k+1)\big(1+\lambda_{k+1}(\xi)\big)\ge k\big(1+\lambda_k(\xi)\big).$$

Consequently, for every integer n with $n\ge k$, we have

$$\lambda_n(\xi)\ge\frac{k\lambda_k(\xi)-n+k}{n}.$$

Theorem 2.4 has been established independently by Schleischitz [19], who also proved a lower estimate of $\lambda_n(\xi)$ in terms of $w_k(\xi)$, for $n\ge k$.

The first assertion of Theorem 2.4 is of interest only when $\lambda_k(\xi)>2/k$. The last assertion is obtained by repeated application of the first one. This shows at once that, if there is equality in (11), then we have

$$\lambda_m(\xi)=\frac{k\lambda_k(\xi)-m+k}{m},\quad m=k,\dots,n.$$

The present paper is organized as follows. We establish two new lower bounds for $\lambda_n(\xi)$ in Section 3. We derive (9) from one of them. The second one is Theorem 2.4 above. Section 4 is devoted to the proof of (10), which follows an original approach inspired by a paper of Davenport and Schmidt [12]. Finally, in Section 5, we give alternative proofs of some earlier results of Schleischitz, including Theorem 1.5.

3. Lower bounds for the exponents λ_n

The key ingredient for the proof of the first assertion of Theorem 2.3 is a new lower bound for $\lambda_n(\xi)$ in terms of a quantity similar to $w_k(\xi)$. For a positive integer n, we denote by w_n^{lead} the exponent of approximation defined as in Definition 1.1, but with the additional requirement that $|x_n|$ is not smaller than $\max\{|x_0|, \ldots, |x_{n-1}|\}$.

Theorem 3.1. *Let k, n be integers with $2 \leq k \leq n$. Let ξ be a real transcendental number. Then, we have*

$$\lambda_n(\xi) \geq \frac{w_k^{\mathrm{lead}}(\xi) - n + k}{(k-1) w_k^{\mathrm{lead}}(\xi) + n}. \tag{12}$$

In the first version of the present paper, we showed that $w_k^{\mathrm{lead}}(\xi)$ can be replaced by $w_k(\xi)$ in (12) when $k = 2$ or when $n = k + 1$. This has been subsequently extended to every k, n with $2 \leq k \leq n$ by Champagne and Roy [11], who made use of the invariance of w_k by linear transformations with rational coefficients.

Proof. Let k, n be integers with $2 \leq k \leq n$. Let ξ be a real transcendental number. We assume for the moment that $w_k^{\mathrm{lead}}(\xi)$ is finite and set $w_k = w_k^{\mathrm{lead}}(\xi)$. Let ε be a positive real number.

For arbitrarily large integers H, there exist integers $a_0, a_1, \ldots, a_k$, not all zero, such that $H = |a_k| = \max\{|a_0|, |a_1|, \ldots, |a_k|\}$ and

$$H^{-w_k - \varepsilon} \leq |a_k \xi^k + \ldots + a_1 \xi + a_0| \leq H^{-w_k + \varepsilon}. \tag{13}$$

Take such an integer H and set

$$\rho := a_k \xi^k + \ldots + a_1 \xi + a_0.$$

Consider the matrix

$$M := \begin{pmatrix} \xi & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \xi^2 & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \xi^k & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 \\ a_0 & a_1 & a_2 & \cdots & a_k & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{k-1} & a_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{k-1} & a_k \end{pmatrix}.$$

One can check that $|\det M| = |a_k|^{n-k}|\rho|$. Therefore, by Minkowski's Theorem, there exist integers $v_0, \ldots, v_n$, not all zero, such that

$$|v_0\xi^j - v_j| \le |a_k|^{(n-k)/k}|\rho|^{1/k}, \quad 1 \le j \le k,$$

$$|a_0 v_i + a_1 v_{i+1} + \ldots + a_k v_{i+k}| < 1, \quad 0 \le i \le n-k.$$

Since the a_j's and v_j's are integers, we get that

$$a_0 v_i + a_1 v_{i+1} + \ldots + a_k v_{i+k} = 0, \quad 0 \le i \le n-k.$$

Using $i = 0$ above, we get

$$\begin{aligned} |\rho v_0| = |(v_0\xi - v_1)a_1 + \ldots + (v_0\xi^k - v_k)a_k| \quad &\le kH|a_k|^{(n-k)/k}|\rho|^{1/k} \\ &= kH^{n/k}|\rho|^{1/k}. \end{aligned}$$

It then follows from (13) that

$$|v_0| \le kH^{n/k}|\rho|^{-(k-1)/k} \le kH^{(n+(k-1)(w_k+\varepsilon))/k}. \tag{14}$$

Furthermore, for $i = 1, \ldots, n-k$, we have

$$\begin{aligned} |v_0\xi^{i+k} - v_{i+k}| \quad = \Bigg| v_0 \left(\frac{a_{k-1}\xi^{i+k-1} + \ldots + a_1\xi^{i+1} + a_0\xi^i - \rho\xi^i}{a_k} \right) \\ - \frac{a_0 v_i + a_1 v_{i+1} + \ldots + a_{k-1} v_{i+k-1}}{a_k} \Bigg|. \end{aligned}$$

Inductively, we derive that

$$|v_0\xi^{i+k} - v_{i+k}| \ll_{n,\xi} H^{(n-k)/k}|\rho|^{1/k} \ll_{n,\xi} H^{(n-k-w_k+\varepsilon)/k}, \quad i = 1, \ldots, n-k. \tag{15}$$

We deduce at once from (14) and (15) that

$$\lambda_n(\xi) \ge \frac{w_k^{\mathrm{lead}}(\xi) - n + k}{(k-1)w_k^{\mathrm{lead}}(\xi) + n}.$$

An inspection of the proof shows that it yields $\lambda_n(\xi) \ge 1/(k-1)$ when $w_k^{\mathrm{lead}}(\xi)$ is infinite, so (12) holds in all cases. □

Proof of the first assertion of Theorem 2.3. Let k, n be integers with $1 \le k \le n$. For $k = 1$, Inequality (9) reduces to (6). For $k \ge 2$ and $\lambda \ge 1/n$, Inequality (12) implies that

$$\{\xi \in \mathbb{R} : \lambda_n(\xi) \ge \lambda\} \supset \left\{ \xi \in \mathbb{R} : w_k^{\mathrm{lead}}(\xi) \ge \frac{(\lambda+1)n - k}{1 - \lambda(k-1)} \right\}. \tag{16}$$

Bernik [3] established that

$$\dim\{\xi \in \mathbb{R} : w_k^{\mathrm{lead}}(\xi) \ge w\} = \frac{k+1}{w+1}, \tag{17}$$

for every real number w with $w \ge k$. The combination of (16) and (17) yields (9). □

Similar ideas as the ones used in the proof of Theorem 3.1 allow us to bound $\lambda_n(\xi)$ from below in terms of $\lambda_k(\xi)$, where $k \le n$.

Proof of Theorem 2.4. Write $\lambda_k = \lambda_k(\xi)$. Assume that λ_k is finite (otherwise, $\lambda_{k+1}(\xi)$ is infinite and we are done). Let ε be a positive real number. There exist arbitrarily large positive integers q such that

$$q^{-\lambda_k-\varepsilon} \le \max\{\|q\xi\|, \dots, \|q\xi^k\|\} \le q^{-\lambda_k+\varepsilon}. \tag{18}$$

Take such an integer q. For $j = 1, \dots, k$, let v_j be the integer such that $\|q\xi^j\| = |q\xi^j - v_j|$. It follows from Siegel's lemma (see Lemma 2.9.1 in [4]) that there exist integers $a_0, a_1, \dots, a_k$, not all zero, such that

$$a_0 q + a_1 v_1 + \dots + a_k v_k = 0$$

and

$$H := \max\{|a_0|, |a_1|, \dots, |a_k|\} \le_{k,\xi} q^{1/k}.$$

Then, we derive from (18) that

$$\begin{aligned} |q(a_k\xi^k + \dots + a_1\xi + a_0) - &(a_k v_k + \dots + a_1 v_1 + a_0 q)| \\ &\le kHq^{-\lambda_k+\varepsilon} \ll_{k,\xi} q^{1/k-\lambda_k+\varepsilon}. \end{aligned} \tag{19}$$

Using triangle inequalities as above, we get from (18) and (19) that

$$\begin{aligned} \|a_k q\xi^{k+1}\| &\le |a_k q\xi^{k+1} + a_{k-1}v_k + a_{k-2}v_{k-1} + \dots + a_1 v_2 + a_0 v_1| \\ &\ll_{n,\xi} q \cdot |a_k\xi^k + \dots + a_1\xi + a_0| + Hq^{-\lambda_k+\varepsilon} \\ &\ll_{n,\xi} q^{1/k-\lambda_k+\varepsilon}. \end{aligned} \tag{20}$$

It now follows from $|a_k q| \ll_{k,\xi} q^{1+1/k}$, (18), and (20) that

$$\lambda_{k+1}(\xi) \ge \frac{\lambda_k(\xi) - 1/k - \varepsilon}{1 + 1/k}.$$

As ε can be chosen arbitrarily close to 0, we deduce that

$$(k+1)\big(1 + \lambda_{k+1}(\xi)\big) \ge k\big(1 + \lambda_k(\xi)\big).$$

This concludes the proof. $\square$

4. Upper bound

Since $\lambda_n(\xi) = \lambda_n(\xi + m)$ for any integer m, we may assume that ξ is in $[1, 2)$ and therefore $\xi \asymp 1$. We investigate the $(n+1)$-tuples $\mathbf{p} := (q, p_1, p_2, \dots, p_n)$ of integers which approximate at least one point $\boldsymbol{\xi} = (\xi, \xi^2, \dots, \xi^n)$ on the Veronese curve, that is, which satisfy

$$|q\xi^i - p_i| \ll q^{-\lambda}, \quad i = 1, \dots, n. \tag{21}$$

Obviously, the condition $\xi \asymp 1$ is equivalent to $q \asymp p_1 \asymp p_2 \asymp \dots \asymp p_n$. For convenience, we will often write p_0 instead of q.

Throughout this section, we extensively make use of matrices of the form

$$\Delta_{m,k} := \begin{pmatrix} p_{k-m+1} & p_{k-m+2} & \cdots & p_k \\ p_{k-m+2} & p_{k-m+3} & \cdots & p_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ p_k & p_{k+1} & \cdots & p_{k+m-1} \end{pmatrix}.$$

Observe that $\Delta_{m,k}$ is an $m \times m$ matrix with p_k in its antidiagonal. Note also that the matrices $\Delta_{m,k}$ are precisely Hankel matrices constructed from the sequence $(p_k)_{k\in\{0,\dots,n\}}$. For a given square matrix A we denote by $|A|$ the absolute value of its determinant.

Proposition 4.1. *Assume that a tuple $\mathbf{p} = (p_0, \dots, p_n)$ in $\mathbb{Z}^{n+1}$ satisfies (21) for some real number ξ with $\xi \asymp 1$. Then, we have*

$$|p_i\xi - p_{i+1}| \ll q^{-\lambda}, \quad \text{for } i \in \{0, \dots, n-1\}, \text{ and}$$

$$|\Delta_{2,i}| \ll q^{1-\lambda}, \quad \text{for } i \in \{0, \dots, n-1\}. \tag{22}$$

Conversely, if an integer tuple $\mathbf{p}$ in $\mathbb{Z}^{n+1}$ with $p_0 \asymp p_1 \asymp \cdots \asymp p_n$ satisfies (22), then there exists a real number ξ for which (21) is true.

Proof. For the first part of the proposition, the triangle inequality gives

$$|p_i\xi - p_{i+1}| = |(q\xi^i + (p_i - q\xi^i))\xi - p_{i+1}| \le |\xi(q\xi^i - p_i)| + |q\xi^{i+1} - p_{i+1}| \ll q^{-\lambda}.$$

For the second inequality, we have

$$\begin{vmatrix} p_{i-1} & p_i \\ p_i & p_{i+1} \end{vmatrix} = \begin{vmatrix} p_{i-1} & p_i \\ p_i - \xi p_{i-1} & p_{i+1} - \xi p_i \end{vmatrix} \ll q^{1-\lambda}.$$

Finally, consider an integer tuple $\mathbf{p}$ which satisfies (22). Then, for $i = 1, \dots, n-1$, we have

$$\left| \frac{p_i}{p_{i-1}} - \frac{p_{i+1}}{p_i} \right| \ll q^{-1-\lambda}.$$

Setting $\xi := p_1/p_0$, these inequalities yield

$$\left| \xi - \frac{p_{i+1}}{p_i} \right| \ll q^{-1-\lambda}, \quad \text{thus} \quad |p_i\xi - p_{i+1}| \ll q^{-\lambda}.$$

Now we use induction on i. For $i = 0$, the statement $|q\xi - p_1| \ll q^{-\lambda}$ follows from the last estimate. Assuming that (21) is true for i, we deduce from

$$|q\xi^{i+1} - p_{i+1}| = |(q\xi^i - p_i)\xi + p_i\xi - p_{i+1}| \ll q^{-\lambda}.$$

that it is also true for $i + 1$. □

Proposition 4.1 allows us to investigate integer $(n + 1)$-tuples $\mathbf{p}$ which satisfy (22), instead of real numbers ξ with $\lambda_n(\xi) \ge \lambda$. The next proposition can be found in [12]. However for the sake of completeness we provide its proof here.

Proposition 4.2. *Let* $\mathbf{p}$ *be in* $\mathbb{Z}^{n+1}$ *which satisfies* (22). *Then, for any positive integers* m, k *with* $k - m + 1 \geq 0$ *and* $k + m - 1 \leq n$, *we have*

$$|\Delta_{m,k}| \ll q^{1-(m-1)\lambda}.$$

Proof. By Proposition 4.1, there exists a real number ξ which satisfies (21) and, in particular, such that $|p_i\xi - p_{i+1}| \ll q^{-\lambda}$, for $i = 1, \ldots, n-1$. Then,

$$|\Delta_{m,k}| = \begin{vmatrix} p_{k-m+1} & \cdots & p_k \\ p_{k-m+2} & \cdots & p_{k+1} \\ \vdots & \ddots & \vdots \\ p_k & \cdots & p_{k+m-1} \end{vmatrix}$$

is equal to

$$\begin{vmatrix} p_{k-m+1} & p_{k-m+2} & \cdots & p_k \\ p_{k-m+2} - p_{k-m+1}\xi & p_{k-m+3} - p_{k-m+2}\xi & \cdots & p_{k+1} - p_k\xi \\ \vdots & \vdots & \ddots & \vdots \\ p_k - p_{k-1}\xi & p_{k+1} - p_k\xi & \cdots & p_{k+m-1} - p_{k+m-2}\xi \end{vmatrix},$$

which, by our assumption, is clearly $\ll q^{1-(m-1)\lambda}$. □

The proof of Proposition 4.2 can easily be adapted to show the next proposition, which is more general.

Proposition 4.3. *Let* $\mathbf{p}$ *be in* $\mathbb{Z}^{n+1}$ *which satisfies* (22) *and* m *a positive integer. For* $i = 0, \ldots, n-m+1$, *let* $\mathbf{y}_i$ *denote the vector* $(p_i, p_{i+1}, \ldots, p_{i+m-1})$. *Then, for any sequence* $c_1, c_2, \ldots, c_m$ *of integers in* $\{0, \ldots, n-k+1\}$, *the determinant* $d(c_1, \ldots, c_m)$ *of the* $m \times m$ *matrix composed of the vectors* $\mathbf{y}_{c_1}, \mathbf{y}_{c_2}, \ldots, \mathbf{y}_{c_m}$ *satisfies*

$$|d(c_1, \ldots, c_m)| \ll q^{1-(m-1)\lambda}.$$

Theorem 4.4 below is a straightforward corollary of Theorem 3 of Davenport and Schmidt [12].

Theorem 4.4. *Let* $a_0, a_1, \ldots, a_h$ *be integers with no common factor throughout. Assume that, for some non-negative integers* t, k *with* $k + h - 1 \leq t$ *and* $t + h \leq n$, *the integers* $p_k, p_{k+1}, \ldots, p_{t+h}$ *are related by the recurrence relation*

$$a_0 p_i + a_1 p_{i+1} + \cdots + a_h p_{i+h} = 0, \quad k \leq i \leq t.$$

Let Z *be the maximum of the absolute values of all the* $h \times h$ *determinants formed from any* h *of the vectors* $\mathbf{y}_i := (p_i, p_{i+1}, \ldots, p_{i+h-1})$, $i = k, \ldots, t+1$. *If* Z *is non-zero, then*

$$\max\{|a_0|, |a_1|, \ldots, |a_h|\} \ll Z^{1/(t-k-h+2)}.$$

We are now in position to establish the second assertion of Theorem 2.3. We use the ideas from [12]. Let $\lambda > 1/\lfloor (n+1)/2 \rfloor$ be a real number and set $m = 1 + \lfloor 1/\lambda \rfloor$. Let ξ be a transcendental real number such that $\lambda_n(\xi) \geq \lambda$

and consider an $(n+1)$-tuple $\mathbf{p}$ for which (21) is satisfied and q is large enough.

Let h be the smallest non-negative integer number such that the matrix

$$P_h := \begin{pmatrix} p_0 & p_1 & \cdots & p_{n-h-1} & p_{n-h} \\ p_1 & p_2 & \cdots & p_{n-h} & p_{n-h+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_h & p_{h+1} & \cdots & p_{n-1} & p_n \end{pmatrix}.$$

has rank at most h. Obviously, $h \le \lceil \frac{n+1}{2} \rceil$, because for $\ell = \lceil \frac{n+1}{2} \rceil$ the matrix P_ℓ has more rows than columns and its rank is at most ℓ. Also, we have $h \ge 1$ since $\mathbf{p}$ is not the zero vector. On the other hand, for $q = p_0$ large enough, we get $h \le m$. Indeed, consider $m+1$ arbitrary columns of the matrix P_m. By Proposition 4.3, the matrix formed from these columns has determinant at most $cq^{1-m\lambda}$ for some absolute positive constant c. Since $\lambda > 1/m$, for q large enough, this determinant is zero. Since $\lambda > 1/\lfloor (n+1)/2 \rfloor$, we have

$$h \le m \le \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{23}$$

By construction of the matrix P_h, there exist integers $a_0, a_1, \ldots, a_h$ with no common factor such that

$$a_0 p_i + a_1 p_{i+1} + \cdots + a_h p_{i+h} = 0, \quad 0 \le i \le n-h. \tag{24}$$

Note that the matrix P_{h-1} has rank h and therefore the value of Z, defined in Theorem 4.4 is non-zero. Moreover, Proposition 4.3 implies that $Z \ll q^{1-(h-1)\lambda}$. From inequality (23) we have $h-1 \le n-h$ and thus all the assumptions of Theorem 4.4 are satisfied. Applied with $k=0$ and $t = n-h$, it yields

$$H := \max\{|a_0|, |a_1|, \ldots, |a_h|\} \le Z^{1/(n-2h+2)} \ll q^{\frac{1-(h-1)\lambda}{n-2h+2}}.$$

Consider the relation (24) for $i = 0$ and divide it by $p_0 = q$. Then, the condition (21) implies that

$$|a_h \xi^h + a_{h-1}\xi^{h-1} + \ldots + a_0| \ll Hq^{-1-\lambda} \ll H^{1-\frac{(1+\lambda)(n-2h+2)}{1-(h-1)\lambda}}.$$

This shows that every good approximation $\mathbf{p}$ of ξ with q large enough provides us with an integer polynomial $Q_{\mathbf{p}}(X)$ of degree at most h such that $|Q_{\mathbf{p}}(\xi)| \ll Hq^{-1-\lambda}$. Then, since ξ is transcendental, we must have infinitely many different polynomials $Q_{\mathbf{p}}(X)$ with this property. In other words,

$$\{\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}} : \lambda_n(\xi) \ge \lambda\} \subset \bigcup_{1 \le h \le m} A_h\left(\frac{(1+\lambda)(n-2h+2)}{1-(h-1)\lambda} - 1\right),$$

where $\overline{\mathbb{Q}}$ denotes the set of algebraic numbers and

$$A_h(w) := \{\xi \in \mathbb{R} \;:\; |P(\xi)| \ll H(P)^{-w} \text{ for i. m. } P \in \mathbb{Z}[x], \deg P \le h\}.$$

It then follows from (2) that

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \geq \lambda\} \leq \max_{1 \leq h \leq m} \left\{ \frac{(h+1)(1-(h-1)\lambda)}{(n-2h+2)(1+\lambda)} \right\}.$$

The proof of the last assertion of Theorem 2.3 is complete.

5. A simple proof of Theorem 1.5

Schleischitz' proof of Theorem 1.5 (see [17] and Theorem 2.5.8 of [8]) is clever, but there is a simpler argument, that we present below. The common ingredient of both proofs is the fact that, if a rational tuple is sufficiently close to the n-tuple $(\xi, \ldots, \xi^n)$, then it must lie on the Veronese curve.

Let $n \geq 2$ be an integer and ξ a real number with $\lambda_n(\xi) > 1$. Let λ be a real number with $1 < \lambda < \lambda_n(\xi)$. Then, there are arbitrarily large integers $q, p_1, \ldots, p_n$ such that

$$|q\xi^j - p_j| < q^{-\lambda}, \quad j = 1, \ldots, n.$$

Set $p_0 = q$. Observe that (as in the previous section, we denote by $|A|$ the absolute value of the determinant of a square matrix A), for $j = 1, \ldots, n-1$, we have

$$\Delta_j := \begin{vmatrix} p_{j-1} & p_j \\ p_j & p_{j+1} \end{vmatrix} = \begin{vmatrix} p_{j-1} & p_j - p_{j-1}\xi \\ p_j & p_{j+1} - p_j\xi \end{vmatrix} = |p_{j-1}(p_{j+1} - p_j\xi) - p_j(p_j - p_{j-1}\xi)|,$$

thus, by the triangle inequality,

$$\Delta_j \ll_\xi |q|^{1-\lambda}.$$

If $|q|$ is sufficiently large, then we get

$$\Delta_1 = \ldots = \Delta_{n-1} = 0,$$

which implies that there exist coprime non-zero integers a, b such that

$$\frac{p_1}{q} = \frac{p_2}{p_1} = \ldots = \frac{p_n}{p_{n-1}} = \frac{a}{b}.$$

We deduce at once that the point

$$\left(\frac{p_1}{q}, \ldots, \frac{p_n}{q}\right) = \left(\frac{a}{b}, \ldots, \left(\frac{a}{b}\right)^n\right)$$

lies on the Veronese curve $x \mapsto (x, x^2, \ldots, x^n)$ and that q is an integer multiple of b^n. In particular, we get

$$\left|\xi - \frac{p_1}{q}\right| = \left|\xi - \frac{a}{b}\right| < q^{-1-\lambda} \leq b^{-n(1+\lambda)}.$$

Since q (and, thus, b) is arbitrarily large, we deduce from the (easy half of the) Jarník–Besicovich theorem that

$$\dim\{\xi \in \mathbb{R} : \lambda_n(\xi) \geq \lambda\} \leq \frac{2}{n(1+\lambda)}.$$

Combined with (6), this gives a full proof of Theorem 1.5.

Similar arguments allow us to give an alternative proof of a result of Schleischitz asserting that the inequality

$$\widehat{\lambda}_n(\xi) \le \max\left\{\frac{1}{n}, \frac{1}{\lambda_1(\xi)}\right\} \tag{25}$$

holds, where $\widehat{\lambda}_n(\xi)$ is the supremum of the real numbers λ for which the inequalities (3) have a non-zero integer solution for all sufficiently large X. We do not claim that the proof below is simpler than the original one.

Since (25) is clearly true for $n = 1$ and for $\lambda_1(\xi) = 1$, we assume that $n \ge 2$ and $\lambda_1(\xi) > 1$. Let q be a large positive integer and v be a real number greater than 1 such that

$$\|q\xi\| = |q\xi - p| \le q^{-v}.$$

In the sequel we will let v tend to $\lambda_1(\xi)$ from below, thus we may assume that p and q are coprime. Then, we check that

$$|q^j\xi^j - p^j| \ll q^{j-1-v}, \quad 1 \le j \le n.$$

Let v' be a real number with $1 < v' < \min\{v, n\}$ and set $X = q^{v'}$. Let x be a positive integer with $x < X$. Assume that $\widehat{\lambda}_n(\xi) > 1/v'$. Then, there are integers $x_1, \ldots, x_n$ such that

$$|x\xi^j - x_j| \ll X^{-1/v'}.$$

We have

$$\begin{vmatrix} q & p \\ x & x_1 \end{vmatrix} = \begin{vmatrix} q & q\xi - p \\ x & x\xi - x_1 \end{vmatrix} \ll Xq^{-v} + qX^{-1/v'} < 1,$$

if q is large enough. As $\gcd(p, q) = 1$, we derive that q divides x. Thus, the determinant

$$\begin{vmatrix} q^2 & p^2 \\ x & x_2 \end{vmatrix}$$

is an integer multiple of q. However, it satisfies

$$\begin{vmatrix} q^2 & p^2 \\ x & x_2 \end{vmatrix} = \begin{vmatrix} q^2 & q^2\xi^2 - p^2 \\ x & x\xi^2 - x_2 \end{vmatrix} \ll Xq^{1-v} + q^2X^{-1/v'} < q.$$

Consequently, we derive that, if q is large enough, the determinant is equal to 0, hence, q^2 divides x. Continuing in the same way, we deduce that q^n divides x, a contradiction with the inequalities $1 \le x < q^n$. Since v' can be chosen arbitrarily close to $\min\{v, n\}$, we conclude that $\widehat{\lambda}_n(\xi) \le \max\{1/n, 1/v\}$. By letting v tend to $\lambda_1(\xi)$, we get (25).

These new proofs of Theorem 1.5 and (25) can be carried out in the p-adic setting to give p-adic analogues of these results, thereby extending Theorem 2.3 of [9]. Details will be given elsewhere.

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References

[1] Beresnevich, Victor. Rational points near manifolds and metric Diophantine approximation. *Ann. of Math.* (2) **175** (2012), no. 1, 187–235. MR2874641, Zbl 1264.11063, arXiv:0904.0474, doi: 10.4007/annals.2012.175.1.5. 365

[2] Beresnevich, Victor; Dickinson, Detta; Velani, Sanju. Diophantine approximation on planar curves and the distribution of rational points. With an Appendix II by Vaughan R. C. *Ann. of Math.* (2) **166** (2007), no. 2, 367–426. MR2373145, Zbl 1137.11048, arXiv:math/0401148, doi: 10.4007/annals.2007.166.367. 364

[3] Bernik, Vasiliĭ I. Application of the Hausdorff dimension in the theory of Diophantine approximations. *Acta Arith.* **42** (1983), no. 3, 219–253. MR0729734, Zbl 0482.10049, doi: 10.4064/aa-42-3-219-253. 363, 369

[4] Bombieri, Enrico; Gubler, Walter. Heights in Diophantine geometry. New Mathematical Monographs, 4. *Cambridge University Press*, 2006. xvi+652 pp. ISBN: 978-0-521-84615-8; 0-521-84615-3. MR2216774, Zbl 1115.11034, doi: 10.1017/CBO9780511542879. 370

[5] Bugeaud, Yann. Approximation by algebraic numbers. Cambridge Tracts in Mathematics, 160. *Cambridge University Press, Cambridge*, 2004. xvi+274 pp. ISBN: 0-521-82329-3. MR2136100, Zbl 1055.11002, doi: 10.1017/CBO9780511542886. 363

[6] Bugeaud, Yann. On simultaneous rational approximation to a real number and its integral powers. *Ann. Inst. Fourier (Grenoble)* **60** (2010), no. 6, 2165–2182. MR2791654, Zbl 1229.11100, doi: 10.5802/aif.2580. 363, 364, 367

[7] Bugeaud Yann. Hausdorff dimension and Diophantine approximation. *Further developments in fractals and related fields*, 35–45. Trends Math, *Birkhäuser/Springer, New York,* 2013. MR3184187, Zbl 1268.28008, doi: 10.1007/978-0-8176-8400-6_3. 363

[8] Bugeaud, Yann. Exponents of Diophantine approximation. *Dynamics and analytic number theory*, 96–135, London Math. Soc. Lecture Note Ser., 437. *Cambridge Univ. Press, Cambridge*, 2016. MR3618788, Zbl 1406.11067. 363, 364, 374

[9] Bugeaud, Yann; Budarina, Natalia; Dickinson, Detta; O'Donnell, Hugh. On simultaneous rational approximation to a p-adic number and its integral powers. *Proc. Edinb. Math. Soc.* (2) **54** (2011), no. 3, 599–612. MR2837469, Zbl 1264.11058, doi: 10.1017/S001309151000060X. 375

[10] Bugeaud, Yann; Laurent, Michel. Exponents of Diophantine approximation and Sturmian continued fractions. *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 3, 773–804. MR2149403, Zbl 1155.11333, arXiv:math/0406064, doi: 10.5802/aif.2114. 363

[11] Champagne, Jérémy; Roy, Damien. A transference inequality for rational approximation to points in geometric progression. Preprint, 2019. arXiv:1910.02817. 368

[12] Davenport, Harold; Schmidt, Wolfgang M. Approximation to real numbers by algebraic integers. *Acta Arith.* **15** (1968/69), 393–416. MR0246822, Zbl 0186.08603, doi: 10.4064/aa-15-4-393-416. 368, 371, 372

[13] Dodson, Maurice M. Hausdorff dimension, lower order and Khintchine's theorem in metric Diophantine approximation. *J. Reine Angew. Math.* **432** (1992), 69–76. MR1184759, Zbl 0749.11036, doi: 10.1515/crll.1992.432.69. 365

[14] Jarník, Vojtěch. Über die simultanen diophantische Approximationen. *Math. Z.* **33** (1931), no. 1, 505–543. MR1545226, Zbl 0001.32403, doi: 10.1007/BF01174368. 363

[15] Mahler, Kurt. Zur Approximation der Exponentialfunktionen und des Logarithmus. I. *J. Reine Angew. Math.* **166** (1932), 118–136. MR1581302, doi: 10.1515/crll.1932.166.118. 362

[16] Mahler, Kurt. Zur Approximation der Exponentialfunktionen und des Logarithmus. II. *J. Reine Angew. Math.* **166** (1932), 137–150. MR1581303, Zbl 0003.38805, doi: 10.1515/crll.1932.166.137. 362

[17] Schleischitz, Johannes. On the spectrum of Diophantine approximation constants. *Mathematika* **62** (2016), no. 1, 79–100. MR3450571, Zbl 1397.11125, arXiv:1409.1472, doi: 10.1112/S0025579315000182. 363, 364, 367, 374

[18] Schleischitz, Johannes. An equivalence principle between polynomial and simultaneous Diophantine approximation. Preprint, 2017. arXiv:1704.00055. 366

[19] Schleischitz, Johannes. Going-up theorems for simultaneous Diophantine approximation. In preparation. 367

[20] Vaughan, Robert C.; Velani, Sanju L. Diophantine approximation on planar curves: the convergence theory. *Invent. Math.* **166** (2006), no. 1, 103–124. MR2242634, Zbl 1185.11047, arXiv:math/0602225, doi: 10.1007/s00222-006-0509-9. 364

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