

# Twisted Alexander polynomials of twisted Whitehead links

Hoang-An Nguyen and Anh T. Tran

ABSTRACT. The twisted Alexander polynomial has been explicitly computed for a few classes of knots and links, including twist knots [11] and genus one two-bridge knots [18]. In this paper we compute the two-variable twisted Alexander polynomial of twisted Whitehead links for  $SL_2(\mathbb{C})$ -representations. As an application, we verify the hyperbolic torsion conjecture in [3, 14] for twisted Whitehead links. We also obtain a formula for the Reidemeister torsion of the 3-manifold obtained from  $S^3$  by  $\frac{p}{1}$ -surgery along one component of twisted Whitehead links.

## CONTENTS

1. Introduction	1241
2. Preliminaries	1244
2.1. Twisted Alexander polynomial	1244
2.2. Hyperbolic torsion conjecture	1245
2.3. Reidemeister torsion	1245
2.4. Chebyshev polynomials	1246
3. Twisted Whitehead links	1247
3.1. A lemma	1248
3.2. Proof of Theorem 1 for $W_{2n-1}$	1248
3.3. Proof of Theorem 1 for $W_{2n}$	1251
4. Proof of Corollaries 1 and 2	1253
4.1. Proof of Corollary 1	1253
4.2. Proof of Corollary 2	1255
Acknowledgements	1257
References	1257

Received June 10, 2019.

2010 *Mathematics Subject Classification*. Primary 57M27, Secondary 57M05, 57M25.

*Key words and phrases*. Reidemeister torsion, twisted Alexander polynomial, twisted Whitehead link.

The second author has been partially supported by a grant from the Simons Foundation (#354595 to AT).

## 1. Introduction

The twisted Alexander polynomial was introduced by Lin [9] for knots in  $S^3$  and by Wada [21] for finitely presented groups. It is a generalization of the classical Alexander polynomial by using linear representations and has become an effective tool in topology. We refer to the survey papers [5, 10] and references therein for details on the twisted Alexander polynomial, Reidemeister torsion and their applications. Finding an explicit formula for the twisted Alexander polynomial of knots and links for all linear representations is a challenging problem. The twisted Alexander polynomial has been explicitly computed for a few classes of knots and links, including twist knots [11] and genus one two-bridge knots [18]. In this paper we compute the two-variable twisted Alexander polynomial of twisted Whitehead links for  $SL_2(\mathbb{C})$ -representations. As an application, we obtain a formula for the Reidemeister torsion of the 3-manifold obtained by  $\frac{p}{1}$ -surgery along one component of twisted Whitehead links.

For a link  $L \subset S^3$  we denote by  $E_L = S^3 \setminus N(L)$  its exterior, where  $N(L)$  is an open tubular neighborhood of  $L$ . The link group of  $L$  is defined to be  $\pi_1(E_L)$  which is the fundamental group of the link exterior. For  $k \geq 0$ , the  $k$ -twisted Whitehead link  $W_k$  is the two-component link depicted in Figure 1. Note that  $W_0$  is the  $(2, 4)$ -torus link and  $W_1$  is the Whitehead link. Moreover,  $W_k$  is the two-bridge link  $C(2, k, 2)$  in the Conway notation and is  $\mathfrak{b}(4k + 4, 2k + 1)$  in the Schubert notation. These links are all hyperbolic except for  $W_0$ , since it is known that a two-bridge link  $\mathfrak{b}(2p, m)$  is non-hyperbolic if and only if  $m = 1$ . The link group of  $W_k$  has a standard two-generator presentation of a two-bridge link group  $\pi_1(E_{W_k}) = \langle a, b \mid aw = wa \rangle$ , where  $a, b$  are meridians depicted in Figure 1 and  $w$  is a word in the letters  $a, b$ . More precisely,  $w = (bab^{-1}a^{-1})^n a(a^{-1}b^{-1}ab)^n$  if  $k = 2n - 1$  and  $w = (bab^{-1}a^{-1})^n bab(a^{-1}b^{-1}ab)^n$  if  $k = 2n$ . For a presentation  $\rho : \pi_1(E_{W_k}) \rightarrow SL_2(\mathbb{C})$  we let  $x, y, z$  denote the traces of the images of  $a, b, ab$  respectively. We also let  $v$  denote the trace of the image of  $bab^{-1}a^{-1}$ . An explicit formula for  $v$  is given by  $v = x^2 + y^2 + z^2 - xyz - 2$ . A representation  $\rho : \pi_1(E_{W_k}) \rightarrow SL_2(\mathbb{C})$  is called nonabelian if its image is a nonabelian subgroup of  $SL_2(\mathbb{C})$ .

To state a formula for the two-variable twisted Alexander polynomial of twisted Whitehead links, we first introduce the Chebyshev polynomials of the second kind. Let  $S_k(v)$  be the Chebyshev polynomials defined by  $S_0(v) = 1$ ,  $S_1(v) = v$  and  $S_k(v) = vS_{k-1}(v) - S_{k-2}(v)$  for all integers  $k$ . We also let  $P_k(v) = \sum_{i=0}^k S_i(v)$  for  $k \geq 0$ . The polynomial  $P_k(v)$  can be expressed in terms of Chebyshev polynomials as  $P_k(v) = (S_{k+1}(v) - S_k(v) - 1)/(v - 2)$ , see Lemma 2.7. Then  $\rho : \pi_1(E_{W_k}) \rightarrow SL_2(\mathbb{C})$  is a nonabelian representation if and only if  $x, y, z$  satisfy the Riley equation  $R_k(x, y, z) = 0$ , where

$$\begin{aligned} R_{2n-1}(x, y, z) &= (xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v))S_{n-1}(v), \\ R_{2n}(x, y, z) &= (zS_n(v) - (xy - z)S_{n-1}(v))(S_n(v) - S_{n-1}(v)). \end{aligned}$$

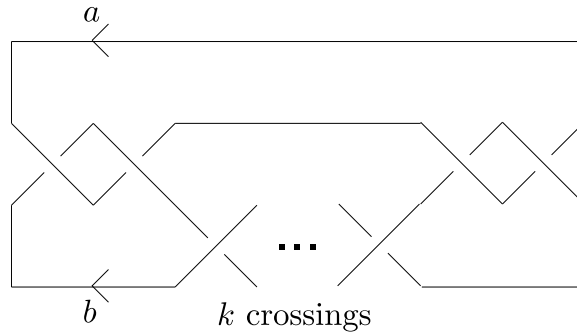


FIGURE 1. The  $k$ -twisted Whitehead link  $W_k$ .

Moreover, by [17], the defining equation for the geometric component of the  $SL_2(\mathbb{C})$ -character variety of  $W_k$  is  $xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v)$  if  $k = 2n - 1$  and is  $zS_n(v) - (xy-z)S_{n-1}(v)$  if  $k = 2n$ . Here a geometric component of the  $SL_2(\mathbb{C})$ -character variety of a hyperbolic link is a component that contains the character of a lift of the holonomy representation of the link.

The following formula for the two-variable twisted Alexander polynomial of twisted Whitehead links is the main result of this paper.

**Theorem 1.** *The two-variable twisted Alexander polynomial of  $W_k$  for a nonabelian representation  $\rho : \pi_1(E_{W_k}) \rightarrow SL_2(\mathbb{C})$  is given by*

$$\begin{aligned} & \Delta_{W_{2n-1}}^\rho(t_1, t_2) \\ = & (P_{n-1}(v) + P_{n-2}(v))(t_1^2 t_2^2 - yt_1^2 t_2 - xt_1 t_2^2 + t_1^2 + t_2^2 - xt_1 - yt_2 + 1) \\ & + 2(zS_{n-1}(v) + xyP_{n-2}(v))t_1 t_2 \end{aligned}$$

and

$$\begin{aligned} & \Delta_{W_{2n}}^\rho(t_1, t_2) \\ = & (P_n(v) + P_{n-1}(v))(t_1^2 t_2^2 + 1) + (P_{n-1}(v) + P_{n-2}(v))(t_1^2 + t_2^2) \\ & - 2P_{n-1}(v)(yt_1^2 t_2 + xt_1 t_2^2 - xy t_1 t_2 + xt_1 + yt_2). \end{aligned}$$

**Example 1.** Consider the arithmetic two-bridge links  $5_1^2$  and  $6_3^2$  in Rolfsen's table, which are the twisted Whiteheads links  $W_1 = C(2, 1, 2)$  and  $W_2 = C(2, 2, 2)$  respectively.

For  $5_1^2 = W_1$ ,  $\rho : \pi_1(E_{W_1}) \rightarrow SL_2(\mathbb{C})$  is a nonabelian representation if and only if  $x, y, z$  satisfy the Riley equation  $R_1(x, y, z) = 0$ , where  $R_1(x, y, z) = xy - zv = xy - (x^2 + y^2 - 2)z + xyz^2 - z^3$ . In this case we have

$$\Delta_{W_1}^\rho(t_1, t_2) = t_1^2 t_2^2 - yt_1^2 t_2 - xt_1 t_2^2 + t_1^2 + t_2^2 - xt_1 - yt_2 + 1 + 2zt_1 t_2.$$

For  $6_3^2 = W_2$ ,  $\rho : \pi_1(E_{W_2}) \rightarrow SL_2(\mathbb{C})$  is a nonabelian representation if and only if  $x, y, z$  satisfy the Riley equation  $R_2(x, y, z) = 0$ , where  $R_2(x, y, z) = (zv - (xy - z))(v - 1) = (z^3 - xyz^2 + (x^2 + y^2 - 1)z - xy)(x^2 + y^2 + z^2 - xyz - 3)$ .

In this case we have

$$\begin{aligned} \Delta_{W_2}^\rho(t_1, t_2) &= (x^2 + y^2 + z^2 - xyz)(t_1^2 t_2^2 + 1) + t_1^2 + t_2^2 \\ &\quad - 2(yt_1^2 t_2 + xt_1 t_2^2 - xyt_1 t_2 + xt_1 + yt_2). \end{aligned}$$

The hyperbolic torsion conjecture of Dunfield, Friedl and Jackson [3] states that for a hyperbolic knot  $K$  in  $S^3$ , the twisted Alexander polynomial associated to a lift of the holonomy representation detects the genus and fiberedness of  $K$ . The conjecture was generalized to hyperbolic links by Morifuji and the second author [14]. This generalized conjecture states that for an oriented hyperbolic link  $L$  in  $S^3$ , the twisted Alexander polynomial associated to a lift of the holonomy representation detects the Thurston norm and fiberedness of  $L$ . The hyperbolic torsion conjecture has been verified for all hyperbolic knots with at most 15 crossings [3] and most double twist knots/links (see [12], [13], [14]). Moreover, Agol and Dunfield [1] showed that the twisted Alexander polynomial detects the genus of libroid hyperbolic knots.

As a corollary of Theorem 1 we will show the following.

**Corollary 1.** *The hyperbolic torsion conjecture holds true for twisted Whitehead links.*

Let  $M_{k,p}$  be the 3-manifold obtained from  $S^3$  by  $\frac{p}{1}$ -surgery along the component of  $W_k$  corresponding the meridian  $a$ . Note that  $M_{1,p}$  is the tunnel number one once-punctured torus bundle studied by Baker and Petersen in [2]. Its character variety and twisted Alexander polynomial (and hence its Reidemeister torsion) were computed in [2]. The following gives a formula for the Reidemeister torsion of  $M_{k,p}$  for all  $k \geq 1$ .

**Corollary 2.** *Suppose  $\rho : \pi_1(E_{W_k}) \rightarrow SL_2(\mathbb{C})$  is a nonabelian representation which extends to a representation  $\rho : \pi_1(M_{k,p}) \rightarrow SL_2(\mathbb{C})$ . Then the Reidemeister torsion of  $M_{k,p}$  for  $\rho$  is given by*

$$\begin{aligned} &\tau_{M_{2n-1,p}}^\rho \\ &= 2[(2-x-y+z)S_{n-1}(v) + (4-2x-2y+xy)P_{n-2}(v)]/(2-x) \end{aligned}$$

and

$$\tau_{M_{2n,p}}^\rho = 2[S_n(v) - S_{n-1}(v) + (4-2x-2y+xy)P_{n-1}(v)]/(2-x).$$

This paper is organized as follows. In Section 2 we review the twisted Alexander polynomial, hyperbolic torsion conjecture, Reidemeister torsion and Chebyshev polynomials. In Section 3 we compute the two-variable twisted Alexander polynomial of twisted Whitehead links and prove Theorem 1. Finally, in Section 4 we prove Corollaries 1 and 2.

## 2. Preliminaries

In this section we review the twisted Alexander polynomial, hyperbolic torsion conjecture, Reidemeister torsion and Chebyshev polynomials. For more details, see [9, 21, 5, 10, 20, 6, 7, 3, 14].

**2.1. Twisted Alexander polynomial.** Let  $L = K_1 \cup \cdots \cup K_m$  be an oriented  $m$ -component link in  $S^3$  and  $E_L = S^3 \setminus N(L)$  its exterior, where  $N(L)$  is an open tubular neighborhood of  $L$ . We choose a Wirtinger presentation of the link group of  $L$ :

$$\pi_1(E_L) = \langle x_1, \dots, x_l \mid r_1, \dots, r_{l-1} \rangle.$$

The abelianization homomorphism

$$\mathfrak{a} : \pi_1(E_L) \rightarrow H_1(E_L; \mathbb{Z}) \cong \mathbb{Z}^{\oplus m} = \langle t_1 \rangle \oplus \cdots \oplus \langle t_m \rangle$$

is given by assigning to each generator  $x_i$  a meridian element  $t_j \in H_1(E_L; \mathbb{Z})$  of its corresponding component  $K_j$ . Here we denote the sum in  $\mathbb{Z}$  multiplicatively.

We consider a linear representation  $\rho : \pi_1(E_L) \rightarrow SL_2(\mathbb{C})$ . The maps  $\rho$  and  $\mathfrak{a}$  naturally induce two ring homomorphisms  $\tilde{\rho} : \mathbb{Z}[\pi_1(E_L)] \rightarrow M_2(\mathbb{C})$  and  $\tilde{\mathfrak{a}} : \mathbb{Z}[\pi_1(E_L)] \rightarrow \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ , where  $\mathbb{Z}[\pi_1(E_L)]$  is the group ring of  $\pi_1(E_L)$  and  $M_2(\mathbb{C})$  is the matrix algebra of degree 2 over  $\mathbb{C}$ . Then the tensor product  $\tilde{\rho} \otimes \tilde{\mathfrak{a}}$  defines a ring homomorphism  $\mathbb{Z}[\pi_1(E_L)] \rightarrow M_2(\mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])$ . Let  $F_l$  denote the free group on generators  $x_1, \dots, x_l$  and  $\Phi : \mathbb{Z}[F_l] \rightarrow M_2(\mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])$  the composition of the surjection  $\tilde{\phi} : \mathbb{Z}[F_l] \rightarrow \mathbb{Z}[\pi_1(E_L)]$  induced by the presentation of  $\pi_1(E_L)$  and the map  $\tilde{\rho} \otimes \tilde{\mathfrak{a}} : \mathbb{Z}[\pi_1(E_L)] \rightarrow M_2(\mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])$ .

Let  $A$  denote the  $(l-1) \times l$  matrix whose  $(i, j)$ -entry is the  $2 \times 2$  matrix  $\Phi \left( \frac{\partial r_i}{\partial x_j} \right) \in M_2(\mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])$ , where  $\frac{\partial}{\partial x}$  denotes the Fox differential. For  $1 \leq j \leq l$ , we denote by  $A_j$  the  $(l-1) \times (l-1)$  matrix obtained from  $A$  by removing the  $j$ th column. We regard  $A_j$  as a  $2(l-1) \times 2(l-1)$  matrix with coefficients in  $\mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ . Then Wada's twisted Alexander polynomial [21] of the link  $L$  for a representation  $\rho : \pi_1(E_L) \rightarrow SL_2(\mathbb{C})$  is defined to be the rational function

$$\Delta_L^\rho(t_1, \dots, t_m) = \frac{\det A_j}{\det \Phi(1 - x_j)}$$

and is well-defined up to multiplication by  $t_1^{k_1} \cdots t_m^{k_m}$  ( $k_i \in \mathbb{Z}$ ). It is known that the rational function  $\Delta_L^\rho(t_1, \dots, t_m)$  will be a Laurent polynomial in the variables  $t_1, \dots, t_m$  if  $L$  is a link with two or more components [21, Proposition 9], or  $L$  is a knot  $K$  and  $\rho$  is a nonabelian representation [8, Theorem 3.1].

When  $t_1 = \cdots = t_m = t$ , we denote  $\Delta_L^\rho(t_1, \dots, t_m)$  by  $\Delta_L^\rho(t)$  and call it the reduced twisted Alexander polynomial of  $L$ .

**2.2. Hyperbolic torsion conjecture.** Let  $M$  be a compact connected orientable 3-manifold and  $\sigma \in H^1(M; \mathbb{Z})$ . The *Thurston norm* of  $\sigma$  is defined as

$$\|\sigma\|_T = \min\{\chi_-(S) \mid S \subset M \text{ properly embedded surface dual to } \sigma\},$$

where for a given surface  $S$  with connected components  $S_1 \cup \dots \cup S_k$ , we define  $\chi_-(S) = \sum_{i=1}^k \max\{-\chi(S_i), 0\}$ . Here  $\chi(S)$  denotes the Euler characteristic of a surface  $S$ . Thurston [15] showed that  $\|\cdot\|_T$  defines a seminorm on  $H^1(M; \mathbb{Z})$ .

For a hyperbolic knot  $K$  in  $S^3$ , Dunfield, Friedl and Jackson [3] studied the twisted Alexander polynomial  $\Delta_{K,\rho_0}(t)$  associated to the holonomy representation  $\rho_0 : \pi_1(E_K) \rightarrow SL_2(\mathbb{C})$  and called it the hyperbolic torsion polynomial of  $K$ . Based on many computations, they conjectured that  $\Delta_{K,\rho_0}(t)$  determines the genus of  $K$ , in the sense that  $\deg \Delta_{K,\rho_0}(t) = 4g(K) - 2$ . Moreover, the knot  $K$  is fibered if and only if  $\Delta_{K,\rho_0}(t)$  is monic. Here, the degree of a Laurent polynomial  $f(t) \in \mathbb{C}[t^{\pm 1}]$  is the difference between the highest degree and the lowest degree of  $f$ .

Morifuji and the second author generalized the conjecture to the case of hyperbolic links. For a  $\mu$ -component oriented link  $L$ , let  $\omega \in H^1(E_L; \mathbb{Z})$  be given by sending each meridian of  $L$  to one. The generalized conjecture in [14] states that for an oriented hyperbolic link  $L$  in  $S^3$ , the twisted Alexander polynomial  $\Delta_{L,\rho_0}(t)$  associated to the holonomy representation  $\rho_0 : \pi_1(E_L) \rightarrow SL_2(\mathbb{C})$  determines the Thurston norm of  $\omega$ , in the sense that  $\deg \Delta_{L,\rho_0}(t) = 2\|\omega\|_T$ . Moreover, the link  $L$  is fibered if and only if  $\Delta_{L,\rho_0}(t)$  is monic. We should remark that  $\deg \Delta_{L,\rho}(t) \leq 2\|\omega\|_T$  holds for any representation  $\rho : \pi_1(E_L) \rightarrow SL_2(\mathbb{C})$ , see [4].

For a knot or a  $\mu$ -component alternating link  $L$  in  $S^3$ , it is known that the Thurston norm  $\|\omega\|_T$  is equal to  $2g(L) + \mu - 2$ , see e.g. [14, Remark 3.4]. Hence the equality in the above conjecture becomes  $\deg \Delta_{L,\rho_0}(t) = 4g(L) + 2(\mu - 2)$  for alternating links.

**2.3. Reidemeister torsion.**

**2.3.1. Torsion of a chain complex.** Let  $C$  be a chain complex of finite dimensional vector spaces over  $\mathbb{C}$ :

$$C = \left( 0 \rightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0 \right)$$

such that for each  $i = 0, 1, \dots, m$  the followings hold

- the homology group  $H_i(C)$  is trivial, and
- a preferred basis  $c_i$  of  $C_i$  is given.

Let  $B_i \subset C_i$  be the image of  $\partial_{i+1}$ . For each  $i$  choose a basis  $b_i$  of  $B_i$ . The short exact sequence of  $\mathbb{C}$ -vector spaces

$$0 \rightarrow B_i \rightarrow C_i \xrightarrow{\partial_i} B_{i-1} \rightarrow 0$$

implies that a new basis of  $C_i$  can be obtained by taking the union of the vectors of  $b_i$  and some lifts  $\tilde{b}_{i-1}$  of the vectors  $b_{i-1}$ . Define  $[(b_i \cup \tilde{b}_{i-1})/c_i]$  to be the determinant of the matrix expressing  $(b_i \cup \tilde{b}_{i-1})$  in the basis  $c_i$ . Note that this scalar does not depend on the choice of the lift  $\tilde{b}_{i-1}$  of  $b_{i-1}$ . The *torsion* of  $C$  is then defined to be

$$\tau(C) := \prod_{i=0}^m [(b_i \cup \tilde{b}_{i-1})/c_i]^{(-1)^{i+1}} \in \mathbb{C} \setminus \{0\}.$$

**Remark 2.1.** Once a preferred basis of  $C$  is given,  $\tau(C)$  is independent of the choice of  $b_0, \dots, b_m$ . See e.g. [20].

**2.3.2. Reidemeister torsion of a CW-complex.** Let  $M$  be a finite CW-complex and  $\rho : \pi_1(M) \rightarrow SL_2(\mathbb{C})$  a representation. Denote by  $\tilde{M}$  the universal covering of  $M$ . The fundamental group  $\pi_1(M)$  acts on  $\tilde{M}$  as deck transformations. Then the chain complex  $C(\tilde{M}; \mathbb{Z})$  has the structure of a chain complex of left  $\mathbb{Z}[\pi_1(M)]$ -modules.

Let  $V$  be the 2-dimensional vector space  $\mathbb{C}^2$  with the canonical basis  $\{e_1, e_2\}$ . Using the representation  $\rho$ ,  $V$  has the structure of a right  $\mathbb{Z}[\pi_1(M)]$ -module which we denote by  $V_\rho$ . Define the chain complex  $C(M; V_\rho)$  to be  $V_\rho \otimes_{\mathbb{Z}[\pi_1(M)]} C(\tilde{M}; \mathbb{Z})$ , and choose a preferred basis of  $C(M; V_\rho)$  as follows. Let  $\{u_1^i, \dots, u_{m_i}^i\}$  be the set of  $i$ -cells of  $M$ , and choose a lift  $\tilde{u}_j^i$  of each cell. Then  $\{\tilde{u}_1^i \otimes e_1, \tilde{u}_1^i \otimes e_2, \dots, \tilde{u}_{m_i}^i \otimes e_1, \tilde{u}_{m_i}^i \otimes e_2\}$  is chosen to be the preferred basis of  $C_i(M; V_\rho)$ .

The Reidemeister torsion  $\tau_M^\rho$  is defined as follows:

$$\tau_M^\rho = \begin{cases} \tau(C(M; V_\rho)) & \text{if } \rho \text{ is acyclic,} \\ 0 & \text{otherwise.} \end{cases}$$

Here a representation  $\rho$  is called *acyclic* if all the homology groups  $H_i(M; V_\rho)$  are trivial.

When  $M$  is the exterior  $E_L$  of a link  $L \subset S^3$  and  $\rho : \pi_1(M) \rightarrow SL_2(\mathbb{C})$  is a representation, we also denote  $\tau_M^\rho$  by  $\tau_L^\rho$  and call it the Reidemeister torsion of  $L$  for  $\rho$ . In which case Johnson showed the following. See also [7].

**Theorem 2.2.** [6] *For any representation  $\rho : \pi_1(E_L) \rightarrow SL_2(\mathbb{C})$  we have*

$$\tau_L^\rho = \Delta_L^\rho(1, \dots, 1).$$

**2.4. Chebyshev polynomials.** Recall that  $S_k(v)$  are the Chebyshev polynomials of the second kind defined by  $S_0(v) = 1$ ,  $S_1(v) = v$  and  $S_k(v) = vS_{k-1}(v) - S_{k-2}(v)$  for all integers  $k$ . Similarly, let  $T_k(v)$  be the Chebyshev polynomials of the first kind defined by  $T_0(v) = 2$ ,  $T_1(v) = v$  and  $T_k(v) = vT_{k-1}(v) - T_{k-2}(v)$  for all integers  $k$ . Note that  $T_k(v) = S_k(v) - S_{k-2}(v)$ . For  $k \geq 0$  we let  $P_k(v) = \sum_{i=0}^k S_i(v)$ .

The following four lemmas are taken from [14].

**Lemma 2.3.** Write  $v = q + q^{-1}$ . Then  $T_k(v) = q^k + q^{-k}$ .

We have  $S_k(2) = k + 1$  and  $S_k(-2) = (-1)^k(k + 1)$ . If  $q \neq \pm 1$  then  $S_k(v) = \frac{q^{k+1} - q^{-(k+1)}}{q - q^{-1}}$ . In particular, if  $v = 2 \cos \beta$ , where  $\frac{\beta}{\pi} \in \mathbb{R} \setminus \mathbb{Z}$ , then  $S_k(v) = \frac{\sin(k+1)\beta}{\sin \beta}$ .

**Lemma 2.4.** (i) If  $k \geq 1$  is even, then

$$T_k(v) - 2 = (v - 2)(v + 2) \prod_{j=1}^{\frac{k}{2}-1} \left( v - 2 \cos \frac{2j\pi}{k} \right)^2.$$

(ii) If  $k \geq 1$  is odd, then

$$T_k(v) - 2 = (v - 2) \prod_{j=1}^{\frac{k-1}{2}} \left( v - 2 \cos \frac{2j\pi}{k} \right)^2.$$

**Lemma 2.5.** (i) If  $k \geq 1$  is odd, then

$$T_k(v) - v = (v - 2)(v + 2) \prod_{j=1}^{\frac{k-3}{2}} \left( v - 2 \cos \frac{2j\pi}{k-1} \right) \prod_{j=1}^{\frac{k-1}{2}} \left( v - 2 \cos \frac{2j\pi}{k+1} \right).$$

(ii) If  $k \geq 1$  is even, then

$$T_k(v) - v = (v - 2) \prod_{j=1}^{\frac{k-2}{2}} \left( v - 2 \cos \frac{2j\pi}{k-1} \right) \prod_{j=1}^{\frac{k}{2}} \left( v - 2 \cos \frac{2j\pi}{k+1} \right).$$

**Lemma 2.6.** For  $k \in \mathbb{Z}$  we have  $S_{k+1}(v)S_{k-1}(v) + 1 = S_k^2(v)$ .

We also need the following two lemmas which are proved in [19].

**Lemma 2.7.** For  $k \geq 0$  we have

- (1)  $P_k(v) = (S_{k+1}(v) - S_k(v) - 1)/(v - 2)$ .
- (2)  $P_{k+1}(v) + P_{k-1}(v) = vP_k(v) + 1$ .
- (3)  $P_k^2(v) + P_{k-1}^2(v) = vP_k(v)P_{k-1}(v) + P_k(v) + P_{k-1}(v)$ .

**Lemma 2.8.** Suppose  $V \in SL_2(\mathbb{C})$  and  $v = \text{tr } V$ . For  $k \geq 0$  we have

$$\sum_{i=0}^k V^i = P_k(v)\mathbf{1} - P_{k-1}(v)V^{-1}$$

where  $\mathbf{1}$  denotes the identity  $2 \times 2$  matrix.

### 3. Twisted Whitehead links

In this section we compute the two-variable twisted Alexander polynomial of twisted Whitehead links and prove Theorem 1. We start with a linear algebra lemma which is useful for computing the twisted Alexander polynomial.



**3.1. A lemma.** For a  $2 \times 2$  matrix  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  we let

$$A^* = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

denote the adjugate of  $A$ . Note that

$$\operatorname{tr} AB^* = \operatorname{tr} A^*B = a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12}$$

$$\text{for } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

**Lemma 3.1.** *Suppose  $A, B, C \in M_2(\mathbb{C})$  and  $x, y \in \mathbb{C}$ . Then*

$$\begin{aligned} \det(Ax + By + C) &= (\det A)x^2 + (\det B)y^2 + (\operatorname{tr} AB^*)xy \\ &\quad + (\operatorname{tr} AC^*)x + (\operatorname{tr} BC^*)y + \det C. \end{aligned}$$

**Proof.** Write  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ ,  $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$  and  $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ .

We have

$$\begin{aligned} &\det(Ax + By + C) \\ &= \det \begin{bmatrix} xa_{11} + yb_{11} + c_{11} & xa_{12} + yb_{12} + c_{12} \\ xa_{21} + yb_{21} + c_{21} & xa_{22} + yb_{22} + c_{22} \end{bmatrix} \\ &= (xa_{11} + yb_{11} + c_{11})(xa_{22} + yb_{22} + c_{22}) \\ &\quad - (xa_{12} + yb_{12} + c_{12})(xa_{21} + yb_{21} + c_{21}) \\ &= (a_{11}a_{22} - a_{12}a_{21})x^2 + (b_{11}b_{22} - b_{12}b_{21})y^2 \\ &\quad + (a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12})xy \\ &\quad + (a_{11}c_{22} + a_{22}c_{11} - a_{12}c_{21} - a_{21}c_{12})x \\ &\quad + (b_{11}c_{22} + b_{22}c_{11} - b_{12}c_{21} - b_{21}c_{12})y \\ &\quad + c_{11}c_{22} - c_{12}c_{21}. \end{aligned}$$

The lemma then follows.  $\square$

**3.2. Proof of Theorem 1 for  $W_{2n-1}$ .** Recall that the link group of  $W_{2n-1}$  has a presentation  $\pi_1(E_{W_{2n-1}}) = \langle a, b \mid wa = aw \rangle$ , where  $a, b$  are meridians depicted in Figure 1 and  $w = (bab^{-1}a^{-1})^n a(a^{-1}b^{-1}ab)^n$ . Suppose  $\rho : \pi_1(E_{W_{2n-1}}) \rightarrow SL_2(\mathbb{C})$  is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix} \quad (1)$$

where  $(u, s_1, s_2) \in (\mathbb{C}^*)^3$  satisfies  $\rho(aw) = \rho(wa)$ . By [17] this matrix equation is equivalent to the following equation

$$(xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v))S_{n-1}(v) = 0. \quad (2)$$

Here  $x, y, z$  and  $v$  are the traces of the images of  $a, b, ab$  and  $bab^{-1}a^{-1}$  respectively.

Let  $r = awa^{-1}w^{-1}$ . We have

$$\frac{\partial r}{\partial b} = a \left( \frac{\partial w}{\partial b} - wa^{-1}w^{-1} \frac{\partial w}{\partial b} \right) = a(1 - wa^{-1}w^{-1}) \frac{\partial w}{\partial b}.$$

Hence  $\Delta_{W_{2n-1}}^\rho(t_1, t_2) = \det \Phi \left( \frac{\partial r}{\partial b} \right) / \det \Phi(1 - a) = \det \Phi \left( \frac{\partial w}{\partial b} \right)$ .

Let  $c = bab^{-1}a^{-1}$  and  $d = a^{-1}b^{-1}ab$ . We have  $w = c^n a d^n$  and

$$\begin{aligned} \frac{\partial w}{\partial b} &= \delta_{n-1}(c) \frac{\partial c}{\partial b} + c^n a \delta_{n-1}(d) \frac{\partial d}{\partial b} \\ &= c^n (\delta_{n-1}(c^{-1})(aba^{-1}b^{-1} - a) + a \delta_{n-1}(d)(a^{-1}b^{-1}a - a^{-1}b^{-1})) \end{aligned}$$

where  $\delta_k(g) = 1 + g + \dots + g^k$ .

For  $g \in \pi_1(E_{W_{2n-1}})$  we denote  $\rho(g)$  by  $\tilde{g}$ . Since  $\text{tr } \tilde{c} = \text{tr } \tilde{d} = v$ , by Lemma 2.8 we have

$$\begin{aligned} \delta_{n-1}(\tilde{c}^{-1}) &= P_{n-1}(v)\mathbf{1} - P_{n-2}(v)\tilde{c}, \\ \delta_{n-1}(\tilde{d}) &= P_{n-1}(v)\mathbf{1} - P_{n-2}(v)\tilde{d}^{-1}. \end{aligned}$$

Since  $\Phi(a) = t_1 \tilde{a}$  and  $\Phi(b) = t_2 \tilde{b}$ , we then obtain

$$\Phi \left( \frac{\partial w}{\partial b} \right) = \tilde{c}^n (P_{n-1}(v)E - P_{n-2}(v)F)$$

where

$$\begin{aligned} E &= \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} - t_1\tilde{a} + t_1t_2^{-1}\tilde{b}^{-1}\tilde{a} - t_2^{-1}\tilde{b}^{-1}, \\ F &= \tilde{c}(\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} - t_1\tilde{a}) + \tilde{a}\tilde{d}^{-1}(t_1t_2^{-1}\tilde{a}^{-1}\tilde{b}^{-1}\tilde{a} - t_2^{-1}\tilde{a}^{-1}\tilde{b}^{-1}) \\ &= \mathbf{1} - t_1\tilde{b}\tilde{a}\tilde{b}^{-1} + t_1t_2^{-1}\tilde{a}\tilde{b}^{-1} - t_2^{-1}\tilde{a}\tilde{b}^{-1}\tilde{a}^{-1}. \end{aligned}$$

By Lemma 3.1 we have

$$\begin{aligned} \det \Phi \left( \frac{\partial w}{\partial b} \right) &= \det(P_{n-1}(v)E - P_{n-2}(v)F) \\ &= (\det E)P_{n-1}^2(v) + (\det F)P_{n-2}^2(v) \\ &\quad - (\text{tr } EF^*)P_{n-1}(v)P_{n-2}(v). \end{aligned}$$

Since  $P_{n-1}^2(v) + P_{n-2}^2(v) = vP_{n-1}(v)P_{n-2}(v) + P_{n-1}(v) + P_{n-2}(v)$  (by Lemma 2.7) we get

$$\begin{aligned} \det \Phi \left( \frac{\partial w}{\partial b} \right) &= (\det F - \det E)P_{n-2}^2(v) \\ &\quad + (v \det E - \text{tr } EF^*)P_{n-1}(v)P_{n-2}(v) \\ &\quad + (\det E)(P_{n-1}(v) + P_{n-2}(v)). \end{aligned} \tag{3}$$

Using the form of  $\rho$  in (1) we have  $E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$  where

$$\begin{aligned} e_{11} &= 1 - s_1 t_1 - s_2^{-1} t_2^{-1} + s_1 s_2^{-1} t_1 t_2^{-1} + (s_1^{-1} s_2^{-1} - s_1 s_2^{-1} + s_1 s_2) u + u^2, \\ e_{12} &= s_1 - s_1 s_2^2 - t_1 + s_2^{-1} t_1 t_2^{-1} - s_2 u, \\ e_{21} &= (-s_2^{-1} + s_1^{-2} s_2^{-1} + t_2^{-1} - s_1 t_1 t_2^{-1}) u + s_1^{-1} u^2, \\ e_{22} &= 1 - s_1^{-1} t_1 - s_2 t_2^{-1} + s_1^{-1} s_2 t_1 t_2^{-1} - (s_1^{-1} s_2 + t_1 t_2^{-1}) u, \end{aligned}$$

and  $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$  where

$$\begin{aligned} f_{11} &= 1 - s_1 t_1 - s_2^{-1} t_2^{-1} + s_1 s_2^{-1} t_1 t_2^{-1} + (s_2 t_1 + s_1^{-1} t_2^{-1} - t_1 t_2^{-1}) u, \\ f_{12} &= -s_2^2 t_1 + s_1 s_2^{-1} t_2^{-1} - s_1 s_2 t_2^{-1} + s_2 t_1 t_2^{-1} - t_2^{-1} u, \\ f_{21} &= (s_1^{-1} s_2^{-1} t_1 - s_1 s_2^{-1} t_1 + s_1^{-2} t_2^{-1} - s_1^{-1} t_1 t_2^{-1}) u + t_1 u^2, \\ f_{22} &= 1 - s_1^{-1} t_1 - s_2 t_2^{-1} + s_1^{-1} s_2 t_1 t_2^{-1} - (s_2 t_1 + s_1^{-1} t_2^{-1}) u. \end{aligned}$$

Then, by direct calculations, we have

$$\det F - \det E = (v - 2)(2z - xy)t_1 t_2^{-1}, \quad (4)$$

$$v \det E - \operatorname{tr} EF^* = (v - 2)(xy - vz)t_1 t_2^{-1}, \quad (5)$$

$$\begin{aligned} \det E &= t_1^2 t_2^{-2} - y t_1^2 t_2^{-1} - x t_1 t_2^{-2} + t_1^2 + t_2^{-2} \\ &\quad + (xy - (v - 2)z)t_1 t_2^{-1} - x t_1 - y t_2^{-1} + 1. \end{aligned} \quad (6)$$

Note that  $x = s_1 + s_1^{-1}$ ,  $y = s_2 + s_2^{-1}$ ,  $z = s_1 s_2 + s_1^{-1} s_2^{-1} + u$  and  $v = x^2 + y^2 + z^2 - xyz - 2$ .

From equations (4) and (5) we have

$$\begin{aligned} &(\det F - \det E)P_{n-2}^2(v) + (v \det E - \operatorname{tr} EF^*)P_{n-1}(v)P_{n-2}(v) \quad (7) \\ &= [(2z - xy)P_{n-2}(v) + (xy - vz)P_{n-1}(v)](v - 2)P_{n-2}(v)t_1 t_2^{-1} \\ &= [(xy - 2z)(P_{n-1}(v) - P_{n-2}(v)) - z(v - 2)P_{n-1}(v)] \\ &\quad \times (v - 2)P_{n-2}(v)t_1 t_2^{-1} \\ &= [(xy - 2z)S_{n-1}(v) - z(S_n(v) - S_{n-1}(v) - 1)] \\ &\quad \times [S_{n-1}(v) - S_{n-2}(v) - 1]t_1 t_2^{-1} \\ &= [(xy - z)S_{n-1}(v) - zS_n(v) + z][S_{n-1}(v) - S_{n-2}(v) - 1]t_1 t_2^{-1}. \end{aligned}$$

Here we use  $(v - 2)P_k(v) = S_{k+1}(v) - S_k(v) - 1$  (by Lemma 2.7) in the third equality.

From equations (3), (6) and (7) we have

$$\begin{aligned} \Delta_{W_{2n-1}}^\rho(t_1, t_2) &= (t_1^2 t_2^{-2} - y t_1^2 t_2^{-1} - x t_1 t_2^{-2} + t_1^2 + t_2^{-2} - x t_1 - y t_2^{-1} + 1) \\ &\quad \times (P_{n-1}(v) + P_{n-2}(v)) + \alpha t_1 t_2^{-1}. \end{aligned}$$

where

$$\alpha = [(xy - z)S_{n-1}(v) - zS_n(v) + z][S_{n-1}(v) - S_{n-2}(v) - 1] + (xy - (v - 2)z)(P_{n-1}(v) + P_{n-2}(v)).$$

Finally, we simplify  $\alpha$  using equation (2). By Lemmas 2.6 and 2.7 we have  $S_n(v)S_{n-2}(v) + 1 = S_{n-1}^2(v)$  and  $(v - 2)(P_{n-1}(v) + P_{n-2}(v)) = S_n(v) - S_{n-2}(v) - 2$ . Hence

$$\begin{aligned} \alpha &= (xy - z)S_{n-1}^2(v) - S_{n-1}(v)[(xy - z)(S_{n-2}(v) + 1) + z(S_n(v) - 1)] \\ &\quad + z(S_n(v) - 1)(S_{n-2}(v) + 1) - z(S_n(v) - S_{n-2}(v) - 2) \\ &\quad + xy(P_{n-1}(v) + P_{n-2}(v)) \\ &= (xy - z)S_{n-1}^2(v) - S_{n-1}(v)[(xy - z)S_{n-2}(v) + zS_n(v)] \\ &\quad - (xy - 2z)S_{n-1}(v) + xy(S_{n-1}(v) + 2P_{n-2}(v)) \\ &\quad + z(S_n(v)S_{n-2}(v) + 1) \\ &= S_{n-1}(v)[xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v)] \\ &\quad + 2zS_{n-1}(v) + 2xyP_{n-2}(v) \\ &= 2zS_{n-1}(v) + 2xyP_{n-2}(v). \end{aligned}$$

Here we use equation (2) in the last equality.

The proof of Theorem 1 for  $W_{2n-1}$  is complete, since  $\Delta_{W_{2n-1}}^\rho(t_1, t_2)$  is well-defined up to multiplication by  $t_1^{k_1}t_2^{k_2}$  ( $k_i \in \mathbb{Z}$ ).

**3.3. Proof of Theorem 1 for  $W_{2n}$ .** Recall that the link group of  $W_{2n}$  has a presentation  $\pi_1(E_{W_{2n}}) = \langle a, b \mid wa = aw \rangle$ , where  $a, b$  are meridians depicted in Figure 1 and  $w = (bab^{-1}a^{-1})^n bab(a^{-1}b^{-1}ab)^n$ . Suppose  $\rho : \pi_1(E_{W_{2n}}) \rightarrow SL_2(\mathbb{C})$  is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix}$$

where  $(u, s_1, s_2) \in (\mathbb{C}^*)^3$  satisfies  $\rho(aw) = \rho(wa)$ . By [17] this matrix equation is equivalent to the following equation

$$(zS_n(v) - (xy - z)S_{n-1}(v))(S_n(v) - S_{n-1}(v)) = 0. \tag{8}$$

As in the case of  $W_{2n-1}$  we have  $\Delta_{W_{2n}}^\rho(t_1, t_2) = \det \Phi \left( \frac{\partial w}{\partial b} \right)$  where

$$\begin{aligned} \frac{\partial w}{\partial b} &= c^n [\delta_{n-1}(c^{-1})(aba^{-1}b^{-1} - a) + bab \delta_{n-1}(d)(a^{-1}b^{-1}a - a^{-1}b^{-1}) \\ &\quad + 1 + ba]. \end{aligned}$$

Since  $\Phi(a) = t_1 \tilde{a}$  and  $\Phi(b) = t_2 \tilde{b}$ , we obtain

$$\Phi \left( \frac{\partial w}{\partial b} \right) = \tilde{c}^n (P_{n-1}(v)G - P_{n-2}(v)H + I)$$

where

$$\begin{aligned} G &= \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} - t_1\tilde{a} + t_1t_2\tilde{b}\tilde{a}\tilde{b}^{-1}\tilde{b}^{-1}\tilde{a} - t_2\tilde{b}\tilde{a}\tilde{b}^{-1}\tilde{b}^{-1}, \\ H &= \mathbf{1} - t_1\tilde{b}\tilde{a}\tilde{b}^{-1} + t_1t_2\tilde{b}\tilde{a} - t_2\tilde{b}, \\ I &= \mathbf{1} + t_1t_2\tilde{b}\tilde{a}. \end{aligned}$$

By Lemmas 3.1 and 2.7 we obtain

$$\begin{aligned} \det \Phi \left( \frac{\partial w}{\partial b} \right) &= \det(P_{n-1}(v)G - P_{n-2}(v)H + I) \tag{9} \\ &= (\det G)P_{n-1}^2(v) + (\det H)P_{n-2}^2(v) + \det I \\ &\quad - (\operatorname{tr} GH^*)P_{n-1}(v)P_{n-2}(v) \\ &\quad + (\operatorname{tr} GI^*)P_{n-1}(v) - (\operatorname{tr} HI^*)P_{n-2}(v) \\ &= (\det G - \det H)P_{n-1}^2(v) + \det I \\ &\quad + (v \det H - \operatorname{tr} GH^*)P_{n-1}(v)P_{n-2}(v) \\ &\quad + (\det H + \operatorname{tr} GI^*)P_{n-1}(v) \\ &\quad + (\det H - \operatorname{tr} HI^*)P_{n-2}(v). \end{aligned}$$

By direct calculations we have

$$\det G - \det H = (v - 2)(vz - xy)t_1t_2, \tag{10}$$

$$v \det H - \operatorname{tr} GH^* = (v - 2)(xy - 2z)t_1t_2, \tag{11}$$

$$\begin{aligned} \det H + \operatorname{tr} GI^* &= (v + 1)t_1^2t_2^2 - 2xt_1t_2^2 - 2yt_1^2t_2 + t_1^2 + t_2^2 + 1 \tag{12} \\ &\quad + (xy + 2(v - 1)z)t_1t_2 - 2xt_1 - 2yt_2 + v, \end{aligned}$$

$$\det H - \operatorname{tr} HI^* = -t_1^2t_2^2 + t_1^2 + t_2^2 + (xy - 2z)t_1t_2 - 1, \tag{13}$$

$$\det I = t_1^2t_2^2 + zt_1t_2 + 1. \tag{14}$$

From equations (10) and (11) we have

$$\begin{aligned} &(\det G - \det H)P_{n-1}^2(v) + (v \det H - \operatorname{tr} GH^*)P_{n-1}(v)P_{n-2}(v) \tag{15} \\ &= [(vz - xy)P_{n-1}(v) + (xy - 2z)P_{n-2}(v)](v - 2)P_{n-1}(v)t_1t_2 \\ &= [z(v - 2)P_{n-1}(v) - (xy - 2z)(P_{n-1}(v) - P_{n-2}(v))](v - 2)P_{n-1}(v)t_1t_2 \\ &= [z(S_n(v) - S_{n-1}(v) - 1) - (xy - 2z)S_{n-1}(v)](v - 2)P_{n-1}(v)t_1t_2 \\ &= [zS_n(v) - (xy - z)S_{n-1}(v) - z][S_n(v) - S_{n-1}(v) - 1]t_1t_2. \end{aligned}$$

From equations (9), (15), (12), (13) and (14) we obtain

$$\begin{aligned} \Delta_{W_{2n}}^\rho(t_1, t_2) &= ((v + 1)P_{n-1}(v) - P_{n-2}(v) + 1)(t_1^2t_2^2 + 1) \\ &\quad + (P_{n-1}(v) + P_{n-2}(v))(t_1^2 + t_2^2) \\ &\quad - 2P_{n-1}(v)(yt_1^2t_2 + xt_1t_2^2 + xt_1 + yt_2) + \beta t_1t_2 \end{aligned}$$

where

$$\begin{aligned} \beta &= [zS_n(v) - (xy - z)S_{n-1}(v) - z][S_n(v) - S_{n-1}(v) - 1] \\ &\quad + (xy + 2(v - 1)z)P_{n-1}(v) + (xy - 2z)P_{n-2}(v) + z. \end{aligned}$$

Note that  $(v + 1)P_{n-1}(v) - P_{n-2}(v) + 1 = P_n(v) + P_{n-1}(v)$ , by Lemma 2.7. Finally, we simplify  $\beta$  using equation (8). Indeed, since

$$(zS_n(v) - (xy - z)S_{n-1}(v))(S_n(v) - S_{n-1}(v)) = 0$$

we have

$$\begin{aligned} \beta &= [z - (zS_n(v) - (xy - z)S_{n-1}(v)) - z(S_n(v) - S_{n-1}(v))] \\ &\quad + (xy + 2(v - 1)z)P_{n-1}(v) + (xy - 2z)(P_{n-1}(v) - S_{n-1}(v)) + z \\ &= 2z - 2zS_n(v) + 2zS_{n-1}(v) + 2(xy + (v - 2)z)P_{n-1}(v) \\ &= 2xyP_{n-1}(v). \end{aligned}$$

Here we use  $(v - 2)P_{n-1}(v) = S_n(v) - S_{n-1}(v) - 1$  (by Lemma 2.7) in the last equality.

This completes the proof of Theorem 1 for  $W_{2n}$ .

#### 4. Proof of Corollaries 1 and 2

**4.1. Proof of Corollary 1.** We prove the hyperbolic torsion conjecture for  $W_{2n-1}$ . The proof for  $W_{2n}$  is similar. Up to mirror image, there are two orientations on  $W_{2n-1}$  which correspond to the two cases  $t_1 = t_2 = t$  and  $t_1 = t_2^{-1} = t$  in the reduced twisted Alexander polynomial  $\Delta_{W_{2n-1}}(t)$ . In both cases, the genus of  $L = W_{2n-1}$  is given by  $g(L) = 1$  and  $L$  is fibered if and only if  $n = 1$ . (These facts can be proved by computing the reduced Alexander polynomial of  $L$  and then applying [16, Theorem 1.1].) Moreover, by Theorem 1, the degree of the reduced twisted Alexander polynomial  $\Delta_{W_{2n-1}}^p(t)$  is 4 and the coefficient of the highest degree term is  $P_{n-1}(v) + P_{n-2}(v)$ . Note that the hyperbolic torsion conjecture holds true for the fibered link  $W_1$  since  $P_{n-1}(v) + P_{n-2}(v) = 1$  when  $n = 1$ . We now consider the case  $n \geq 2$ .

For a link  $L$  in  $S^3$ , a non-abelian representation  $\rho : \pi_1(E_L) \rightarrow SL_2(\mathbb{C})$  is called parabolic if the images of all the meridians of  $L$  by  $\rho$  are matrices with trace 2. When  $L$  is hyperbolic, it is known that the holonomy representation  $\rho_0$  is a parabolic representation. Moreover,  $\rho_0$  lies on the geometric component of the  $SL_2(\mathbb{C})$ -character variety of  $L$ .

To prove Corollary 1 for  $W_{2n-1}$ , with  $n \geq 2$ , it suffices to show that  $P_{n-1}(v) + P_{n-2}(v) \neq 0$  and  $P_{n-1}(v) + P_{n-2}(v) \neq 1$  for any parabolic representation on the geometric component of  $W_{2n-1}$ . By [17] the defining equation for the geometric component of  $W_{2n-1}$  is  $xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v) = 0$ . Hence, for a parabolic representation on the geometric component we have  $x = y = 2$  and  $4S_{n-1}(v) - (4 - z)S_{n-2}(v) - zS_n(v) = 0$ , where  $v = x^2 + y^2 + z^2 - xyz - 2 = (z - 2)^2 + 2$ . By Lemma 2.7 we have

$$P_{n-1}(v) + P_{n-2}(v) = (S_n(v) - S_{n-2}(v) - 2)/(v - 2) = (T_n(v) - 2)/(v - 2).$$

**4.1.1. Genus.** We show that  $P_{n-1}(v) + P_{n-2}(v) \neq 0$  for any parabolic representation on the geometric component of  $W_{2n-1}$ , where  $n \geq 2$ .

By Lemma 2.4 we have  $\frac{T_n(v)-2}{v-2} = 0$  if and only if  $v = 2 \cos \frac{2j\pi}{n}$  for some  $1 \leq j \leq \frac{n}{2}$ . In particular,  $v \in \mathbb{R}$  and  $-2 \leq v < 2$ .

Suppose  $v = 2 \cos \frac{2j\pi}{n}$  for some  $1 \leq j < \frac{n}{2}$ . Then, by Lemma 2.3,

$$S_{n-2}(v) = \frac{\sin(n-1)\frac{2j\pi}{n}}{\sin \frac{2j\pi}{n}} = -1, \quad S_{n-1}(v) = \frac{\sin n \frac{2j\pi}{n}}{\sin \frac{2j\pi}{n}} = 0,$$

$$S_n(v) = \frac{\sin(n+1)\frac{2j\pi}{n}}{\sin \frac{2j\pi}{n}} = 1.$$

Equation  $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$  then implies that  $z = 2$ . Hence  $v = (z-2)^2 + 2 = 2$ . This contradicts  $-2 \leq v < 2$ .

Suppose  $v = -2$  (in this case  $n$  must be even). Then

$$S_{n-2}(v) = (-1)^{n-2}(n-1) = n-1, \quad S_{n-1}(v) = (-1)^{n-1}n = -n$$

and  $S_n(v) = (-1)^n(n+1) = n+1$ . Equation  $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$  then implies that  $z = 4n-2$ . Hence  $v = (z-2)^2 + 2 = (4n-4)^2 + 2 \geq 2$ . This contradicts  $-2 \leq v < 2$ .

**4.1.2. Fiberedness.** We show that  $P_{n-1}(v) + P_{n-2}(v) \neq 1$  for any parabolic representation on the geometric component of  $W_{2n-1}$ , where  $n \geq 2$ .

By Lemma 2.5 we have  $\frac{T_n(v)-2}{v-2} = 1$  if and only if  $v = 2 \cos \frac{2j\pi}{n-1}$  for some  $1 \leq j \leq \frac{n-1}{2}$ , or  $v = 2 \cos \frac{2k\pi}{n+1}$  for some  $1 \leq k \leq \frac{n+1}{2}$ . In particular,  $v \in \mathbb{R}$  and  $-2 \leq v < 2$ .

Suppose  $v = 2 \cos \frac{2j\pi}{n-1}$  for some  $1 \leq j < \frac{n-1}{2}$ . Then, by Lemma 2.3, we have

$$S_{n-2}(v) = \frac{\sin(n-1)\frac{2j\pi}{n-1}}{\sin \frac{2j\pi}{n-1}} = 0, \quad S_{n-1}(v) = \frac{\sin n \frac{2j\pi}{n-1}}{\sin \frac{2j\pi}{n-1}} = 1,$$

$$S_n(v) = \frac{\sin(n+1)\frac{2j\pi}{n-1}}{\sin \frac{2j\pi}{n-1}} = v.$$

Equation  $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$  then implies that  $0 = 4 - zv = 4 - 6z + 4z^2 - z^3 = 0$ . This is equivalent to  $(2-z)(z^2 - 2z + 2) = 0$ . Then  $z \in \{2, 1 \pm i\}$ . Hence  $v = (z-2)^2 + 2 \in \{2, 2 \mp 2i\}$ . This contradicts  $-2 \leq v < 2$ .

Suppose  $v = 2 \cos \frac{2j\pi}{n+1}$  for some  $1 \leq j < \frac{n+1}{2}$ . Then, by Lemma 2.3, we have

$$S_{n-2}(v) = \frac{\sin(n-1)\frac{2j\pi}{n+1}}{\sin \frac{2j\pi}{n+1}} = -v, \quad S_{n-1}(v) = \frac{\sin n \frac{2j\pi}{n+1}}{\sin \frac{2j\pi}{n+1}} = -1,$$

$$S_n(v) = \frac{\sin(n+1)\frac{2j\pi}{n+1}}{\sin \frac{2j\pi}{n+1}} = 0.$$

Equation  $4S_{n-1}(v) - (4 - z)S_{n-2}(v) - zS_n(v) = 0$  then implies that  $0 = -4 + (4 - z)v = 20 - 22z + 8z^2 - z^3 = 0$ . This is equivalent to  $(2 - z)(z^2 - 6z + 10) = 0$ . Then  $z \in \{2, 3 \pm i\}$ . Hence  $v = (z - 2)^2 + 2 \in \{2, 2 \pm 2i\}$ . This contradicts  $-2 \leq v < 2$ .

Suppose  $v = -2$  (in this case  $n$  must be odd). Then

$$S_{n-2}(v) = (-1)^{n-2}(n - 1) = -(n - 1), \quad S_{n-1}(v) = (-1)^{n-1}n = n$$

and  $S_n(v) = (-1)^n(n + 1) = -(n + 1)$ . Equation  $4S_{n-1}(v) - (4 - z)S_{n-2}(v) - zS_n(v) = 0$  then implies that  $z = 4n - 2$ . Hence  $v = (z - 2)^2 + 2 = (4n - 4)^2 + 2 \geq 2$ . This contradicts  $-2 \leq v < 2$ .

**4.2. Proof of Corollary 2.** We prove Corollary 2 for  $W_{2n-1}$ . The proof for  $W_{2n}$  is similar. Let  $M_{2n-1,p}$  be the 3-manifold obtained from  $S^3$  by  $\frac{p}{1}$ -surgery along the component of  $W_{2n-1}$  corresponding the meridian  $a$ . Suppose  $\rho : \pi_1(E_{W_{2n-1}}) \rightarrow SL_2(\mathbb{C})$  is a nonabelian representation of the form

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix}$$

where

$$(xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v))S_{n-1}(v) = 0.$$

Here  $x = s_1 + s_1^{-1}$ ,  $y = s_2 + s_2^{-1}$ ,  $z = s_1s_2 + s_1^{-1}s_2^{-1} + u$  and  $v = x^2 + y^2 + z^2 - xyz - 2$  are the traces of the images of  $a$ ,  $b$ ,  $ab$  and  $bab^{-1}a^{-1}$  respectively.

The following determines when the representation  $\rho : \pi_1(E_{W_{2n-1}}) \rightarrow SL_2(\mathbb{C})$  extends to a representation  $\rho : \pi_1(M_{2n-1,p}) \rightarrow SL_2(\mathbb{C})$ .

**Lemma 4.1.**  $\rho$  extends to a representation  $\rho : \pi_1(M_{2n-1,p}) \rightarrow SL_2(\mathbb{C})$  if and only if one of the following holds:

- $S_{n-1}(v) = 0$ ,  $s_1^p = 1$  and  $s_1 \neq \pm 1$ ,
- $xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v) = 0$ ,  $s_1 = -1$ ,  $\frac{2y}{y+z} = -p$  and  $p$  is odd,
- $xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v) = 0$ ,  $\frac{s_1y-z}{-s_1^{-1}y+z} = s_1^{-p}$  and  $s_1 \neq \pm 1$ .

In all cases we have  $x \neq 2$  (i.e.  $s_1 \neq 1$ ).

**Proof.** Let  $\lambda_a$  be the canonical longitude corresponding to the meridian  $a$  of  $W_{2n-1}$ . By [17] we have

$$\rho(\lambda_a) = \begin{bmatrix} l_a & * \\ 0 & l_a^{-1} \end{bmatrix},$$

where

$$l_a = \begin{cases} 1 & \text{if } S_{n-1}(v) = 0, \\ \frac{s_1y-z}{-s_1^{-1}y+z} & \text{if } xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v) = 0, \end{cases} \quad (16)$$



and

$$* = \begin{cases} 0 & \text{if } S_{n-1}(v) = 0, \\ \frac{y(xy-2z)}{xyz-y^2-z^2} & \text{if } xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v) = 0. \end{cases} \quad (17)$$

Note that there is a small error in [17]: the canonical longitude in [17] is actually the inverse of the canonical longitude.

The nonabelian representation  $\rho : \pi_1(E_{W_{2n-1}}) \rightarrow SL_2(\mathbb{C})$  extends to a representation  $\rho : \pi_1(M_{2n-1,p}) \rightarrow SL_2(\mathbb{C})$  if and only if  $a^p\lambda_a = \mathbf{1}$ . Note that  $\rho(a^p) = \begin{bmatrix} s_1^p & S_{p-1}(x) \\ 0 & s_1^{-p} \end{bmatrix}$ , where  $S_{p-1}(x) = \frac{s_1^p - s_1^{-p}}{s_1 - s_1^{-1}}$  if  $s_1 \neq \pm 1$  and  $S_{p-1}(x) = (\pm 1)^{p-1}p$  if  $s_1 = \pm 1$ . Hence the matrix equation  $a^p\lambda_a = \mathbf{1}$  is equivalent to  $l_a = s_1^{-p}$  and  $* = -S_{p-1}(x)$ .

Case 1: If  $S_{n-1}(v) = 0$ , then  $l_a = 1$  and  $* = 0$  by (16) and (17) respectively. Hence  $a^p\lambda_a = \mathbf{1}$  is equivalent to  $s_1^{-p} = 1$  and  $S_{p-1}(x) = 0$ , which means that  $s_1^p = 1$  and  $s_1 \neq \pm 1$ .

Case 2: If  $xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v) = 0$ , then

$$l_a = \frac{s_1y - z}{-s_1^{-1}y + z} \quad \text{and} \quad * = \frac{y(xy - 2z)}{xyz - y^2 - z^2}$$

by (16) and (17) respectively. Hence  $a^p\lambda_a = \mathbf{1}$  is equivalent to

$$\frac{s_1y - z}{-s_1^{-1}y + z} = s_1^{-p} \quad \text{and} \quad \frac{y(xy - 2z)}{xyz - y^2 - z^2} = -S_{p-1}(x). \quad (18)$$

If  $s_1 = 1$ , then  $\frac{s_1y-z}{-s_1^{-1}y+z} = s_1^{-p}$  does not hold. Hence  $a^p\lambda_a \neq \mathbf{1}$ .

If  $s_1 = -1$ , then  $\frac{s_1y-z}{-s_1^{-1}y+z} = s_1^{-p}$  holds if and only if  $p$  is odd. In which case, the second equation of (18) is equivalent to  $\frac{2y}{y+z} = -p$ .

If  $s_1 \neq \pm 1$  then the first equation of (18) implies the second one. Indeed, if  $\frac{s_1y-z}{-s_1^{-1}y+z} = s_1^{-p}$  then

$$S_{p-1}(x) = \frac{s_1^p - s_1^{-p}}{s_1 - s_1^{-1}} = \frac{\frac{-s_1^{-1}y+z}{s_1y-z} - \frac{s_1y-z}{-s_1^{-1}y+z}}{s_1 - s_1^{-1}} = -\frac{y(xy - 2z)}{xyz - y^2 - z^2}.$$

In this case  $a^p\lambda_a = \mathbf{1}$  is equivalent to  $\frac{s_1y-z}{-s_1^{-1}y+z} = s_1^{-p}$ . □

We now finish the proof of Corollary 2 for  $W_{2n-1}$ . Suppose the non-abelian representation  $\rho : \pi_1(E_{W_{2n-1}}) \rightarrow SL_2(\mathbb{C})$  extends to a representation  $\rho : \pi_1(M_{2n-1,p}) \rightarrow SL_2(\mathbb{C})$ . By Lemma 4.1,  $x \neq 2$ . Since the core of the attaching solid torus is homotopic to  $a$ , by the gluing formula for the Reidemeister torsion (see e.g. [6]) we have

$$\tau_{M_{2n-1,p}}^\rho = \frac{\tau_{W_{2n-1}}^\rho}{2 - \text{tr } \rho(a)} = \frac{\tau_{W_{2n-1}}^\rho}{2 - x}. \quad (19)$$

By Theorem 2.2 we have  $\tau_{W_{2n-1}}^\rho = \Delta_{W_{2n-1}}^\rho(1, 1)$ . Theorem 1 then implies that

$$\tau_{W_{2n-1}}^\rho = 2[(2 - x - y + z)S_{n-1}(v) + (4 - 2x - 2y + xy)P_{n-2}(v)].$$

This, together with equation (19), completes the proof of Corollary 2 for  $W_{2n-1}$ .

## Acknowledgements

The authors would like to thank the referee for helpful comments and suggestions.

## References

- [1] AGOL, IAN; DUNFIELD, NATHAN M. Certifying the Thurston norm via  $SL(2, \mathbb{C})$ -twisted homology. arXiv:1501.02136. 1243
- [2] BAKER, KENNETH L.; PETERSEN, KATHLEEN L. Character varieties of once-punctured torus bundles with tunnel number one. *Internat. J. Math.* **24** (2013), no. 6, 1350048, 57 pp. MR3078072, Zbl 1275.57016, arXiv:1211.4479, doi:10.1142/S0129167X13500481. 1243
- [3] DUNFIELD, NATHAN M.; FRIEDL, STEFAN; JACKSON, NICHOLAS. Twisted Alexander polynomials of hyperbolic knots. *Exp. Math.* **21** (2012), no. 4, 329–352. MR3004250, Zbl 1266.57008, arXiv:1108.3045, doi:10.1080/10586458.2012.669268. 1240, 1243, 1244, 1245
- [4] FRIEDL, STEFAN; KIM, TAEHEE. The Thurston norm, fibered manifolds and twisted Alexander polynomials. *Topology* **45** (2006), no. 6, 929–953. MR2263219, Zbl 1105.57009, arXiv:math/0505594, doi:10.1016/j.top.2006.06.003. 1245
- [5] FRIEDL, STEFAN; VIDUSSI, STEFANO. A survey of twisted Alexander polynomials. *The mathematics of knots*, 45–94, Contrib. Math. Comput. Sci., 1. Springer, Heidelberg, 2011. ISBN: 978-3-642-15636. MR2777847, Zbl 1223.57012, arXiv:0905.0591, doi:10.1007/978-3-642-15637-3. 1241, 1244
- [6] JOHNSON, DENNIS. A geometric form of Casson invariant and its connection to Reidemeister torsion. Unpublished lecture notes. 1244, 1246, 1256
- [7] KITANO, TERUAKI. Twisted Alexander polynomial and Reidemeister torsion. *Pacific J. Math.* **174** (1996), 431–442. MR1405595, Zbl 0863.57001 doi:10.2140/pjm.1996.174.431. 1244, 1246
- [8] KITANO, TERUAKI; MORIFUJI, TAKAYUKI. Divisibility of twisted Alexander polynomials and fibered knots. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **4** (2005), no. 1, 179–186. MR2165406, Zbl 1117.57004. 1244
- [9] LIN, XIAO S. Representations of knot groups and twisted Alexander polynomials. *Acta Math. Sin. (Engl. Ser.)* **17** (2001), no. 3, 361–380. MR1852950, Zbl 0986.57003, doi:10.1007/s101140100122. 1241, 1244
- [10] MORIFUJI, TAKAYUKI. Representations of knot groups into  $SL_2(\mathbb{C})$  and twisted Alexander polynomials. Handbook of Group Actions, Adv. Lect. Math. (ALM) **31**, Int. Press, Somerville, MA, 2015, 527–576. MR3380341. 1241, 1244
- [11] MORIFUJI, TAKAYUKI. Twisted Alexander polynomials of twist knots for non-abelian representations. *Bull. Sci. Math.* **132** (2008), no. 5, 439–453. MR2426646, Zbl 1147.57018, doi:10.1016/j.bulsci.2008.04.001. 1240, 1241
- [12] MORIFUJI, TAKAYUKI. On a conjecture of Dunfield, Friedl and Jackson. *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 19–20, 921–924. MR2990904, Zbl 1253.57007, arXiv:1209.4239, doi:10.1016/j.crma.2012.10.013. 1243

- [13] MORIFUJI, TAKAYUKI; TRAN ANH T. Twisted Alexander polynomials of 2-bridge knots for parabolic representations. *Pacific J. Math.* **269** (2014), no. 2, 433–451. MR3238485, Zbl 1305.57027, arXiv:1301.1101, doi: 10.2140/pjm.2014.269.433. 1243
- [14] MORIFUJI, TAKAYUKI; TRAN ANH T. Twisted Alexander polynomials of hyperbolic links. *Publ. Res. Inst. Math. Sci.* **53** (2017), no. 3, 395–418. MR3669741, Zbl 1379.57021, arXiv:1606.06360, doi: 10.4171/PRIMS/53-3-3. 1240, 1243, 1244, 1245, 1246
- [15] THURSTON, WILLIAM P. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.* **59** (1986), no. 39, i–vi and 99–130. MR0823443, Zbl 0585.57006. 1245
- [16] MURASUGI, KUNIO. On a certain subgroup of the group of an alternating link. *Amer. J. Math.* **85** (1963), 544–550. MR157375, Zbl 0117.17201, doi: 10.2307/2373107. 1253
- [17] TRAN ANH T. The A-polynomial 2-tuple of twisted Whitehead links. *Internat. J. Math.* **29** (2018), no. 2, 1850013, 14 pp. MR3770936, Zbl 1388.57016, arXiv:1608.01381, doi: 10.1142/S0129167X18500131. 1242, 1248, 1251, 1253, 1255, 1256
- [18] TRAN ANH T. Twisted Alexander polynomials of genus one two-bridge knots. *Kodai Math. J.* **41** (2018), no. 1, 86–97. MR377388, Zbl 1396.57035, arXiv:1506.05035, doi: 10.2996/kmj/1521424825. 1240, 1241
- [19] TRAN ANH T. Reidemeister torsion and Dehn surgery on twist knots. *Tokyo J. Math.*, **39** (2016), 517–526. MR3599506, Zbl 1367.57013, arXiv:1506.02896, doi: 10.3836/tjm/1484903134. 1247
- [20] TURAEV, VLADIMIR. Introduction to combinatorial torsions. Lectures in Mathematics ETH Zürich. *Birkhäuser Verlag, Basel*, 2001. viii+123 pp. ISBN: 3-7643-6403-3, 978-3-0348-8321-4. MR1809561, Zbl 0970.57001, doi: 10.1007/978-3-0348-8321-4. 1244, 1246
- [21] WADA, MASAAKI. Twisted Alexander polynomial for finitely presentable groups. *Topology* **33** (1994), no. 2, 241–256. MR1273784, Zbl 0822.57006, doi: 10.1016/0040-9383(94)90013-2. 1241, 1244

(Hoang-An Nguyen) DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242, USA

hoang-an-nguyen@uiowa.edu

(Anh T. Tran) DEPARTMENT OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TEXAS AT DALLAS, RICHARDSON, TX 75080, USA

att140830@utdallas.edu

This paper is available via <http://nyjm.albany.edu/j/2019/25-51.html>.