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Compact intertwining relations for composition operators on H^∞ and the Bloch space

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Abstract. We study how certain Volterra type operators compactly intertwine with composition operators on the space of bounded analytic functions and composition operators on the Bloch space of the unit disk.

Contents

1. Introduction

If X and Y are two Banach spaces, the symbol $\mathscr{B}(X,Y)$ denotes the collection of all bounded linear operators from X to Y. Let $\mathcal{K}(X,Y)$ be the collection of all compact elements of $\mathscr{B}(X,Y)$, and let $\mathscr{Q}(X,Y)$ be the quotient set $\mathscr{B}(X,Y)/\mathcal{K}(X,Y)$.

For linear operators $A \in \mathscr{B}(X,X)$, $B \in \mathscr{B}(Y,Y)$ and $T \in \mathscr{B}(X,Y)$, the phrase "T intertwines A and B in $\mathscr{Q}(X,Y)$" (or "T intertwines A and B compactly") means that

$$TA = BT \mod \mathcal{K}(X,Y) \quad \text{with} \quad T \neq 0. \tag{1.1}$$

The notation $A \propto_K B\ (T)$ represents the relation in equation (1.1). In fact, if T is an invertible operator on X, then the relation $\propto_K$ is symmetric.

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Recall that the essential norm of a bounded linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}.$$

Notice that $\|T\|_e = 0$ if and only if T is compact. So estimates on $\|T\|_e$ lead to conditions for T to be compact.

Let $\mathbb{D}$ be the unit disk in the complex plane. Denote by $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $S(\mathbb{D})$ the collection of all the holomorphic self-mappings of $\mathbb{D}$. Every $\varphi \in S(\mathbb{D})$ induces a composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$.

Let $g \in H(\mathbb{D})$. The Volterra operator J_g is defined by

$$J_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta,\ z \in \mathbb{D},\quad f \in H(\mathbb{D}).$$

Another integral operator I_g is defined by

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta,\ z \in \mathbb{D},\quad f \in H(\mathbb{D}).$$

The notation $H^\infty(\mathbb{D})$ represents the algebra of bounded holomorphic functions with $\|\cdot\|_\infty$ as its supreme norm. For $\alpha > 0, \beta \in \mathbb{R}$, the Bloch type space $\mathcal{B}_{\alpha,\log^\beta}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_* := \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha \log^\beta \frac{2}{1 - |z|^2} |f'(z)| < \infty.$$

Then $\|\cdot\|_*$ is a complete semi-norm on $\mathcal{B}_{\alpha,\log^\beta}$. We denote the Banach space associated to $\mathcal{B}_{\alpha,\log^\beta}$ by $\tilde{\mathcal{B}}_{\alpha,\log^\beta}$, where the norm is given by the formula

$$\|f\|_{\tilde{\mathcal{B}}_{\alpha,\log^\beta}} = |f(0)| + \|f\|_*.$$

The little Bloch space, denoted by $\mathcal{B}_{\alpha,\log^\beta;0}$, consists of $f \in \mathcal{B}_{\alpha,\log^\beta}$ for which

$$\lim_{|z|\to 1} (1 - |z|^2)^\alpha \log^\beta \frac{2}{1 - |z|^2} |f'(z)| = 0.$$

In this paper, we abbreviate $\mathcal{B} = \mathcal{B}_{1,\log^0}$ and $\mathcal{B}_{\log} = \mathcal{B}_{1,\log^1}$.

Composition operators were studied intensively in the past few decades. A lot of efforts have been made on characterizing bounded and compact composition operators on spaces of analytic functions, for example, [Sha87] for Hardy space and [MM95] for Bloch spaces. Interested readers may refer to books [CM95, Sha93, Zhu05] and some recent papers [ZC08, ZS02, ZZ12] to learn more details on this subject.

The discussion of J_g first arose in connection with semigroups of composition operators, and readers may refer to [SZ98] for the background. Recently, the boundedness and compactness of J_g and I_g on various spaces of analytic functions has attracted considerable attention. For example, the boundedness of J_g on Hardy spaces, Bergman spaces, BMOA space, Bloch space and

$\mathcal{Q}_p$ space are characterized in [AC01, AS97, SZ98, Xia01, Xia04, Xia08], respectively. The same problems for the product of composition and Volterra operators on some function spaces in $\mathbb{D}$ have also been discussed. See [LS08c, LS08j, LS09] for example.

Based on these results, we consider the composition operator $C_\varphi : X \to X$, and the integral-type operator V_g $(= J_g$ or $I_g) : X \to \mathcal{B}$ where X represents H^∞ or $\mathcal{B}$. We are interested in the compact intertwining relations

$$C_\varphi \,|_X \propto_K C_\varphi \,|_{\mathcal{B}} \quad (V_g \,|_{X\to\mathcal{B}}). \tag{1.2}$$

To be more intuitive, we consider the following commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{C_\varphi} & X \\ \downarrow{\scriptstyle V_g} & & \downarrow{\scriptstyle V_g} \\ \mathcal{B} & \xrightarrow{C_\varphi} & \mathcal{B} \end{array} \qquad \text{mod } \mathcal{K}(X, \mathcal{B}).$$

If (1.2) holds for some $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, we may also say that C_φ *essentially commute* with V_g. In this paper, we make our effort to answer the following two main questions.

(Q1): What are the sufficient and necessary conditions on g, so that

$$C_\varphi \,|_X \propto_K C_\varphi \,|_{\mathcal{B}} \quad (V_g \,|_{X\to\mathcal{B}})$$

holds for every $\varphi \in S(\mathbb{D})$?

(Q2): What are the sufficient and necessary conditions on $\varphi \in S(\mathbb{D})$, so that

$$C_\varphi \,|_X \propto_K C_\varphi \,|_{\mathcal{B}} \quad (V_g \,|_{X\to\mathcal{B}})$$

holds for every bounded V_g?

It is of particular interest to consider the above two "global" questions. We omit the symbols, which induce the operators, to reflect the "global" essential commutativity. That is, we write $C \,|_X \propto_K C \,|_{\mathcal{B}}\ (V_g)$ to indicate that the operator V_g can commute with every composition operator C_φ essentially. Analogously, we write $C_\varphi \,|_X \propto_K C_\varphi \,|_{\mathcal{B}}\ (V \,|_{X\to\mathcal{B}})$ to indicate that the operator C_φ can commute with every bounded $V_g : X \to \mathcal{B}$ essentially. By the way, the collections of g satisfying conditions similar as **(Q1)** was called the *universal set* of V_g by the authors in [TZ14, TZ13]. Our use of the term "universal set" should not be confused with the notion of "universal set" which appears in the dynamical theory of linear operators.

In the following discussion, we write $A \lesssim B$ if there exists an absolute constant $C > 0$ such that $A \leq C \cdot B$, and $A \approx B$ represents $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

Before the discussion of our main results, we need some preliminary propositions. For $\varphi \in S(\mathbb{D})$, denote the *Schwarz derivative* of φ by

$$\varphi^{\#}(z) := \frac{1-|z|^2}{1-|\varphi(z)|^2}\varphi'(z).$$

From the Schwarz Lemma we know that $|\varphi^{\#}(z)| \leq 1$ for any $z \in \mathbb{D}$, and the equality holds if and only if φ is an automorphism of the unit disk. The following lemma characterizes bounded and compact composition operators on $\mathcal{B}$ and H^∞(see [MM95] and [CM95]).

Lemma 2.1. *If $\varphi \in S(\mathbb{D})$, then*

(1) *Every φ induces an bounded composition operator on $\mathcal{B}$ and H^∞.*
(2) *$C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if C_φ is bounded and*

$$\lim_{|\varphi(z)|\to 1} |\varphi^{\#}(z)| = 0.$$

(3) *$C_\varphi : H^\infty \to H^\infty$ is compact if and only if $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$.*

The following criterion for compactness follows from standard arguments and its proof is similar to the method of Proposition 3.11 in [CM95]. Hence we omit the details.

Lemma 2.2. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $C_\varphi V_g - V_g C_\varphi$ is compact from X to $\mathcal{B}$ if and only if for any bounded sequence $\{f_k\}$, $k = 1, 2, ...$ in X which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|(C_\varphi V_g - V_g C_\varphi) f_k\|_* \to 0$ as $k \to \infty$.*

Recall that the notation $\mathbb{C}^N$ represents the N dimensional complex Euclidean space. Denote the unit ball of $\mathbb{C}^N$ by B_N. If $z, w \in B_N$, we define Möbius transform by

$$\Phi_w(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle},$$

where $P_a(z) = \frac{\langle z,a\rangle}{\langle a,a\rangle} a$, $Q_a(z) = z - P_a(z)$ and $s_a = \sqrt{1-|a|^2}$. The following Lemma will be used in Section 4, which was first presented by Berndtsson in [Ber85].

Lemma 2.3. *Let $\{x_i\}$ be a sequence in the ball B_N satisfying*

$$\prod_{j:j\neq k} |\Phi_{x_j}(x_k)| \geq d > 0 \quad \textit{for any} \quad k. \tag{2.1}$$

Then there exists a number $M = M(d) < \infty$ and a sequence of functions $h_k \in H^\infty(B_N)$ such that

$$(a)\ \ h_k(x_j) = \delta_{kj}; \quad (b) \ \sum_k |h_k(z)| \leq M \quad \textit{for} \quad |z| < 1. \tag{2.2}$$

(The symbol δ_{kj} is equal to 1 if $k = j$ and 0 otherwise.)

The next lemma was proved by Carl Toews in [Toe04].

Lemma 2.4. *Let* $\{z_n\} \subset B_N$ *be a sequence with* $|z_n| \to 1$ *as* $n \to \infty$. *Then for any given* $d \in (0,1)$ *there is a subsequence such that* $\{x_i\} := \{z_{n_i}\}$ *satisfies* (2.1).

From this lemma, there is always a subsequence which satisfies (2.1) for every sequence converging to the boundary of B_N, and Lemma 2.2 holds for this subsequence. We just need the result in one dimension.

To get some simple consequences of our main problems, we will consider the situation in the little Bloch setting. The next lemma is well known, see [OSZ03].

Lemma 2.5. *A closed set* K *in* $\mathcal{B}_0$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)|f'(z)| = 0.$$

In our discussion, we will use the boundedness of operators J_g and I_g, from H^∞ or $\mathcal{B}$ to $\mathcal{B}$. Several characterizations are listed below.

Lemma 2.6. *Suppose that* $g \in H(\mathbb{D})$. *Then*

(a) $J_g : H^\infty \to \mathcal{B}$ *is bounded if and only if* $g \in \mathcal{B}$;
(b) $J_g : \mathcal{B} \to \mathcal{B}$ *is bounded if and only if* $g \in \mathcal{B}_{\log}$;
(c) $I_g : H^\infty$ *or* $\mathcal{B} \to \mathcal{B}$ *is bounded if and only if* $g \in H^\infty$.

Part (a) above was Corollary 4 in [LS08c], part (b) is from Theorem 3 in [LS09], and part (c) follows from Corollary 1 in [LS08c] and Theorem 14 in [LS09].

Some definitions and results in Geometric Function Theory are needed, and interested readers can refer to [Gar81] and [Kra06]. For $\zeta \in \partial\mathbb{D}$ and $M > 1$ the *nontangential approaching region* at ζ is defined by

$$\Gamma(\zeta, M) = \{z \in \mathbb{D} : |z-\zeta| < M(1-|z|^2)\}.$$

A function f is said to have a *nontangential limit* at ζ if $\lim_{z\to\zeta} f(z)$ exists in each nontangential region $\Gamma(\zeta, M)$, and we denote it by $\angle - \lim_{z\to\zeta} f(z)$. If $\varphi \in S(\mathbb{D})$ and $\zeta \in \partial\mathbb{D}$, we will call ζ a *boundary fixed point* of φ if

$$\angle - \lim_{z\to\zeta} \varphi(z) = \zeta.$$

We say φ has a *finite angular derivative* at $\zeta \in \partial\mathbb{D}$ if there is $\eta \in \partial\mathbb{D}$ so that$(\varphi(z)-\eta)/(z-\zeta)$ has finite nontangential limit as $z \to \zeta$. When it exists as a finite complex number, this limit is denoted $\varphi'(\zeta)$. A $\varphi \in S(\mathbb{D})$ is said to be *parabolic type* if φ has a boundary fixed point ζ with $\varphi'(\zeta) = 1$. If φ is parabolic type, $\varphi(z) \to \zeta$ and $\varphi'(z) \to 1$ as $z \to \zeta$ unrestricted in the unit disk, we say ζ is a C^1 parabolic boundary fixed point of φ.

3. The case of intertwining operator I_g

First we consider $C_\varphi I_g - I_g C_\varphi$ as an operator from the Bloch space to itself.

Theorem 3.1. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $C_\varphi I_g - I_g C_\varphi$ is bounded on the Bloch space if and only if*

$$\sup_{z\in\mathbb{D}} |\varphi^{\#}(z)| \cdot |g(\varphi(z)) - g(z)| < \infty. \tag{3.1}$$

Proof. Suppose (3.1) holds, we will show that $C_\varphi I_g - I_g C_\varphi$ is bounded on $\mathcal{B}$ by a direct computation. For any $f \in \mathcal{B}$,

$$\begin{aligned} ((C_\varphi I_g - I_g C_\varphi)f)(z) &= C_\varphi \int_0^z f'(\zeta)g(\zeta)d\zeta - I_g f(\varphi(z)) \\ &= \int_0^{\varphi(z)} f'(\zeta)g(\zeta)d\zeta - \int_0^z (f\circ\varphi)'(\zeta)g(\zeta)d\zeta. \end{aligned}$$

It follows that

$$\begin{aligned} &\|(C_\varphi I_g - I_g C_\varphi)f\|_* \\ = &\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\varphi'(z)f'(\varphi(z))g(\varphi(z)) - \varphi'(z)f'(\varphi(z))g(z)\right| \\ = &\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\varphi'(z)f'(\varphi(z))(g(\varphi(z)) - g(z))\right| \\ = &\sup_{z\in\mathbb{D}}|\varphi^{\#}(z)|\,|g(\varphi(z)) - g(z)|\,(1-|\varphi(z)|^2)\left|f'(\varphi(z))\right| \\ \le &\sup_{z\in\mathbb{D}}|\varphi^{\#}(z)|\,|g(\varphi(z)) - g(z)|\cdot\|f\|_*. \end{aligned}$$

From which we obtain that $C_\varphi I_g - I_g C_\varphi$ is bounded by (3.1).

Conversely, suppose that $C_\varphi I_g - I_g C_\varphi$ is bounded on $\mathcal{B}$, then there is a constant $C > 0$ such that

$$\|(C_\varphi I_g - I_g C_\varphi)f\|_* \le \|(C_\varphi I_g - I_g C_\varphi)f\|_{\mathcal{B}} < C$$

for $\|f\|_{\mathcal{B}} \le 1$. We will prove condition (3.1). Suppose not, there exists a sequence $\{w_n\}$ in $\mathbb{D}$ such that

$$\lim_{n\to\infty} |\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)| = \infty.$$

Let $\alpha_n(z) = \frac{\varphi(w_n)-z}{1-\overline{\varphi(w_n)}z}$, and $\alpha_n'(z) = -\frac{1-|\varphi(w_n)|^2}{(1-\overline{\varphi(w_n)}z)^2}$ for $n = 1, 2, \ldots$. It is easy to check that $\|\alpha_n\|_* = 1$.

$$\begin{aligned} &\|(C_\varphi I_g - I_g C_\varphi)\alpha_n\|_* \\ = &\sup_{z\in\mathbb{D}}|\varphi^{\#}(z)||g(\varphi(z)) - g(z)|\cdot(1-|\varphi(z)|^2)|\alpha_n'(\varphi(z))| \\ \ge &(1-|\varphi(w_n)|^2)\frac{1-|\varphi(w_n)|^2}{|(1-\overline{\varphi(w_n)}\varphi(w_n))^2|}\cdot|\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)| \\ = &|\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)| \to \infty. \end{aligned}$$

That contradicts to the boundedness of $C_\varphi I_g - I_g C_\varphi$. Hence (3.1) holds. □

When we investigate essential commutativity of C_φ and I_g, we need to add the condition $g \in H^\infty$ to ensure the boundedness of I_g on the Bloch space, see Lemma 2.6.

Theorem 3.2. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H^\infty(\mathbb{D})$. Then C_φ and I_g are essentially commutative on $\mathcal{B}$ if and only if*

$$\lim_{|\varphi(z)|\to 1} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| = 0. \tag{3.2}$$

Proof. *Sufficiency.* Note that $g \in H^\infty$ and $\|\varphi^{\#}\|_\infty \le 1$, we have

$$\sup_{z\in\mathbb{D}} |\varphi^{\#}(z)| \cdot |g(\varphi(z)) - g(z)| < \infty.$$

Hence $C_\varphi I_g - I_g C_\varphi$ is a bounded operator by Theorem 3.1. For any bounded sequence $\{f_k\}$ in $\mathcal{B}$ converging to zero uniformly on compact subsets of $\mathbb{D}$, we firstly find a positive $M > 0$ so that $\|f_k\|_{\mathcal{B}} \le M$. It follows from (3.2) that for any small $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\varphi^{\#}(z)||g(\varphi(z)) - g(z)| < \frac{\varepsilon}{M} \tag{3.3}$$

whenever $\varphi(z) \in \mathbb{D} \setminus (1-\delta)\mathbb{D}$. Then we can compute as follows

$$\begin{aligned} & \|(C_\varphi I_g - I_g C_\varphi) f_k\|_* \\ = & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))| \\ \le & \ (A) + (B), \end{aligned}$$

where

$$(A) = \sup_{\varphi(z)\in(1-\delta)\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))|,$$

and

$$(B) = \sup_{\varphi(z)\in\mathbb{D}\setminus(1-\delta)\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))|.$$

Since $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, it is obvious that $(A) < \varepsilon$ for sufficiently large k. And by (3.3) we have

$$(B) < \frac{\varepsilon}{M} \cdot \sup_{z\in\mathbb{D}} (1 - |\varphi(z)|^2)|f_k'(\varphi(z))| = \frac{\varepsilon}{M}\|f_k\|_* < \varepsilon.$$

Thus $C_\varphi I_g - I_g C_\varphi$ is compact on $\mathcal{B}$.

Necessity. If $C_\varphi I_g - I_g C_\varphi$ is compact on $\mathcal{B}$, it is certainly a bounded operator. Therefore

$$\sup_{z\in\mathbb{D}} |\varphi^{\#}(z)| \cdot |g(\varphi(z)) - g(z)| < \infty$$

by Theorem 3.1. Suppose that

$$\sup_{|\varphi(z)|\to 1} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| \ne 0,$$

then there is a sequence $\{w_n\}$ in $\mathbb{D}$ with $|\varphi(w_n)| \to 1$ and an $\epsilon > 0$ such that

$$|\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)| > \epsilon \quad \forall n. \tag{3.4}$$

Let

$$h_n(z) = \frac{1 - |\varphi(w_n)|^2}{1 - \overline{\varphi(w_n)}z}. \tag{3.5}$$

A simple computation shows that

$$h_n'(z) = (1 - |\varphi(w_n)|^2)\frac{\overline{\varphi(w_n)}}{(1 - \overline{\varphi(w_n)}z)^2}.$$

So $\|h_n\|_* \le 1$ and $\{h_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. It follows from Lemma 2.2 that

$$\|(C_\varphi I_g - I_g C_\varphi)h_n\|_* \to 0. \tag{3.6}$$

On the other hand, by (3.4), we have

$$\begin{aligned} & \|(C_\varphi I_g - I_g C_\varphi)h_n\|_* \\ = & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| \cdot (1 - |\varphi(z)|^2)|h_n'(\varphi(z))| \\ = & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| \cdot (1 - |\varphi(z)|^2)\frac{|\varphi(w_n)|(1 - |\varphi(w_n)|^2)}{|1 - \overline{\varphi(w_n)}\varphi(z)|^2} \\ \ge & |\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)||\varphi(w_n)| > \epsilon \end{aligned}$$

for any n. It contradicts to (3.6). Hence (3.2) holds when C_φ and I_g are essentially commutative. □

Corollary 3.3. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, then $C_\varphi I_g - I_g C_\varphi : H^\infty \to \mathcal{B}$ is bounded if and only if (3.1) holds; $C_\varphi I_g - I_g C_\varphi : H^\infty \to \mathcal{B}$ is compact if and only if (3.2) holds.*

Proof. The proof of this corollary is analogous to those of Theorem 3.1 and 3.2. We only need to change the test function (3.5) to

$$f_n(z) = \frac{1 - |\varphi(w_n)|^2}{1 - \overline{\varphi(w_n)}z} \cdot \frac{\varphi(w_n) - z}{1 - \overline{\varphi(w_n)}z}.$$

The rest of the arguments are highly similar to those of Theorems 3.1 and 3.2. Hence we omit the details. □

Using Lemma 2.4, we can easily get the following corollary on the little Bloch space. The method is the same as before and we omit its proof.

Corollary 3.4. *Suppose $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Let X represent the space H^∞, $\mathcal{B}$ or $\mathcal{B}_0$. The following three statements are equivalent:*

(a) *$C_\varphi I_g - I_g C_\varphi : X \to \mathcal{B}_0$ is bounded;*
(b) *$C_\varphi I_g - I_g C_\varphi : X \to \mathcal{B}_0$ is compact;*
(c) *$\lim_{|z|\to 1} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| = 0$.*

Now we consider **(Q1)** raised in the first section:

- When does $C\mid_X \propto_K C\mid_{\mathcal{B}}\ (V_g)$ hold?

Theorem 3.5. *The compact intertwining relation $C\mid_X \propto_K C\mid_{\mathcal{B}}\ (V_g)$ holds if and only if g is a constant.*

Proof. Sufficiency is obvious from a direct computation. To verify the necessity, we just need to consider φ as an automorphism of $\mathbb{D}$ in condition (3.2). Then maximum modulus theorem implies that g must be a constant. □

The following proposition gives a sufficient condition which answers **(Q2)** in part.

Proposition 3.6. *Suppose $\varphi \in S(\mathbb{D})$ and $g \in H^\infty$. Let X denote either $\mathcal{B}$ or H^∞. If φ satisfies*

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi(z) - z|}{(1 - \max\{|z|, |\varphi(z)|\})^2} \left|\varphi^\#(z)\right| = 0, \tag{3.7}$$

then $C_\varphi\mid_X \propto_K C_\varphi\mid_{\mathcal{B}} \quad (I\mid_{X\to\mathcal{B}})$.

Proof. By the Cauchy integral formula, one finds that

$$|g'(z)| \le \frac{\|g\|_\infty}{(1 - |z|)^2} \quad (\forall z \in \mathbb{D})$$

for $g \in H^\infty$. The proposition will be proved immediately by noting that

$$\begin{aligned}
& |\varphi^\#(z)||g(\varphi(z)) - g(z)| \\
= \ & |\varphi^\#(z)| \left| \int_z^{\varphi(z)} g'(\zeta) \mathrm{d}\zeta \right| \\
\le \ & |\varphi^\#(z)| \frac{\|g\|_\infty}{(1 - \max\{|\varphi(z)|, |z|\})^2} \int_z^{\varphi(z)} |\mathrm{d}\zeta| \\
\le \ & |\varphi^\#(z)| \|g\|_\infty \frac{|\varphi(z) - z|}{(1 - \max\{|\varphi(z)|, |z|\})^2}.
\end{aligned}$$

□

According to Lemma 2.6, the operator $I_g : X \to \mathcal{B}$ is bounded if g is analytic in $\mathbb{D}$ and continuous to the boundary. The next theorem will answer **(Q2)** in part. We will get a necessary and sufficient condition of the compact intertwining relation, when g is the function in the disk algebra.

Theorem 3.7. *Let $\mathcal{A}$ be the disk algebra, that is the subspace of H^∞ whose elements are analytic in $\mathbb{D}$ and continuous to the unit circle. Then*

$$C_\varphi\mid_X \propto_K C_\varphi\mid_{\mathcal{B}} \quad (I_g\mid_{X\to\mathcal{B}})$$

holds for all $g \in \mathcal{A}$ if and only if

$$\lim_{|\varphi(z)|\to 1} |\varphi^\#(z)||\varphi(z) - z| = 0. \tag{3.8}$$

Proof. Necessity can be proved immediately by plugging $g = \mathrm{id}$ into equation (3.2). We just need to prove sufficiency. Condition (3.8) is sufficient to make

$$C_\varphi\,|_X \propto_K C_\varphi\,|_{\mathcal{B}} \quad (I_g)$$

hold for any monomial $g(z) = z^n$. For each $h \in \mathcal{A}$, there is a polynomial sequence $\{g_n\}$ such that $g_n \to h$ in supreme norm. One has

$$\begin{aligned}
&\|C_\varphi I_h - I_h C_\varphi\|_{e,X\to\mathcal{B}} \\
= \;& \inf\{\|C_\varphi I_h - I_h C_\varphi - K\|_{X\to\mathcal{B}} :\ K \in \mathcal{K}(X,\mathcal{B})\} \\
\le \;& \|C_\varphi I_h - I_h C_\varphi - (C_\varphi I_{g_n} - I_{g_n} C_\varphi)\|_{X\to\mathcal{B}} \\
\le \;& 2\|C_\varphi\|_{X\to\mathcal{B}} \cdot \|I_h - I_{g_n}\|_{X\to\mathcal{B}} \\
= \;& 2\|C_\varphi\|_{X\to\mathcal{B}} \cdot \sup_{\|f\|_X \le 1} \left\| \int_0^z f'(\zeta)(h-g_n)(\zeta)\mathrm{d}\zeta \right\|_{\mathcal{B}} \\
\le \;& 2\|C_\varphi\|_{X\to\mathcal{B}} \cdot \|h - g_n\|_\infty \cdot \sup_{\|f\|_X \le 1} \left\| \int_0^z f'(\zeta)\mathrm{d}\zeta \right\|_{\mathcal{B}} \\
= \;& 2\|C_\varphi\|_{X\to\mathcal{B}} \cdot \|h - g_n\|_\infty
\end{aligned}$$

Hence $\|C_\varphi I_h - I_h C_\varphi\|_{e,X\to\mathcal{B}} = 0$ by taking $n \to \infty$. □

4. The case of intertwining operator J_g

First, we consider the case when $X = H^\infty$ in the main question.

Theorem 4.1. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then*

(1) *$C_\varphi J_g - J_g C_\varphi$ is bounded from H^∞ to $\mathcal{B}$ if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| < \infty; \tag{4.1}$$

(2) *$C_\varphi J_g - J_g C_\varphi$ is compact from H^∞ to $\mathcal{B}$ if and only if (4.1) holds and*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| = 0. \tag{4.2}$$

Proof. Being similar to the proofs of Theorems 3.1 and 3.2, we just need some modifications. We firstly see that

$$\begin{aligned}
&\|(C_\varphi J_g - J_g C_\varphi)f\|_* \\
= \;& \left\| \textstyle\int_0^{\varphi(z)} f(\zeta)g'(\zeta)d\zeta - \int_0^z f(\varphi(\zeta))g'(\zeta)d\zeta \right\|_* \\
= \;& \sup_{z\in\mathbb{D}}(1-|z|^2)|f(\varphi(z))g'(\varphi(z))\varphi'(z) - f(\varphi(z))g'(z)| \\
= \;& \sup_{z\in\mathbb{D}}(1-|z|^2)|(g\circ\varphi)'(z) - g'(z)||f(\varphi(z))|.
\end{aligned} \tag{4.3}$$

Sufficiency of the two items in the theorem is obvious from the last formula in (4.3).

Necessity of boundedness can be proved by computing test functions

$$f_n(z) = 1 - \frac{\varphi(w_n) - z}{1 - \overline{\varphi(w_n)}z},$$

where sequence $\{w_n\}$ violates equation (4.1).

Necessity of compactness can also be proved by contradiction. Suppose that we can find a sequence $\{\varphi(w_n)\}$ converging to the boundary of $\mathbb{D}$ and $\epsilon_0 > 0$ such that

$$\lim_{n\to\infty}(1 - |w_n|^2)|(g \circ \varphi)'(w_n) - g'(w_n)| > \epsilon_0. \tag{4.4}$$

Further we may assume that $\{\varphi(w_n)\}$ is interpolating. Then there exist functions $\{h_n\}$ in H^∞ for $\{\varphi(w_n)\}$ such that

$$h_n(\varphi(w_k)) = \begin{cases} 1 & n = k, \\ 0 & n \neq k. \end{cases} \tag{4.5}$$

and

$$\sum_n |h_n(z)| \leq M < \infty, \tag{4.6}$$

by Lemma 2.4 and 2.3, or see [Gar81]. Equation (4.6) guarantees that $\{h_n\}$ is bounded in H^∞ and converges to zero on compact subsets of $\mathbb{D}$. Then we have

$$\begin{aligned} & \|(C_\varphi J_g - J_g C_\varphi)h_n\|_* \\ = \; & \sup_{z\in\mathbb{D}}(1 - |z|^2)|(g \circ \varphi'(z) - g'(z)||h_n(\varphi(z))| \\ \geq \; & (1 - |w_n|^2)|(g \circ \varphi'(w_n) - g'(w_n)||h_n(\varphi(w_n))| \\ = \; & (1 - |w_n|^2)|(g \circ \varphi)'(w_n) - g'(w_n)|, \end{aligned}$$

where the last equation follows by (4.5). Letting $n \to \infty$, we find a contradiction by (4.4). $\square$

Corollary 4.2. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. The following three conditions are equivalent:*

(a) $C_\varphi J_g - J_g C_\varphi : H^\infty \to \mathcal{B}_0$ *is bounded;*
(b) $C_\varphi J_g - J_g C_\varphi : H^\infty \to \mathcal{B}_0$ *is compact;*
(c) $\lim_{|z|\to 1}(1 - |z|^2)|(g \circ \varphi)'(z) - g'(z)| = 0$.

When we consider **(Q1)** for composition operator and J_g, the result turns out to be interesting. Note that $g \in \mathcal{B}$ implies that $\sup_{z\in\mathbb{D}}(1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| < \infty$, thus $C_\varphi J_g - J_g C_\varphi$ is a bounded operator from H^∞ to $\mathcal{B}$. And

$$C_\varphi |_{H^\infty} \propto_K C_\varphi |_{\mathcal{B}} \quad (J_g |_{H^\infty\to\mathcal{B}})$$

if and only if (4.2) holds. Now, we can answer **(Q1)**.

Corollary 4.3. *If $g \in \mathcal{B}_0$, then*

$$C\,|_{H^\infty} \propto_K C\,|_{\mathcal{B}} \quad (J_g\,|_{H^\infty \to \mathcal{B}}). \tag{4.7}$$

Proof. Since g is in the little Bloch space, $(1-|z|^2)|g'(z)|$ tends to 0 whenever z tends to the boundary of the disk. Then we have

$$\begin{aligned} & \lim_{|\varphi(z)|\to 1} (1-|z|^2)|g'(\varphi(z))\varphi'(z) - g'(z)| \\ \leq\ & \lim_{|\varphi(z)|\to 1} |\varphi^{\#}(z)|(1-|\varphi(z)|^2)|g'(\varphi(z))| + \lim_{|\varphi(z)|\to 1} (1-|z|^2)|g'(z)|. \end{aligned}$$

Conditions $g \in \mathcal{B}$ and $|\varphi^{\#}| \leq 1$ imply that $C_\varphi J_g - J_g C_\varphi$ is compact from H^∞ to $\mathcal{B}$ for every self-mapping φ. □

In contrast to the case of intertwining operator I_g, there are many nonconstant functions g, which are actually the little Bloch functions, so that J_g can commute with any composition operator essentially. Naturally, we are going to ask the inverse question: if J_g commute with every composition operator essentially, does that imply $g \in \mathcal{B}_0$? The answer is positive. The next lemma is the key lemma to investigate the necessary condition on g in **(Q1)**. The method of proof is the same as in our recent papers [TZ14, TZ13].

Lemma 4.4. *If g is a Bloch function on the unit disk with the property that, for any rotation $\tau(z) = e^{\mathbf{i}t}z$, $g \circ \tau - g$ is in the little Bloch space, then g itself must be in the little Bloch space.*

Proof. Since $g \circ \tau - g$ is in the little Bloch space, we have

$$\lim_{|z|\to 1} (1-|z|^2)\left|g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t} - g'(z)\right| = 0. \tag{4.8}$$

We estimate the upper bound of left hand side in equation (4.8) as follows,

$$\begin{aligned} & (1-|z|^2)\left|g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t} - g'(z)\right| \\ \leq\ & (1-|z|^2)\left|g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t}\right| + (1-|z|^2)\left|g'(z)\right| \\ =\ & (1-|e^{\mathbf{i}t}z|^2)\left|g'(e^{\mathbf{i}t}z)\right| + (1-|z|^2)\left|g'(z)\right| \\ \leq\ & 2\|g\|_*. \end{aligned}$$

Thus $g \in \mathcal{B}$ implies that the left hand side in equation (4.8) is finite uniformly in t.

Suppose that $g(z)=\sum\limits_{n=0}^{\infty}a_nz^n$, then $g'(z)=\sum\limits_{n=1}^{\infty}na_nz^{n-1}$. Integrating with respect to t from 0 to 2π, we have that

$$\begin{aligned}&\int_0^{2\pi}\lim_{|z|\to 1}(1-|z|^2)\left|g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t}-g'(z)\right|\mathrm{d}t\\ =\;&\lim_{|z|\to 1}(1-|z|^2)\left|\int_0^{2\pi}g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t}-g'(z)\mathrm{d}t\right|\\ =\;&\lim_{|z|\to 1}(1-|z|^2)\left|\int_0^{2\pi}\sum_{n=1}^{\infty}\left(na_n(e^{\mathbf{i}t}z)^{n-1}e^{\mathbf{i}t}-na_nz^{n-1}\right)\mathrm{d}t\right|\\ =\;&\lim_{|z|\to 1}(1-|z|^2)\left|\sum_{n=1}^{\infty}na_nz^{n-1}\int_0^{2\pi}(e^{\mathbf{i}nt}-1)\mathrm{d}t\right|\\ =\;&2\pi\cdot\lim_{|z|\to 1}(1-|z|^2)|g'(z)|,\end{aligned}$$

where the equation in the second line is true by the dominated convergence theorem. Thus we get $g\in\mathcal{B}_0$ from (4.8). □

Theorem 4.5. *$C\mid_{H^\infty}\propto_K C\mid_{\mathcal{B}}$ $(J_g\mid_{H^\infty\to\mathcal{B}})$ if and only if $g\in\mathcal{B}_0$.*

Proof. Sufficiency is stated in Corollary 4.3. To prove necessity, just consider the rotation of the disk. That is to say, by putting $\varphi(z)=\tau(z)=e^{\mathbf{i}t}z$ in the condition (4.2), and we have $g\in\mathcal{B}_0$ by Lemma 4.4. □

Considering the composition operator C_φ and Volterra operator J_g, we can obtain a necessary condition (Proposition 4.6) and a sufficient condition (Proposition 4.7) to answer **(Q2)** partially.

Proposition 4.6. *Let $\zeta\in\partial\mathbb{D}$, $\varphi\in S(\mathbb{D})$ and $\varphi(z)$ tend to the unit circle when z converges to ζ nontangentially. If*

$$C_\varphi\mid_{H^\infty}\propto_K C_\varphi\mid_{\mathcal{B}}\quad(J\mid_{H^\infty\to\mathcal{B}}),$$

then $\angle-\lim\limits_{z\to\zeta}\varphi(z)=\zeta$.

Proof. By Theorem 4.5, $\lim\limits_{|\varphi(z)|\to 1}(1-|z|^2)|(g\circ\varphi-g)'(z)|=0$ holds for every $g\in\mathcal{B}$. Suppose that $\varphi(z)\to\partial\mathbb{D}$ as $z\to\zeta$ in some nontangential approaching region, such that $\eta\neq\zeta$ and $\varphi(z_n)\nrightarrow\eta$. let $g(z)=-\log(1-\bar{\zeta}z)$ in equation (4.7), then

$$\begin{aligned}&\lim_{z\to\zeta}(1-|z|^2)\left|\frac{\bar{\zeta}\varphi'(z)}{1-\bar{\zeta}\varphi(z)}-\frac{\bar{\zeta}}{1-\bar{\zeta}z}\right|\\ =\;&\lim_{z\to\zeta}\frac{1-|z|^2}{|1-\bar{\zeta}z|}\left|\frac{\zeta-z}{\zeta-\varphi(z)}\varphi'(z)-1\right|=0.\end{aligned}$$

For any $M>0$, if $z\in\Gamma(\zeta,M)=\{z\in\mathbb{D}:|\zeta-z|<M(1-|z|^2)\}$,

$$\angle-\lim_{z\to\zeta}\left|\frac{\zeta-z}{\zeta-\varphi(z)}\varphi'(z)-1\right|=0.$$

By noting that $|\zeta-z||\varphi'(z)|=\frac{|\zeta-z|}{1-|z|^2}\cdot(1-|\varphi(z)|^2)|\varphi^{\#}(z)|\to 0$, one has

$$\angle-\lim_{z\to\zeta}\varphi(z)=\zeta. \tag{4.9}$$

□

Proposition 4.7. *Let $\varphi\in S(\mathbb{D})$ be of parabolic type with C^1 boundary fixed point $\zeta\in\partial\mathbb{D}$. If*

$$\sup_{z\in\mathbb{D}}\frac{|\varphi(z)-z|}{(1-\max\{|z|,|\varphi(z)|\})^2}\leq\infty, \tag{4.10}$$

then $C_\varphi\,|_{H^\infty}\propto_K C_\varphi\,|_{\mathcal{B}}\ (J\,|_{H^\infty\to\mathcal{B}})$.

Proof. It is well known that $g\in\mathcal{B}$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^n|g^{(n)}(z)|<\infty.$$

Note that

$$\begin{aligned}
& (1-|z|^2)|(g\circ\varphi)'(z)-g'(z)| \\
\leq\ & (1-|z|^2)|\varphi'(z)||g'(\varphi(z))-g'(z)|+(1-|z|^2)|g'(z)||\varphi'(z)-1| \\
=\ & (1-|z|^2)|\varphi'(z)|\left|\int_z^{\varphi(z)}g''(\zeta)\mathrm{d}\zeta\right|+(1-|z|^2)|g'(z)||\varphi'(z)-1| \\
\leq\ & (1-|z|^2)|\varphi'(z)|\sup_{z\in\mathbb{D}}(1-|z|^2)^2|g''(z)|\int_z^{\varphi(z)}\frac{|\mathrm{d}\zeta|}{(1-|\zeta|)^2} \\
& +(1-|z|^2)|g'(z)||\varphi'(z)-1| \\
\leq\ & (1-|z|^2)|\varphi'(z)|\sup_{z\in\mathbb{D}}(1-|z|^2)^2|g''(z)|\frac{|\varphi(z)-z|}{(1-\max\{|z|,|\varphi(z)|\})^2} \\
& +(1-|z|^2)|g'(z)||\varphi'(z)-1|
\end{aligned}$$

where the integral path is chosen to be the segment from z to $\varphi(z)$. Then $C_\varphi\,|_{H^\infty}\propto_K C_\varphi\,|_{\mathcal{B}}\ (J\,|_{H^\infty\to\mathcal{B}})$ follows immediately from those conditions in the proposition. □

The next several propositions concern C_φ and J_g as maps from $\mathcal{B}$ to itself.

Proposition 4.8. *Assume that $\varphi\in S(\mathbb{D})$ and $g\in H(\mathbb{D})$. Then $C_\varphi J_g-J_gC_\varphi$ is bounded from $\mathcal{B}$ to itself if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(g\circ\varphi)'(z)-g'(z)|\log\frac{2}{1-|\varphi(z)|^2}<\infty. \tag{4.11}$$

Proof. Sufficiency can be verified by some straightforward computations and inequalities. Necessity will be proved by choosing the test function

$$f_w(z) = \log \frac{2}{1-\overline{w}z}.$$

□

Proposition 4.9. *Assume that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then the following statements are equivalent:*

(a) $C_\varphi J_g - J_g C_\varphi : \mathcal{B} \to \mathcal{B}$ *is compact and* (4.11) *holds;*
(b) $C_\varphi J_g - J_g C_\varphi : \mathcal{B}_0 \to \mathcal{B}_0$ *is compact;*
(c) $C_\varphi J_g - J_g C_\varphi : \mathcal{B}_0 \to \mathcal{B}_0$ *is weakly compact;*
(d) *Condition* (4.11) *holds and*

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| \log \frac{2}{1-|\varphi(z)|^2} = 0; \tag{4.12}$$

(e) $C_\varphi J_g - J_g C_\varphi : \mathcal{B} \to \mathcal{B}_0$ *is compact;*
(f) $C_\varphi J_g - J_g C_\varphi : \mathcal{B} \to \mathcal{B}_0$ *is bounded;*

The compactness of operators $C_\varphi J_g$ and $J_g C_\varphi$ are characterized in Theorem 4 and Theorem 11 in [LS09]. The proofs are similar to those in Theorem 4 in [LS09], so we omit them.

To continue our discussion, we need an upper bound on the modulus of $\varphi(z)$ (see Corollary 2.40 in [CM95]).

Lemma 4.10. *If $\varphi \in S(\mathbb{D})$, then $|\varphi(z)| \leq \frac{|z|+|\varphi(0)|}{1+|z||\varphi(0)|}$.*

Now we are ready to consider **(Q1)** for C_φ and J_g, where both of them are operators acting on $\mathcal{B}$.

Theorem 4.11. *Assume $g \in H(\mathbb{D})$ is such that J_g is bounded on the Bloch space. Then*

$$C\,|_{\mathcal{B}} \propto_K C\,|_{\mathcal{B}} \quad (J_g\,|_{H^\infty\to\mathcal{B}})$$

if and only if $g \in \mathcal{B}_{\log,0}$, that is

$$\lim_{|z|\to 1} (1-|z|^2)|g'(z)| \log \frac{2}{1-|z|^2} = 0. \tag{4.13}$$

Proof. Following the method in the proof of Lemma 4.10, the necessity can be proved similarly.

Now suppose that (4.13) holds. We have

$$\begin{aligned}
&(1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| \log \frac{2}{1-|\varphi(z)|^2} \\
\leq\ & |\varphi^{\#}(z)|(1-|\varphi(z)|^2)|g'(\varphi(z)| \log \frac{2}{1-|\varphi(z)|^2} \\
&+ (1-|z|^2)|g'(z)| \log \frac{2}{1-|z|^2} \cdot \frac{\log 2 - \log(1-|\varphi(z)|^2)}{\log 2 - \log(1-|z|^2)}.
\end{aligned}$$

Just consider those z's tending to $\partial\mathbb{D}$ with $|\varphi(z)| \to 1$.

$$\begin{aligned}
&\lim_{|z|\to 1^-} \frac{\log 2 - \log(1-|\varphi(z)|^2)}{\log 2 - \log(1-|z|^2)} \\
= &\lim_{|z|\to 1^-} \frac{-\log(1-|\varphi(z)|)}{-\log(1-|z|)} \\
\le &\lim_{|z|\to 1^-} \frac{-\log(1-\frac{|z|+|\varphi(0)|}{1+|z||\varphi(0)|})}{-\log(1-|z|)} \\
= &\lim_{|z|\to 1^-} \frac{\log(1-|z|)(1-|\varphi(0)|) - \log(1+|z||\varphi(0)|)}{\log(1-|z|)} \\
= &\lim_{|z|\to 1^-} \frac{-\frac{1}{1-|z|} - \frac{|\varphi(0)|}{1+|z||\varphi(0)|}}{-\frac{1}{1-|z|}} = 1
\end{aligned}$$

where Lemma 4.8 and L'Hospital Law are applied. Thus (4.12) holds for every $\varphi \in S(\mathbb{D})$ by (4.13). The proof is completed by Proposition 4.9. □

By the same method as in Proposition 4.6, we can obtain a necessary condition for **(Q2)** when $X = \mathcal{B}$. Hence we omit the proof.

Proposition 4.12. *Let $\zeta \in \partial\mathbb{D}$, $\varphi \in S(\mathbb{D})$ and $\varphi(z)$ tend to the unit circle when z converges to ζ nontangentially. If*

$$C_\varphi\,|_{\mathcal{B}} \propto_K C_\varphi\,|_{\mathcal{B}} \quad (J\,|_{\mathcal{B}\to\mathcal{B}}),$$

then $\angle - \lim_{z\to\zeta} \varphi(z) = \zeta$.

The following proposition gives a sufficient condition on g to answer **(Q2)**.

Proposition 4.13. *Let $\varphi \in S(\mathbb{D})$ be parabolic type with C^1 boundary fixed point $\zeta \in \partial\mathbb{D}$. If*

$$\sup_{z\in\mathbb{D}} \frac{|\varphi(z)-z|}{1-\max\{|z|, |\varphi(z)|\}} \le \infty, \tag{4.14}$$

then $C_\varphi\,|_{\mathcal{B}} \propto_K C_\varphi\,|_{\mathcal{B}}$ $(J\,|_{\mathcal{B}})$.

Proof. The proposition follows by $\mathcal{B}_{\log} \subset \mathcal{B}$. In fact,

$$\begin{aligned}
&\log\frac{2}{1-|\varphi(z)|^2} \approx \log\frac{2}{1-|\varphi(z)|} \le \log\frac{2(1+|\varphi(z)||z|)}{(1-|z|)(1-|\varphi(z)|)} \\
\lesssim\ &\log\frac{2}{1-|z|} \approx \log\frac{2}{1-|z|^2}
\end{aligned}$$

for z close enough to the unit circle. We can conduct the following estimate

$$\begin{aligned}
&(1-|z|^2)|(g\circ\varphi)'(z)-g'(z)|\log\frac{2}{1-|\varphi(z)|^2}\\
\lesssim\ &(1-|z|^2)|(g\circ\varphi)'(z)-g'(z)|\log\frac{2}{1-|z|^2}\\
\leq\ &(1-|z|^2)\log\frac{2}{1-|z|^2}|\varphi'(z)||g'(\varphi(z))-g'(z)|\\
&+(1-|z|^2)\log\frac{2}{1-|z|^2}|g'(z)||\varphi'(z)-1|.
\end{aligned}$$

The proof will be completed by the same argument as in the proof of Proposition 4.7. □

Remark. Question 1 concerns the subclass of the bounded Volterra operators, whose elements' essential commutants contain all the composition operators. Question 2 concerns the subclass of the composition operators, whose elements' essential commutants contain all the bounded Volterra operators. Answers to **(Q1)** are complete, but we can only find some sufficient or necessary conditions for **(Q2)**. It seems very difficult to answer **(Q2)** completely, since the boundary behavior of a function either in H^∞ or $\mathcal{B}$ can be rather wild.

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