

The Mittag-Leffler Theorem for regular functions of a quaternionic variable

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ABSTRACT. We prove a version of the classical Mittag-Leffler Theorem for regular functions over quaternions. Our result relies upon an appropriate notion of principal part, that is inspired by the recent definition of spherical analyticity.

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1. Introduction

The class of (slice) regular functions of a quaternionic variable was introduced in [8], [9], and proved to be a good counterpart of the class of holomorphic functions, in the quaternionic setting. Regular functions have nice new features, when compared with the classical quaternionic Fueter regular functions: for instance natural polynomials and power series are regular, and regular functions can be expanded in power series on special classes of domain in the space of quaternions \mathbb{H} .

This theory is having a fast development in several directions, and is by now already well established; it has interesting applications to the construction of a noncommutative functional calculus, [3], and to the classification of Orthogonal Complex Structures in subdomains of the space \mathbb{H} , [5]. An exhaustive presentation of this theory can be found in [7].

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Many results that concern regular functions reflect the structure of their complex analogues, other are surprisingly different: for example the zero sets of regular functions (and the sets of poles of *semiregular* functions) consist of isolated points and isolated 2-dimensional spheres.

One of the fundamental results in the theory of holomorphic functions is the celebrated Mittag-Leffler Theorem, that has been used in many different contexts, and in particular in that of sheaves of meromorphic functions.

Theorem 1.1. *Let Ω be an open subset of the complex plane \mathbb{C} , and let $A \subset \Omega$. Let us suppose that A has no accumulation point in Ω and, for any $a \in A$, choose an integer $m(a) \in \mathbb{N}$ and a rational function*

$$P_a(z) = \sum_{j=1}^{m(a)} (z - a)^{-j} c_{j,a}.$$

Then there exists a meromorphic function $f : \Omega \rightarrow \mathbb{C}$, whose principal part at every $a \in A$ is P_a , having no other pole in Ω .

The search for an analogous result for regular functions, connected with the under-construction theory of sheaves of regular and semiregular functions, [4], inspired this work. Since in the new environment of regular functions there are several, inequivalent notions of analyticity, [6], [13], an important step is the choice of the “right” notion of principal part. We adopt here the approach suggested by spherical series, [13], which, together with the quaternionic version of the Runge Theorem, [2], leads to the aimed result.

2. On quaternionic analyticity

With the usual notations, let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ denote the four dimensional noncommutative real algebra of quaternions. For any $q = x_0 + x_1i + x_2j + x_3k \in \mathbb{H}$ let $\operatorname{Re}(q) = x_0$ and $\operatorname{Im}(q) = x_1i + x_2j + x_3k$ denote its real and imaginary parts and let $|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ be its modulus. The definition of regular function is given in terms of the elements of the 2-sphere $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$ of *quaternion imaginary units*.

Definition 2.1. Let Ω be a domain in \mathbb{H} and let $f : \Omega \rightarrow \mathbb{H}$ be a function. For all $I \in \mathbb{S}$, let us set $L_I = \mathbb{R} + I\mathbb{R}$, $\Omega_I = \Omega \cap L_I$ and $f_I = f|_{\Omega_I}$. The function f is called (*slice*) *regular* if, for all $I \in \mathbb{S}$, the restriction f_I is holomorphic, i.e., the function $\bar{\partial}_I f : \Omega_I \rightarrow \mathbb{H}$ defined by

$$\bar{\partial}_I f(x + Iy) = \frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy)$$

vanishes identically.

One of the reasons of the immediate interest for regular functions stays in the fact that an analog of Abel’s Theorem holds: any power series

$$f(q) = \sum_{n \in \mathbb{N}} q^n a_n$$

with quaternionic coefficients $\{a_n\}$ defines a regular function on its ball of convergence $B(0, R) = \{q \in \mathbb{H} : |q| < R\}$. The set of such series inherits the classical multiplication $*$ defined for quaternionic polynomials (or, more in general, for polynomials with coefficients in a noncommutative ring):

$$(1) \quad \left(\sum_{n \in \mathbb{N}} q^n a_n \right) * \left(\sum_{n \in \mathbb{N}} q^n b_n \right) = \sum_{n \in \mathbb{N}} q^n \sum_{k=0}^n a_k b_{n-k}.$$

Let now $(q - q_0)^{*n} = (q - q_0) * \dots * (q - q_0)$ denote the $*$ -product of n copies of $q \mapsto q - q_0$. In [6] series of the form

$$(2) \quad f(q) = \sum_{n \in \mathbb{N}} (q - q_0)^{*n} a_n$$

are studied, whose sets of convergence are balls with respect to the distance $\sigma : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$ defined in the following fashion.

Definition 2.2. For all $p, q \in \mathbb{H}$, we set

$$(3) \quad \sigma(q, p) = \begin{cases} |q - p| & \text{if } p, q \text{ lie on the same complex plane } \mathbb{R} + I\mathbb{R} \\ \omega(q, p) & \text{otherwise} \end{cases}$$

where

$$(4) \quad \omega(q, p) = \sqrt{[Re(q) - Re(p)]^2 + [Im(q)| + |Im(p)]|^2}.$$

A new notion of analyticity can be given in terms of the distance σ :

Definition 2.3. If Ω is a domain in \mathbb{H} , a function $f : \Omega \rightarrow \mathbb{H}$ is called σ -analytic if it admits at every $q_0 \in \Omega$ an expansion of type (2) that is valid in a σ -ball $\Sigma(q_0, R) = \{q \in \mathbb{H} : \sigma(q, q_0) < R\}$ of positive radius R .

Regularity and σ -analyticity turn out to be the same notion, as it appears in the following result proved in [6].

Theorem 2.4. *If Ω is a domain in \mathbb{H} , a function $f : \Omega \rightarrow \mathbb{H}$ is regular if and only if it is σ -analytic.*

The meaning of Theorem 2.4 is not as strong as in the complex case, since σ -analyticity has not the features one may imagine at a first glance. In fact the topology induced by the distance σ is finer than the Euclidean: if $q_0 = x_0 + Iy_0$ does not lie on the real axis then for $R < y_0$ the σ -ball $\Sigma(q_0, R)$ reduces to a (2-dimensional) disc $\{z \in L_I : |z - q_0| < R\}$ in the complex plane L_I through q_0 (see [6] for a presentation of the shape of σ -balls). Hence the behavior of f in a Euclidean neighborhood of q_0 cannot be envisaged by the series expansion (2), which, in general, will only represent f along the complex plane L_I containing q_0 . To understand this phenomenon we will present what we believe to be a meaningful example (see, e.g., [7]).

Example 2.5. Let Δ be the open unit disc centered at the origin of $L_i = \mathbb{R} + i\mathbb{R} = \mathbb{C}$ and let $f : \Delta \rightarrow \mathbb{C}$, $f(z) = \sum_{n \in \mathbb{N}} z^n a_n$ be a holomorphic function whose maximal domain of definition is Δ . Then the power series

$$f(q) = \sum_{n \in \mathbb{N}} \left(q - \frac{3}{4}i \right)^n a_n$$

does not converge on a Euclidean neighborhood of $\frac{3}{4}i$ but only in a 2-dimensional disc of \mathbb{C} containing $\frac{3}{4}i$.

As explained in [1], the situation is much better if the domain Ω is carefully chosen. Consider the following class of domains:

Definition 2.6. Let Ω be a domain in \mathbb{H} . If

$$\Omega = \bigcup_{x+Iy \in \Omega} x + y\mathbb{S}$$

then Ω is called an (*axially*) *symmetric domain*. If the domain Ω intersects the real axis and is such that for all $I \in \mathbb{S}$, $\Omega_I = \Omega \cap L_I$ is a domain in $L_I \simeq \mathbb{C}$ then Ω is called a *slice domain*.

Regular functions f on symmetric slice domains are affine when restricted to a single 2-sphere $x + y\mathbb{S}$ (see, e.g., [1, Theorem 3.2], [13, Theorem 1.10]). As a consequence, if f is a regular function on a symmetric slice domain then its values can all be recovered from those of one of its restrictions f_I . This last fact leads to the definition of a stronger form of analyticity than the one presented in Theorem 2.4, which is related to a different type of series expansion valid in Euclidean open sets. If we denote as $R_{q_0}f : \Omega \rightarrow \mathbb{H}$ the function such that

$$f(q) = f(q_0) + (q - q_0) * R_{q_0}f(q),$$

then the following result holds (see [13, Theorem 4.1]).

Theorem 2.7. *Let f be a regular function on a symmetric slice domain Ω , and let $x_0, y_0 \in \mathbb{R}$ and $R > 0$ be such that*

$$U(x_0 + y_0\mathbb{S}, R) = \{q \in \mathbb{H} : |(q - x_0)^2 + y_0^2| < R^2\} \subseteq \Omega.$$

For all $q_0 \in x_0 + y_0\mathbb{S}$, setting

$$A_{2n} = (R_{\bar{q}_0}R_{q_0})^n f(q_0)$$

and

$$A_{2n+1} = R_{q_0}(R_{\bar{q}_0}R_{q_0})^n f(\bar{q}_0),$$

we have that

$$(5) \quad f(q) = \sum_{n \in \mathbb{N}} [(q - x_0)^2 + y_0^2]^n [A_{2n} + (q - q_0)A_{2n+1}]$$

for all $q \in U(x_0 + y_0\mathbb{S}, R)$.

Here is the announced notion of analyticity, [13].

Definition 2.8. Let f be a regular function on a symmetric slice domain Ω . We say that f is *symmetrically analytic* if it admits at any $q_0 \in \Omega$ an expansion of type (5) valid in a Euclidean neighborhood of q_0 .

Thanks to the previous theorem, we obtain:

Corollary 2.9. Let Ω be a symmetric slice domain. A function $f : \Omega \rightarrow \mathbb{H}$ is regular if, and only if, it is symmetrically analytic.

3. Principal part of a semiregular function

Definition 3.1. Let f be a regular function on a symmetric slice domain Ω . We say that a point $p = x + yI_p \in \mathbb{H}$ is a *singularity* for f if $f_{I_p} : \Omega_{I_p} \rightarrow \mathbb{H}$ has a singularity at p . In other words if there exists $R > 0$ such that f has the Laurent expansion $f(z) = \sum_{n \in \mathbb{Z}} (z - p)^n a_n$ converging for any $z \in L_{I_p}$ with $0 < |z| < R$.

As proven in [12], if $p = x + yI_p$ is a singularity for a regular function f , then f admits a regular Laurent expansion

$$(6) \quad f(q) = \sum_{n \in \mathbb{Z}} (q - p)^{*n} a_n,$$

converging in $\Sigma(p, R) \setminus \{x + y\mathbb{S}\}$, whose restriction to L_{I_p} coincides with the Laurent expansion of f_{I_p} at p . It is clear that, as it happens for regular power series of type (2), the domains of convergence of regular Laurent series are not always open sets. Inessential singularities are defined as follows.

Definition 3.2. Let p be a singularity for f . We say that p is a *removable singularity* if f extends to a neighborhood of p as a regular function. Otherwise consider the expansion

$$(7) \quad f(q) = \sum_{n \in \mathbb{Z}} (q - p)^{*n} a_n.$$

We say that p is a *pole* for f if there exists an $m \geq 0$ such that $a_{-k} = 0$ for all $k > m$.

We can now recall the notion of semiregular function, analogue to that of meromorphic function in the complex setting.

Definition 3.3. A function f is *semiregular* in a symmetric slice domain Ω if it is regular in a symmetric slice domain $\Omega' \subseteq \Omega$ such that every point of $\mathcal{S} = \Omega \setminus \Omega'$ is a pole (or a removable singularity) for f .

If f is semiregular in Ω then the set \mathcal{S} of its nonremovable poles consists of isolated real points or isolated 2-spheres of type $x + y\mathbb{S}$.

The following result shows how we can “extract” a pole from a semiregular function, see [12].

Theorem 1. *Let $f : \Omega \rightarrow \mathbb{H}$ be a semiregular function on a symmetric slice domain with a pole at $x_0 + y_0\mathbb{S} \subset \Omega$. Then there exist $k \in \mathbb{N}$ and a unique semiregular function g on Ω , regular on a symmetric slice neighborhood of $x_0 + y_0\mathbb{S}$, such that*

$$f(q) = ((q - x_0)^2 + y_0^2)^{-k} g(q).$$

In this case, the *spherical order* of the pole is $2k$ at every point of $x_0 + y_0\mathbb{S}$ with the possible exception of one single point, where the spherical pole has lesser order.

Using the spherical series expansion (5) for regular functions we can give the following Definition (see also [10]):

Definition 3.4. Let $\Omega \subset \mathbb{H}$ be a symmetric slice domain, let $f : \Omega \rightarrow \mathbb{H}$ be a semiregular function with a pole of spherical order $2k$ at the sphere $x_0 + y_0\mathbb{S}$, and let q_0 be any point of $x_0 + y_0\mathbb{S}$. Then the *spherical Laurent series* of f at the sphere $x_0 + y_0\mathbb{S}$ is:

$$\begin{aligned} f(q) &= \sum_{j \geq 0} ((q - x_0)^2 + y_0^2)^{j-k} [A_{2j} + (q - q_0)A_{2j+1}] \\ &= \sum_{n \geq -k} ((q - x_0)^2 + y_0^2)^n [A_{2(n+k)} + (q - q_0)A_{2(n+k)+1}] \end{aligned}$$

converging in a symmetric slice open set $U(x_0 + y_0\mathbb{S}, R) \setminus \{x_0 + y_0\mathbb{S}\}$. Moreover, the *principal part* of f at the spherical pole $x_0 + y_0\mathbb{S}$ is defined as

$$P_{x_0+y_0\mathbb{S}}(q) = \sum_{n=1}^k ((q - x_0)^2 + y_0^2)^{-n} [A_{2(k-n)} + (q - q_0)A_{2(k-n)+1}].$$

The use of the spherical Laurent series approach to the Mittag-Leffler Theorem is motivated by the fact that a principal part defined using the apparently simpler regular Laurent series could vary for points of a same spherical pole $x + y\mathbb{S}$.

4. The Mittag-Leffler Theorem

We can now prove the announced result, that states that we can find a semiregular function having prescribed poles and prescribed principal parts. Denote by $\hat{\mathbb{H}}$ the Alexandrov compactification of \mathbb{H} .

Theorem 2. *Let $\Omega \subseteq \mathbb{H}$ be a symmetric slice domain and let*

$$S = \{x_\alpha + y_\alpha\mathbb{S}\}_{\alpha \in A}$$

be a closed and discrete set of two dimensional spheres (or real points) contained in Ω . For every $\alpha \in A$ let $q_\alpha = x_\alpha + y_\alpha I$, with I any imaginary unit, $m(\alpha) \in \mathbb{N}$ and

$$P_\alpha(q) = \sum_{n=1}^{m(\alpha)} ((q - x_\alpha)^2 + y_\alpha^2)^{-n} [A_{2n} + (q - q_\alpha)A_{2n+1}]$$

with $A_j \in \mathbb{H}$ for any $j = 2, \dots, 2m(\alpha) + 1$. Then there exists f semiregular on Ω such that for every $\alpha \in A$ the principal part of f at $x_\alpha + y_\alpha \mathbb{S}$ is $P_\alpha(q)$ and such that f does not have other poles in Ω .

Proof. Let $I \in \mathbb{S}$. Thanks to known results in the complex case (see, e.g., Theorem 13.3 in [11]) we can find a covering $\{K_n^I\}_{n \in \mathbb{N}}$ of Ω_I such that: K_n^I is a compact set, K_n^I is contained in the interior of K_{n+1}^I , every compact subset of Ω_I is contained in K_n^I for some $n \in \mathbb{N}$ and every connected component of $\hat{L}_I \setminus K_n^I$ contains a connected component of $\hat{L}_I \setminus \Omega_I$. The fact that Ω is a symmetric domain yields that setting, for any $n \in \mathbb{N}$, K_n to be the symmetrization of K_n^I , we obtain a covering of Ω such that K_n is a compact set, K_n is contained in the interior of K_{n+1} , every compact subset of Ω is contained in K_n for n sufficiently large, and every connected component of $\hat{\mathbb{H}} \setminus K_n$ contains a connected component of $\hat{\mathbb{H}} \setminus \Omega$. Moreover, since Ω is a slice domain, we can suppose that K_n is also slice for any $n \in \mathbb{N}$. Let us set

$$S_1 := S \cap K_1 \quad \text{and} \quad S_n := S \cap (K_n \setminus K_{n-1}).$$

The compactness of K_n guarantees that S_n is a finite set of spheres (or real points). For any $n \in \mathbb{N}$ define

$$Q_n(q) = \sum_{\alpha \in S_n} P_\alpha(q).$$

Notice that, for every $n \in \mathbb{N}$, Q_n is a rational function, regular on an open neighborhood of K_{n-1} . Thanks to the Runge Theorem for regular functions (see Theorem 4.10 in [2]), for any $n \in \mathbb{N}$ we can find a rational function R_n having (prescribed) poles outside Ω and such that

$$(8) \quad |R_n(q) - Q_n(q)| < 2^{-n} \quad \text{for any } q \in K_{n-1}.$$

Consider now the semiregular function $f : \Omega \rightarrow \mathbb{H}$ defined by

$$f(q) := Q_1(q) + \sum_{n \geq 2} (Q_n(q) - R_n(q)).$$

We aim to show that f is the desired function. Fix $N \in \mathbb{N}$ and split f as

$$f(q) = Q_1(q) + \sum_{n=2}^N (Q_n(q) - R_n(q)) + \sum_{n \geq N+1} (Q_n(q) - R_n(q)).$$

The last term is an infinite sum of functions which are regular in the interior of K_N . Thanks to equation (8), we get that it converges uniformly to a regular function on the interior of K_N (see, e.g., [10, Remark 3.3]). Hence the function

$$f(q) - Q_1(q) - \sum_{n=2}^N (Q_n(q) - R_n(q))$$

is regular in the interior of K_N as well, which means that the principal parts of f at the poles contained in K_N are exactly the prescribed $P_\alpha(q)$

for $\alpha \in \bigcup_{n=1}^N S_n$. Since N was arbitrary, we conclude that f is the desired function. \square

As we already noticed, unlike the case of holomorphic functions, the poles of a regular function over quaternions can be either isolated real points or isolated 2-spheres of the form $x + y\mathbb{S}$. To conclude, we present two simple, meaningful examples of the Mittag-Leffler phenomenon in the case of semiregular functions.

Example 4.1. We calculate a semiregular function defined in the entire space of quaternions, such that:

- (1) Its only poles are all the 2-spheres $n + \mathbb{S}$, centered at $n \in \mathbb{Z}$ with radius 1.
- (2) At each such sphere, the principal part is

$$P_{n+\mathbb{S}}(q) = ((q - n)^2 + 1)^{-1}$$

with minimum possible spherical order equal to 2.

Since, for any $N \in \mathbb{N}$, both

$$\sum_{n \geq N+1} \frac{1}{(q - n)^2 + 1}$$

and

$$\sum_{n \geq N+1} \frac{1}{(q + n)^2 + 1}$$

converge uniformly to a regular function inside the open ball centered at the origin and having radius N , we get that the function

$$f(q) = \sum_{n \in \mathbb{Z}} \frac{1}{(q - n)^2 + 1}$$

is the desired semiregular function.

Example 4.2. A phenomenon peculiar to the quaternionic setting, is that of a semiregular function having infinitely many spheres of poles with spherical order 2 at each point, except for one point (on every sphere) which has lesser order. To give an example of this phenomenon we compute a semiregular function, defined on the entire space of quaternions, such that:

- (1) Its only poles are all the 2-spheres $n + \mathbb{S}$, centered at $n \in \mathbb{Z}$ with radius 1.
- (2) At each such sphere, the principal part is

$$P_{n+\mathbb{S}}(q) = ((q - n)^2 + 1)^{-1}(q - n - i).$$

In this case it is immediate to see that the series

$$\sum_{n \in \mathbb{Z}} P_{n+\mathbb{S}}(q) = \sum_{n \in \mathbb{Z}} \frac{q - n - i}{((q - n)^2 + 1)}$$

does not converge and hence does not define a semiregular function on \mathbb{H} . However if we sum up the two terms

$$\frac{q-n-i}{((q-n)^2+1)} + \frac{q+n-i}{((q+n)^2+1)} = \frac{2(q^3 - q^2i + q(1-n^2) - (n^2+1)i)}{((q+n)^2+1)((q-n)^2+1)}$$

we get, arguing as in the first example, that

$$\frac{q-i}{q^2+1} + \sum_{n \geq 1} \frac{2(q^3 - q^2i + q(1-n^2) - (n^2+1)i)}{((q+n)^2+1)((q-n)^2+1)}$$

defines the semiregular function we were looking for.

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