New York Journal of Mathematics

New York J. Math. **22** (2016) 1135–1137.

Automorphisms of free groups. I — erratum

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Abstract. I report an error in Theorem A of *Automorphisms of free groups. I*, New York J. Math. **19** (2013), 395–421, where it was claimed that two filtrations of the group of IA automorphisms of a free group coincide up to torsion.

In fact, using a recent result by Day and Putman, I show that, for a free group of rank 3, the opposite conclusion holds, namely that the two series differ rationally.

1. Introduction

Let F denote a free group of rank r. Filter F by its lower central series $(F_n)_{n\geq 1}$, defined by $F_1 = F$ and $F_n = [F, F_{n-1}]$. Let A denote the automorphism group of F, and let A_1 denote the kernel of the natural map $A \to \mathsf{GL}_r(\mathbb{Z}) = \mathsf{Aut}(F/F')$. The group A_1 has two natural filtrations: on the one hand, its lower central series, defined as above by $\gamma_1 = A_1$ and $\gamma_n = [A_1, \gamma_{n-1}]$, and on the other hand $A_n = \ker(A_1 \twoheadrightarrow \mathsf{Aut}(F/F_{n+1}))$. We have $\gamma_n \leq A_n$ for all n.

Andreadakis conjectures [1, page 253] that $A_n = \gamma_n$, and proves his assertion for $r = 3, n \leq 3$ and for $r = 2$. This is further developed by Pettet [7], who proves that γ_3 has finite index in A_3 for all r, building her work on Johnson's homomorphism [6].

It was noted in [3, Theorem A] that, if $r = 3$, the groups γ_7 and A_7 differ, disproving Andreadakis's conjecture. It was however also erroneously claimed there that A_n/γ_n is a finite group for all n. The "proof" relied on the unproven assertion that the filtrations $(\gamma_n)_{n\geq 1}$ and $(A_n)_{n\geq 1}$ define the same topology on A_1. Theorem A should be replaced by the following statement:

Theorem $\overline{\text{A}}$. *The filtrations $(\gamma_n)_{n\geq 1}$ and $(A_n)_{n\geq 1}$ differ rationally at $n = 4$ for $r = 3$, and we have*

$$(A_4/\gamma_4) \otimes \mathbb{Q} \cong \mathbb{Q}^3.$$

Received September 20, 2016.

2010 *Mathematics Subject Classification.* 20E36, 20F28, 20E05, 20F40.

Key words and phrases. Lie algebra; Automorphism groups; Lower central series.

ISSN 1076-9803/2016

Let $\widehat{A_1} = \operatorname{proj\,lim} A_1/A_n$ *denote the completion of* A_1 *under the filtration* $(A_n)_{n\ge1}$*, let* $(\widehat{\gamma_n})_{n\ge1}$ *denote its closed lower central series, and let* $(\widehat{A_n})_{n\ge1}$ *denote the closure of* $(A_n)_{n\ge1}$ *in* $\widehat{A_1}$*. Then* $\widehat{A_7}/\widehat{\gamma}_7 \cong \mathbb{Z}/3$*.*

Proof. In [8], Day and Putman give explicit presentations of A_1 for all r, by generators, relators and endomorphisms (see [2]). Here is a small adaptation of their result. Let E be the free group generated by the set

$$S := \{M_{i,[j,k]} : 1 \le j \ne i \ne k \le r, j < k\} \cup \{C_{i,j} : 1 \le i \ne j \le r\}.$$

These are the Magnus generators of A_1, and act on F respectively by

$$x_i \mapsto x_i[x_j, x_k] \quad \text{and} \quad x_i \mapsto x_i^{x_j},$$

all other generators being fixed.

Day and Putman give explicit finite sets $R \subset E'$ (of size around 30) and $\Theta \subset \operatorname{End}(E)$ (of size around 4) such that

$$A_1 \cong \langle S \mid w^\theta \text{ for all } w \in R \text{ and all } \theta \in \Theta^*\rangle.$$

Furthermore, Θ induces automorphisms of A_1 that generate the conjugation action of $\mathsf{Aut}(F)$ on A_1.

Using the algorithm described in [4], implemented in [5], it is possible to compute nilpotents quotients of A_1 of arbitrary class. I entered Day and Putman's presentation in GAP for $r = 3$, and computed (in about 1 minute) its class-4 quotient. The result, atop the calculations in [3] gives (with a^b for $(\mathbb{Z}/a\mathbb{Z})^b$)

$n =$	1	2	3	4
γ_n/γ_{n+1}	$\mathbb{Z}^9$	$\mathbb{Z}^{18}$	$\mathbb{Z}^{43} \times 2^{14} \times 3^9$	$\mathbb{Z}^{123} \times 2^{50} \times 4^3 \times 8^3 \times 3^{45} \times 9^9$.
A_n/A_{n+1}	$\mathbb{Z}^9$	$\mathbb{Z}^{18}$	$\mathbb{Z}^{43}$	$\mathbb{Z}^{120}$

We deduce $A_5/\gamma_5 \cong \mathbb{Z}^3 \times \text{torsion}$.

For the second claim, it suffices to note that the computer calculations described in [3] actually manipulate (approximations of) the group $\widehat{A_1}$ rather than A_1. □

Acknowledgments

I am grateful to Matt Day and Andy Putman for their generous insights, discussions and patience in resolving the discrepancy between their work and the original Theorem A.

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This paper is available via http://nyjm.albany.edu/j/2016/22-52.html.