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A short proof that $\mathrm{Diff}_c(M)$ is perfect

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Abstract. In this note, we follow the strategy of Haller, Rybicki and Teichmann to give a short, self contained, and elementary proof that $\mathrm{Diff}_0(M)$ is a perfect group, given a theorem of Herman on diffeomorphisms of the circle.

Contents

1. Introduction

Let M be a smooth manifold of dimension $n > 1$ and let $\mathrm{Diff}_c(M)$ denote the group of diffeomorphisms supported on compact sets and isotopic to the identity through a compactly supported isotopy. Note that if M is compact, then $\mathrm{Diff}_c(M) = \mathrm{Diff}_0(M)$, the group of isotopically trivial diffeomorphisms. That $\mathrm{Diff}_c(M)$ is a perfect group was first proved by Thurston, as announced in [Th74]. The proof relies on the relationship between the homology of $\mathrm{Diff}_c(M)$ and certain classifying spaces of foliations. Recently, Haller, Rybicki and Teichmann gave a fundamentally different proof in [HT03] and [HRT13]. In fact, they prove a stronger form of "smooth perfection" and give bounds on commutator width of $\mathrm{Diff}_c(M)$ for some manifolds.

Bounds on commutator width have also been given in [BIP08], [Ts09] and [Ts12]. In particular, *given* the result that $\mathrm{Diff}_c(\mathbb{R}^n)$ is perfect, Burago, Ivanov, and Polterovich show in [BIP08, Lemma 2.2] that any element of $\mathrm{Diff}_c(\mathbb{R}^n)$ can be written as a product of two commutators; an isotopically trivial diffeomorphism of a compact 3-manifold can be written as a product of 10 commutators, and an element of $\mathrm{Diff}_0(S^n)$ as a product of 4 commutators. Related results were obtained by Tsuboi in [Ts08], who later gave general bounds on commutator width of $\mathrm{Diff}_0(M)$, depending on M ([Ts09], [Ts12]).

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The purpose of this note is to show that if one only wants to show that $\mathrm{Diff}_c(M)$ is perfect, then the techniques of Haller, Rybicki and Teichmann provide a remarkably simple proof. Our exposition closely follows the strategy of [HT03], but avoids discussion of the tame Fréchet manifold structure on $\mathrm{Diff}_0(M)$. As the perfectness of these diffeomorphism groups is widely cited, we thought it worthwhile to make available this short and widely accessible proof. We show the following.

Theorem 1.1. *Let M be a smooth manifold, $M \neq \mathbb{R}$. Then $\mathrm{Diff}_c(M)$ is perfect. In fact, any compactly supported diffeomorphism g can be written as a product of commutators $g = [g_1, f_1][g_2, f_2] \dots [g_r, f_r]$ where each f_i is the time one map of a (time independent) vector field X_i on M.*

In particular, this result can then be fed into Lemma 2.2 of [BIP08] to obtain their bounds on commutator width. The assumption $M \neq \mathbb{R}$ seems essential to this proof, although $\mathrm{Diff}_c(\mathbb{R})$ is also perfect. The proof uses only one deep theorem, a result of Herman on circle diffeomorphisms.

Theorem 1.2 ([He79])**.** *There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_0(S^1)$ and a dense set of rotations R_θ by angles $\theta \in [0, 2\pi)$ such that any $g \in \mathcal{U}$ can be written as $R_\lambda[g_0, R_\theta]$ for some rotation R_λ and some $g_0 \in \mathrm{Diff}_0(S^1)$. Moreover, λ and g_0 can be chosen to vary smoothly in g, with $\lambda = 0$ and $g_0 = \mathrm{id}$ at $g = \mathrm{id}$.*

"Vary smoothly in g" can be made precise with reference to the Fréchet structure on $\mathrm{Diff}_0(M)$, but for our purposes the reader may take it to mean the following.

Definition 1.3. A *smooth family* in $\mathrm{Diff}_c(M)$ is a family $\{g_t : t \in [0,1]\}$ such that the map $(x,t) \mapsto (g_t(x), t)$ is a smooth diffeomorphism of $M \times [0,1]$. A map $\phi : \mathrm{Diff}_c(M) \to \mathrm{Diff}_c(N)$ *varies smoothly* if it maps smooth families to smooth families.

A more general version of Herman's theorem on diffeomorphisms of the n-dimensional torus is used in both Thurston's original proof and the Haller-Rybciki-Teichmann proof, though Haller, Rybciki and Teichmann state that their methods work using only Herman's theorem for S^1. This note provides the details.

2. Reduction to $M = \mathbb{R}^n$ and diffeomorphisms near identity

Recall that the *support* of a diffeomorphism g is the closure of the set $\{x \in M \mid g(x) \neq x\}$. The first step in the proof of Theorem 1.1 is to reduce it to the case of compactly supported diffeomorphisms on $M = \mathbb{R}^n$. This reduction is a consequence of the well-known fragmentation property. For simplicity, we assume M is compact.

Lemma 2.1 (Fragmentation)**.** *Let $\{U_i\}$ be a finite open cover of M. Then any $g \in \mathrm{Diff}_0(M)$ can be written as a composition $g_1 \circ g_2 \circ \dots \circ g_n$ of* *diffeomorphisms where each g_i has support contained in some element of $\{U_i\}$.*

Proof. The proof is straightforward, for completeness we outline it here, following [Ba97, Ch. 2]. Let g_t be an isotopy from $g_0 = id$ to $g_1 = g$. By writing

$$g = g_{1/r} \circ (g_{1/r}^{-1}\, g_{2/r}) \circ \cdots \circ (g_{r-1/r}^{-1}\, g_1)$$

for r large, and working separately with each factor $g_{k-1/r}^{-1}\, g_{k/r}$, we may assume that g and g_t lie in an arbitrarily small neighborhood of the identity.

Take a partition of unity λ_i subordinate to $\{U_i\}$ and define

$$\mu_k := \sum_{i \le k} \lambda_i.$$

Now define $\psi_k(x) := g_{\mu_k(x)}(x)$. This is a C^∞ map, and can be made as close to the identity as we like by taking g_t close to the identity. Although ψ_k is not *a priori* invertible, being invertible with smooth inverse is an open condition. Thus, ψ_k being sufficiently close to the identity *implies* that it is a diffeomorphism. By definition, ψ_k agrees with ψ_{k-1} outside of U_k, and hence $g = (\psi_0^{-1}\psi_1)(\psi_1^{-1}\psi_2)\dots(\psi_{n-1}^{-1}\psi_n)$ is the desired decomposition of g with each diffeomorphism $\psi_{k-1}^{-1}\psi_k$ supported on U_k. □

To prove Theorem 1.1, it is also sufficient to prove that some neighborhood of the identity in $\mathrm{Diff}_c(\mathbb{R}^n)$ is perfect, because any neighborhood of the identity generates $\mathrm{Diff}_c(\mathbb{R}^n)$. The strategy is to first prove perfectness of a neighborhood of the identity for S^1, move to $\mathbb{R}^2$, and then induct on dimension.

3. Proof for S^1 and diffeomorphisms preserving vertical lines

Perfectness of $\mathrm{Diff}_0(S^1)$ is a consequence of Herman's theorem together with the fact that $\mathrm{PSL}(2,\mathbb{R})$ is perfect, so any rotation can be written as a commutator.

Lemma 3.1 (Perfectness for S^1)**.** *There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_0(S^1)$ and $f_1, \dots, f_4 \in \mathrm{Diff}_0(S^1)$ such that any $g \in \mathcal{U}$ can be written $g = [g_1, f_1] \dots [g_4, f_4]$, with each g_i depending smoothly on g.*

Furthermore, we may take $f_i = \exp(X_i)$ to be the time one map of a vector field, and may take $g_i = \mathrm{id}$ when $g = \mathrm{id}$.

Proof. Let $\mathcal{U}$ be a neighborhood of the identity as in Herman's theorem and let $g \in \mathcal{U}$. Then g can be written as $R_\lambda[g_0, R_\theta]$ with λ and g_0 depending smoothly on g. Let $f_4 = R_\theta$; this is indeed the time one map of a (constant) vector field. We need to write the rotation R_λ as a product of commutators $[g_1, f_1][g_2, f_2][g_3, f_3]$ with g_i depending smoothly on λ, and will do this working inside of $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Diff}_0(S^1)$. This can be done completely explicitly: take $f_1 = f_3 = \exp\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)$ and $f_2 = \exp\left(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right)$, and define $g_\alpha = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$.

Then

$$[g_\alpha, f_1][g_\beta, f_2][g_\alpha, f_3] = \begin{pmatrix} 1 & \alpha^2-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta^{-2}-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha^2-1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+(\alpha^2-1)(\beta^{-2}-1) & 2(\alpha^2-1)+(\alpha^2-1)^2(\beta^{-2}-1) \\ \beta^{-2}-1 & 1+(\alpha^2-1)(\beta^{-2}-1) \end{pmatrix}.$$

This is the matrix of rotation by $\lambda := \sin^{-1}(\beta^{-2}-1)$ provided that $-(\beta^{-2}-1) = 2(\alpha^2-1)+(\alpha^2-1)^2(\beta^{-2}-1)$. If α is close to 1, then there exists β (close to 1) satisfying this equation, namely

$$\beta = \left(\frac{-2(\alpha^2-1)}{1+(\alpha^2-1)^2} + 1 \right)^{-1/2}.$$

In fact, the inverse function theorem implies that $\alpha \mapsto \left(\frac{-2(\alpha^2-1)}{1+(\alpha^2-1)^2} + 1 \right)^{-1/2}$ is a local diffeomorphism of $\mathbb{R}$ at $\alpha = 1$. Since $\beta \mapsto \sin^{-1}(\beta^{-2}-1)$ is also a local diffeomorphism at $\beta = 1$ onto a neighborhood of 0, this shows that α and β can be chosen to smoothly depend on λ, and approach 1 as $\lambda \to 0$.

Alternatively, one can see that such f_i exist from the fact that $\mathrm{PSL}(2,\mathbb{R})$ is a three dimensional perfect Lie group. See [HT03, Sect. 4] for details and further generalizations. □

Remark 3.2. Above, we showed that every diffeomorphism in a neighborhood of the identity could be written as a product of four commutators of a specific form. Relaxing this condition allows one to (easily) write every element g in a neighborhood of id in $\mathrm{Diff}(S^1)$ as a product of two commutators, $g = [a_1, b_1][a_2, b_2]$ with a_i and b_i depending smoothly on g. To do so, Herman's theorem again implies that it suffices to write a rotation R_λ as $[a_1, b_1]$, with a_1 and b_1 depending smoothly on λ, and this can be done in $\mathrm{PSL}(2,\mathbb{R})$, either explicitly or with an elementary argument using hyperbolic geometry as in [Gh01, Prop 5.11].

As a consequence, we now prove a perfectness result for compactly supported diffeomorphisms of $\mathbb{R}^n$ that preserve vertical lines.

Proposition 3.3. *Let $U \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ be precompact, and V a neighborhood of the closure of U. There exist vector fields $Y_1, \ldots, Y_4$ supported on V with the following property:*

- *If $g \in \mathrm{Diff}_c(\mathbb{R}^n)$ is supported on U, sufficiently close to the identity, and preserves each vertical line $\mathbb{R}^{n-1} \times \{x\}$, then g can be decomposed as*
$$g = [g_1, \exp(Y_1)] \ldots [g_4, \exp(Y_4)]$$
with g_i supported on V and depending smoothly on g.

Proof. Let B^{n-1} be a ball in $\mathbb{R}^{n-1}$. There exists an embedding ϕ of $S^1 \times B^{n-1}$ in $\mathbb{R}^n$ with $U \subset \phi(S^1 \times \{b\}) \subset V$, and such that for each $b \in B^{n-1}$ the image $\phi(S^1 \times \{b\}) \cap U$ is a vertical line segment as in Figure 1.

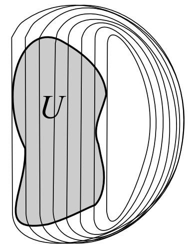

FIGURE 1. An embedding of $S^1 \times B^1$ in $\mathbb{R}^2$ giving a vertical foliation of U

If g preserves vertical lines, then we can consider it as a diffeomorphism $\mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}$ of the form $(x, y) \mapsto (x, \hat{g}(x, y))$. For each $x \in \mathbb{R}^{n-1}$ let $g_x(y)$ denote $\hat{g}(x, y)$. Then g_x has support on a vertical line in U so we can consider it as a diffeomorphism of S^1 by pulling it back to $S^1 \times \{b\}$ via ϕ. Using Lemma 3.1, write the pullback $\phi^*(g_x) = [g_{x,1}, \exp(X_1)] \ldots [g_{x,4}, \exp(X_4)]$. Now push the vector fields X_i on each $S^1 \times \{b\}$ forward to $\mathbb{R}^n$ to get vector fields on $\phi(S^1 \times B)$ tangent to $\phi(S^1 \times \{b\})$ and extend these smoothly to vector fields Y_i with support in V. The smooth dependence of $g_{x,i}$ on g_x and hence on x means that the functions $\phi g_{x,i} \phi^{-1}$ on the vertical lines $\phi(S^1 \times \{b\})$ piece together to form smooth functions g_i on the image of ϕ. Since $g = \mathrm{id}$ on the boundary of the image of ϕ, Lemma 3.1 implies that $g_i = \mathrm{id}$ as well, so it can be extended (trivially) to a diffeomorphism of $\mathbb{R}^n$. Now $g = [g_1, \exp(Y_1)] \ldots [g_4, \exp(Y_4)]$ on the image of ϕ by construction, and both are equal to the identity everywhere else. □

4. Proof for $\mathbb{R}^n$

The proof of Theorem 1.1 for $\mathbb{R}^n$ will follow from a short inductive argument using Proposition 3.3 and the following lemma.

Lemma 4.1. *There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_c(\mathbb{R}^n)$ such that any $f \in \mathcal{U}$ can be written as $g \circ h$, where h preserves each vertical line and g preserves each horizontal hyperplane. Moreover, g and h can be chosen to depend smoothly on f.*

In other words, if $x = (x_1, \ldots, x_{n-1})$, this Lemma says that we may take h to be of the form $h(x, y) = (x, \hat{h}(x, y))$ and g of the form $g(x, y) = (\hat{g}(x, y), y)$.

Proof. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denote projection to the i^{th} coordinate. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is compactly supported and sufficiently C^∞ close to the identity. Then for any point $(x, y) = (x_1, \ldots, x_{n-1}, y)$ the map $f_x : \mathbb{R} \to \mathbb{R}$ given by $f_x(y) = \pi_n f(x, y)$ is a diffeomorphism. (Injectivity follows from the fact that tangent vectors to vertical lines remain nearly vertical under a diffeomorphism close to the identity – if $\pi_n f(x, y_1) = \pi_n f(x, y_2)$ for some $y_1 \neq y_2$, then the image of f_x has horizontal tangent at some point $y \in [y_1, y_2]$.)

Now given f, define h and $g : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1} \times \mathbb{R}$ by

$$h(x, y) = (x, f_x(y)),$$
$$g(x, y) = (g_1(x, y), \dots g_{n-1}(x, y), y),$$

where $g_i(x, y) = \pi_i(x, f_x^{-1}(y)) \in \mathbb{R}$. Then $f = g \circ h$ and g and h vary smoothly with f. □

Proof of Theorem 1.1. We induct on the dimension n. The case $n = 2$ follows from Lemma 4.1 using $n = 2$, together with Proposition 3.3 applied to g and h in the decomposition (Proposition 3.3 works just as well for the diffeomorphism g, which preserves horizontal rather than vertical lines).

Now suppose Theorem 1.1 holds for $n = k$, and let $f \in \mathrm{Diff}_c(\mathbb{R}^{k+1})$ be close to the identity. By Lemma 4.1, $f = g \circ h$, where h preserves each vertical line and g preserves each horizontal hyperplane in $\mathbb{R}^{k+1}$, and g and h are close to the identity. By our inductive assumption, there are smooth vector fields $X_1, \dots, X_{r(k)}$ tangent to each horizontal hyperplane such that $g = [g_1, \exp(X_1)] \dots [g_r, \exp(X_{r(k)})]$ where the g_i preserve horizontal hyperplanes as well. Technically speaking, our hypothesis gives vector fields X_i and diffeomorphisms g_i defined separately on each $\mathbb{R}^k$–hyperplane, but the proof of Proposition 3.3 allows us to choose them so that they vary smoothly and form global vector fields or diffeomorphisms on $\mathbb{R}^{k+1}$. By Proposition 3.3, there are also vector fields $Y_1, \dots, Y_4$ supported on a neighborhood of $\mathrm{supp}(h)$ so that $h = [h_1, \exp(Y_1)] \dots [h_4, \exp(Y_4)]$. Thus, $f = g \circ h$ is a product of commutators as desired. □

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