New York Journal of Mathematics

New York J. Math. **21** (2015) 801–809.

Infinitesimal local boundary dilatation attained by asymptotical extremal

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Abstract. In this paper, we prove the existence of an asymptotical extremal in an infinitesimal equivalence class as a locally extremal representative at a boundary point.

Contents

1. Introduction

Let S be a plane domain with at least two boudary points. The Teichmüller space $T(S)$ is the space of equivalence classes of quasiconformal maps f from S to a variable domain $f(S)$. Two quasiconformal maps f from S to $f(S)$ and g from S to $g(S)$ are equivalent if there is a conformal map c from $f(S)$ onto $g(S)$ and a homotopy through quasiconformal maps h_t mapping S onto $g(S)$ such that $h_0 = c\circ f$, $h_1 = g$ and $h_t(p) = c\circ f(p) = g(p)$ for every $t \in [0,1]$ and every p in the boundary of S. Denote by $[f]$ the Teichmüller equivalence class of f; also sometimes denote the equivalence class by $[\mu]$ where μ is the Beltrami differential of f.

Denote by $\mathrm{Bel}(S)$ the Banach space of Beltrami differentials

$$\mu = \mu(z)d\bar{z}/dz$$

on S with finite L^∞-norm and by $M(S)$ the open unit ball in $\mathrm{Bel}(S)$.

Received July 28, 2014.

2010 *Mathematics Subject Classification.* Primary 30C75, 30C62.

Key words and phrases. Teichmüller space, Beltrami differential, asymptotical extremal, boundary dilatation.

The author was supported by the National Natural Science Foundation of China (Grant No. 11271216).

ISSN 1076-9803/2015

The cotangent space to $T(S)$ at the basepoint is the Banach space $Q(S)$ of integrable holomorphic quadratic differentials on S with L^1-norm

$$\|\varphi\| = \iint_S |\varphi(z)|\, dxdy < \infty.$$

In what follows, let $Q^1(S)$ denote the unit sphere of $Q(S)$.

Two Beltrami differentials μ and ν in $\mathrm{Bel}(S)$ are said to be infinitesimally equivalent if

$$\iint_S (\mu - \nu)\varphi\, dxdy = 0, \text{ for any } \varphi \in Q(S).$$

The tangent space $Z(S)$ of $T(S)$ at the basepoint is defined as the quotient space of $\mathrm{Bel}(S)$ under the equivalence relations. Denote by $[\mu]_Z$ the equivalence class of μ in $Z(S)$.

$Z(S)$ is a Banach space and its standard sup-norm is defined by

$$\|[\mu]_Z\| := \sup_{\varphi \in Q^1(S)} \mathrm{Re} \iint_S \mu\varphi\, dxdy = \inf\{\|\nu\|_\infty : \nu \in [\mu]_Z\}.$$

Define the (infinitesimal) boundary dilatation $b([\mu]_Z)$ of $[\mu]_Z$ to be the infimum over all elements in the equivalence class $[\mu]_Z$ of the quantity $b^*(\nu)$. Here $b^*(\nu)$ is the infimum over all compact subsets E contained in S of the essential supremum of the the Beltrami differential ν as z varies over $S - E$.

Define $h^*(\mu)$ to be the infimum over all compact subsets E contained in S of the essential supremum norm of the Beltrami differential $\mu(z)$ as z varies over $S\backslash E$ and $h([\mu])$ to be the infimum of $h^*(\nu)$ taken over all representatives ν of the class $[\mu]$.

Let p be a point on ∂S and let $\mu \in M(S)$. Define

$$h_p([\mu]) = \inf\left\{h_p^*(\nu) : \nu \in [\mu]\right\},$$

to be the boundary dilatation of $[\mu]$ at p, where

$$h_p^*(\mu) = \inf\left\{ \operatorname*{esssup}_{z \in U \bigcap S} |\mu(z)| : \right.$$
$$\left. U \text{ is an open neighborhood in } \mathbb{C} \text{ containing } p \right\}.$$

If $\mu \in M(S)$, define

$$h_p([\mu]) = \inf\{h_p^*(\nu) : \nu \in [\mu]\}.$$

It was proved by Fehlmann [3] for the unit disk and by Lakic [5] for the plane domains that

$$h([\mu]) = \max_{p \in \partial S} h_p([\mu]).$$

For $\mu \in \mathrm{Bel}(S)$, we use $b_p^*(\mu)$ to denote $h_p^*(\mu)$ and define

$$b_p([\mu]_Z]) = \inf\{b_p^*(\nu) : \nu \in [\mu]_Z\}$$

to be the boundary dilatation of $[\mu]_Z$ at p. The parallel result

$$b([\mu]_Z) = \max_{p \in \partial S} b_p([\mu]_Z)$$

for the plane domains was proved by Lakic in [5].

The following problem was proposed by F. Gardiner and N. Lakic in ([4], page 335) as an open problem.

Problem 1. *Let p be a boundary point of a plane domain S, and let $\tau \in T(S)$. Is there a locally extremal Beltrami differential μ representing the class τ at the point p? That is, can we find a Beltrami differential $\mu \in M(S)$ such that $\tau = [\mu]$ and $h_p^*(\mu) = h_p(\tau)$?*

The problem was partly solved by Cui and Qi in [2] and the answer is affirmative when S is the unit disk Δ. Recently, the author strengthened their result in [7] by showing that $h_p^*(\mu) = h_p(\tau)$ can be attained by an asymptotically extremal representative $\mu \in \tau$. Naturally, the problem has its counterpart in the infinitesimal case. That is:

Problem 2. *Let p be a boundary point of a plane domain S, and let $\tau \in Z(S)$. Is there a locally extremal Beltrami differential μ representing the class τ at the point p? That is, can we find a Beltrami differential $\mu \in \mathrm{Bel}(S)$ such that $\tau = [\mu]_Z$ and $b_p^*(\mu) = b_p(\tau)$?*

Generally, $\mu \in \mathrm{Bel}(S)$ is called an asymptotical extremal in $[\mu]_Z$ if

$$b^*(\mu) = b([\mu]_Z).$$

In this paper, we prove that the local boundary dilatation can be attained by an asymptotical extremal which gives an affirmative answer to Problem 2 in a stronger sense.

Theorem 1. *Let p be a boundary point of the unit disk Δ and let $\tau \in Z(\Delta)$. Then for any given $\epsilon > 0$, there is an asymptotically extremal Beltrami differential $\mu \in \tau$ such that $\|\mu\|_\infty < \|\tau\| + \epsilon$ and $b_p^*(\mu) = b_p(\tau)$.*

The method used here can also be used to deal with some more general cases.

2. Deformation of Beltrami differentials

In this section, we deform a Beltrami differential to obtain a new equivalent Beltrami differential whose essential supremum can be controlled properly. The following infinitesimal main inequality (see [1]) is needed.

Theorem A. *Let μ, $\nu \in M(S)$. Suppose μ and ν are infinitesimally equivalent. Then*

$$\iint_S |\varphi|(1 - |\mu|^2)dxdy \le \iint_S |\varphi| \left|1 - \mu\frac{\varphi}{|\varphi|}\right|^2 \frac{\left|1 + \nu\frac{\varphi}{|\varphi|}\frac{1-\bar{\mu}\frac{\overline{\varphi}}{|\varphi|}}{1-\mu\frac{\varphi}{|\varphi|}}\right|^2}{1 - |\nu|^2} dxdy \tag{2.1}$$

for all $\varphi \in Q(S)$.

The following lemma is Proposition 1 of Chapter 15 in [4].

Lemma 2.1. *For every $\tau \in Z(S)$ and every $\epsilon > 0$ there exists a representative η in τ such that $\|\eta\|_\infty < \|\tau\| + \epsilon$ and $b^*(\eta) = b(\tau)$.*

Suppose $\{J_n : n \in \mathbb{N}\}$ is a sequence of Jordan domains in Δ with the properties:

(1) $\Delta \backslash \bar{J}$ is simply-connected where $J = J_0$.
(2) $J_{n+1} \subsetneqq J_n$ and $J_n \backslash \bar{J}_{n+1}$ is simply-connected for all $n \geq 0$.
(3) $\lim_{n\to\infty} \bar{J}_n$ is a boundary point $\zeta \in \partial\Delta$.

Set $U_n = \Delta \backslash \bar{J}_n$ for $n \in \mathbb{N}$. It is easy to see that U_n is simply-connected.

Theorem 2. *Let $\nu \in \mathrm{Bel}(\Delta)$ and let J, J_n given as the above. Then for every given $\epsilon > 0$, there exists some $n \in \mathbb{N}$ and $\mu \in \mathrm{Bel}(\Delta)$ such that:*

(1) *$\mu \in [\nu]_Z$.*
(2) *$\mu(z) = \nu(z)$ restricted on J_n.*
(3) *$\|\mu|_{U_n}\|_\infty \leq \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon$.*

Proof. Since $Z(\Delta)$ is a Banach space, without loss of generality, we can assume that $\|\nu\|_\infty < 1 - \epsilon$ for small $\epsilon > 0$. Regard $[\nu|_{U_n}]_Z$ as a point in the space $Z(U_n)$. Then there is an infinitesimal extremal μ_n in $[\nu|_{U_n}]_Z$ such that $\|\mu_n\|_\infty = \|[\nu|_{U_n}]_Z\|$. It is obvious that $\|\mu_n\|_\infty \leq \|\nu\|_\infty < 1$. If for some n, $\|\mu_n\|_\infty \leq \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon$, then

$$\widetilde{\mu}_n(z) := \begin{cases} \mu_n(z), & z \in U_n, \\ \nu(z), & z \in \bar{J}_n \end{cases} \tag{2.2}$$

is the required Beltrami differential.

Now, we assume that $\|\mu_n\|_\infty > \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon$ holds for all $n \in \mathbb{N}$. Then $\|[\nu|_{U_n}]_Z\| > b([\nu|_{U_n}]_Z)$ and consequently by the infinitesimal frame mapping theorem (see Theorem 2.4 in [6]) of Reich, μ_n is a Teichmüller differential, i.e., $\mu_n = k_n\overline{\varphi_n}/|\varphi_n|$ $(0 < k_n < 1)$, where $\varphi_n \in Q^1(U_n)$.

Claim. φ_n converges to 0 uniformly on any compact subset of Δ as $n \to \infty$.

Note the condition $\lim_{n\to\infty} \bar{J}_n = \zeta \in \partial\Delta$. We may assume, by contradiction, that there is $\varphi_0 \in Q(\Delta)$, $\varphi_0 \not\equiv 0$ and a subsequence $\{n_j\}$ of $\mathbb{N}$ with $n_j < n_{j+1}$ such that $\varphi_{n_j} \to \varphi_0$ as $j \to \infty$. We may choose a subsequence of μ_{n_j}, also denoted by itself, such that $k_{n_j} \to k_0$ as $j \to \infty$. Thus, the Teichmüller differential μ_{n_j} converges to $\mu_0 = k_0\overline{\varphi_0}/|\varphi_0|$ in Δ.

Observe that $\|\widetilde{\mu}_{n_j}\|_\infty \leq \|\nu\|_\infty$ for all $j > 0$. We have $\mu_0 \in [\nu]_Z$ and hence μ_0 is a Teichmüller extremal in $[\nu]_Z$. On the other hand, the assumption that $\|\mu_n\|_\infty > \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon$ holds for all $n \in \mathbb{N}$ implies $k_0 \geq \|[\nu]_Z\| + \epsilon$. This gives rise to a contradiction. The proof of Claim is completed.

Fix a positive integer N. By the definition of boundary dilatation, we have

$$b([\nu|_{U_N}]_Z) \leq \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\}.$$

By Lemma 2.1, there exists a Beltrami differential $\nu_N \in [\nu|_{U_N}]_Z$ such that $b^*(\nu_N) = b([\nu|_{U_N}]_Z)$. So, there is a compact subset $E \subset U_N$ such that

$$|\nu_N(z)| \leq \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \frac{\epsilon}{2}$$

for almost all $z \in U_N \backslash E$.

For any $n > N$, let

$$\widetilde{\nu}_n(z) := \begin{cases} \nu_N(z), & z \in U_N, \\ \nu(z), & z \in U_n \backslash U_N. \end{cases}$$

Then $\widetilde{\nu}_n \in [\nu|_{U_n}]_Z$ $(= [\mu_n|_{U_n}]_Z)$. We apply the infinitesimal main inequality (2.1) on U_n and get

$$\begin{aligned} &\iint_{U_n} |\varphi_n|(1 - |\mu_n|^2) dxdy \\ &\leq \iint_{U_n} |\varphi_n| \left|1 - \mu_n \frac{\varphi_n}{|\varphi_n|}\right|^2 \frac{\left|1 + \widetilde{\nu}_n \frac{\varphi_n}{|\varphi_n|} \frac{1 - \bar{\mu}_n \frac{\overline{\varphi_n}}{|\varphi_n|}}{1 - \mu_n \frac{\varphi_n}{|\varphi_n|}}\right|^2}{1 - |\widetilde{\nu}_n|^2} dxdy \\ &\leq \iint_{U_n} |\varphi_n| \left|1 - \mu_n \frac{\varphi_n}{|\varphi_n|}\right|^2 \frac{1 + |\widetilde{\nu}_n|}{1 - |\widetilde{\nu}_n|} dxdy. \end{aligned}$$

Notice that $\mu_n = k_n \overline{\varphi_n}/|\varphi_n|$. We have

$$\iint_{U_n} |\varphi_n|(1 - k_n^2) dxdy \leq \iint_{U_n} |\varphi_n|(1 - k_n)^2 \frac{1 + |\widetilde{\nu}_n|}{1 - |\widetilde{\nu}_n|} dxdy.$$

Thus,

$$\begin{aligned} \frac{1 + k_n}{1 - k_n} &\leq \iint_{U_n} |\varphi_n| \frac{1 + |\widetilde{\nu}_n|}{1 - |\widetilde{\nu}_n|} dxdy \\ &\leq \iint_{E} |\varphi_n| \frac{1 + |\widetilde{\nu}_n|}{1 - |\widetilde{\nu}_n|} dxdy + \iint_{U_n \backslash E} |\varphi_n| \frac{1 + |\widetilde{\nu}_n|}{1 - |\widetilde{\nu}_n|} dxdy. \end{aligned}$$

Choose $\widetilde{\epsilon} > 0$ such that

$$\frac{1 + (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon/2)}{1 - (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon/2)} + \widetilde{\epsilon} \leq \frac{1 + (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon)}{1 - (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon)}.$$

Since φ_n converges to 0 on E as $n \to \infty$,

$$\iint_{E} |\varphi_n| \frac{1 + |\widetilde{\nu}_n|}{1 - |\widetilde{\nu}_n|} dxdy \leq \widetilde{\epsilon}$$

holds for all sufficiently large n. On the other hand, by the definition of $\widetilde{\nu}_n$, we have

$$\iint_{U_n \backslash E} |\varphi_n| \frac{1 + |\widetilde{\nu}_n|}{1 - |\widetilde{\nu}_n|} dxdy \leq \frac{1 + (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon/2)}{1 - (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon/2)}.$$

Hence we get

$$\frac{1+k_n}{1-k_n} \le \frac{1+(\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\}+\epsilon/2)}{1-(\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\}+\epsilon/2)} + \widetilde{\epsilon}$$

and consequently,

$$k_n \le \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon,$$

which completes the proof of Theorem 2. □

Unlike the Teichmüller equivalence class, the notion of the boundary map is lost for the infinitesimal equivalence classes. The gluing method used in [2] does not apply to prove our Theorem 2.

3. Proof of Theorem 1

We prove Theorem 1 by gluing Beltrami differentials in a suitable way. By Lemma 2.1, we only need to prove Theorem 1 in the case $b_p(\tau) < b(\tau) := b$. Put $\delta = b(\tau) - b_p(\tau)$. Define $J_r = \{z \in \Delta : |z-p| < r\}$ for small $r \in (0, 2)$ and $U_r = \Delta \backslash \bar{J}_r$.

Step 1. By the definition of boundary dilatation, there is a Beltrami differential $\nu_1 \in \tau$ such that

$$b_p^*(\nu_1) \le b_p(\tau) + \frac{\delta}{2^3}.$$

By the definition of $b_p^*(\nu_1)$, there is some $r_1 > 0$ such that

$$|\nu_1(z)| \le b_p(\tau) + \frac{\delta}{2} < b, \ a.e.\ z \in J_{r_1}.$$

Applying Theorem 2, we can find some $r_1' < r_1$ and a Beltrami differential $\mu_1 \in \tau$ such that, $\mu_1(z) = \nu_1(z)$ restricted on $J_{r_1'}$, $\|\nu_1|_{J_{r_1'}}\|_\infty \le b_p(\tau) + \frac{\delta}{2^2}$ and

$$\|\mu_1|_{U_{r_1'}}\|_\infty < \max\{\|\tau\|, \|\nu_1|_{J_{r_1}}\|_\infty\} + \frac{\epsilon}{2} = \|\tau\| + \frac{\epsilon}{2}.$$

It is not hard to see that $b^*([\mu_1|_{U_{r_1'}}]_Z) = b$. By Lemma 2.1, we can choose $\eta_1 \in [\mu_1|_{U_{r_1'}}]_Z$ such that $b^*(\eta_1) = b$ and $\|\eta_1\|_\infty < \|\tau\| + \epsilon$.

Step 2. Consider $\nu_1(z)$ on $J_{r_1'}$ and choose a Beltrami differential $\nu_2 \in [\nu_1|_{J_{r_1'}}]_Z$ such that

$$b_p^*(\nu_2) \le b_p(\tau) + \frac{\delta}{2^4}.$$

There is some $r_2 < r_1'$ such that

$$|\nu_2(z)| \le b_p(\tau) + \frac{\delta}{2^2}, \ a.e.\ z \in J_{r_2}.$$

Again applying Theorem 2 on $J_{r_1'}$, we can find some $r_2' < r_2$ and a Beltrami differential $\mu_2 \in [\nu_2|_{J_{r_1'}}]_Z$ such that, $\mu_2(z) = \nu_2(z)$ restricted on $J_{r_2'}$ and

$$\begin{aligned} \|\nu_2|_{J_{r_2'}}\|_\infty &\leq b_p(\tau) + \frac{\delta}{2^3}, \\ \|\mu_2|_{J_{r_1'} \backslash J_{r_2'}}\|_\infty &\leq \max\{\|[\nu_2|_{J_{r_1'}}]_Z\|, \|\nu_2|_{J_{r_2}}\|_\infty\} + \frac{\delta}{2^2} \\ &= \max\{\|[\nu_1|_{J_{r_1'}}]_Z\|, \|\nu_2|_{J_{r_2}}\|_\infty\} + \frac{\delta}{2^2} \\ &\leq b_p(\tau) + \frac{\delta}{2^2} + \frac{\delta}{2^2} = b_p(\tau) + \frac{\delta}{2}. \end{aligned}$$

Step 3. Following the construction in Step 2, we get two sequences $\{r_n\}$ and $\{r_n'\}$ and two sequences of Beltrami differentials $\{\mu_n\}$ and $\{\nu_n\}$ $(n \geq 2)$ with the following conditions:

(i) $r_n < r_{n-1}' < r_{n-1}$ and $\lim_{n\to\infty} r_n = \lim_{n\to\infty} r_n' = 0$.

(ii) $\nu_n \in [\nu_{n-1}|_{J_{r_{n-1}'}}]_Z$ and

$$b_p^*(\nu_n) \leq b_p(\tau) + \frac{\delta}{2^{n+2}}, \tag{3.1}$$

$$|\nu_n(z)| \leq b_p(\tau) + \frac{\delta}{2^n}, \ \ a.e.\ z \in J_{r_n}. \tag{3.2}$$

(iii) $\mu_n \in [\nu_{n-1}|_{J_{r_{n-1}'}}]_Z$, $\mu_n(z) = \nu_n(z)$ restricted on $J_{r_n'}$ and

$$\|\nu_n|_{J_{r_n'}}\|_\infty \leq b_p(\tau) + \frac{\delta}{2^{n+1}}, \tag{3.3}$$

$$\begin{aligned} \|\mu_n|_{J_{r_{n-1}'} \backslash J_{r_n'}}\|_\infty &\leq \max\{\|[\nu_n|_{J_{r_{n-1}'}}]_Z\|, \|\nu_n|_{J_{r_n}}\|_\infty\} + \frac{\delta}{2^n} \\ &= \max\{\|[\nu_{n-1}|_{J_{r_{n-1}'}}]_Z\|, \|\nu_n|_{J_{r_n}}\|_\infty\} + \frac{\delta}{2^n} \\ &\leq b_p(\tau) + \frac{\delta}{2^n} + \frac{\delta}{2^n} = b_p(\tau) + \frac{\delta}{2^{n-1}}. \end{aligned} \tag{3.4}$$

Finally, we define

$$\mu(z) := \begin{cases} \eta_1(z), & z \in \Delta \backslash \bar{J}_{r_1'}, \\ \mu_2(z), & z \in \bar{J}_{r_1'} \backslash J_{r_2'}, \\ \vdots & \\ \mu_n(z), & z \in \bar{J}_{r_{n-1}'} \backslash J_{r_n'}, \\ \vdots & \end{cases}$$

Then $\mu \in \tau$. Inequality (3.4) indicates that $b_p^*(\mu) = b_p(\tau)$. The choice of η_1 together with (3.4) gives $\|\mu\|_\infty < \|\tau\| + \epsilon$. It is clear that $b^*(\mu) = b(\tau)$ and hence μ is an asymptotical extremal.

The proof of Theorem 1 is completed. □

At last, we note that the following corollary follows from the above proof readily.

Corollary 3.1. *Let $p_1, p_2, \cdots, p_n$ be boundary points of the unit disk Δ and let $\tau \in Z(\Delta)$. Then for any given $\epsilon > 0$, there is an asymptotically extremal Beltrami differential $\mu \in \tau$ such that $\|\mu\|_\infty < \|\tau\| + \epsilon$ and $b^*_{p_j}(\mu) = b_{p_j}(\tau)$ for all $1 \le j \le n$.*

There even exists an asymptotical extremal in $[\mu]_Z$ assuming local extremal boundary dilatations at infinitely many boundary points whose essential supremum is properly controlled as well.

Theorem 3. *Let $\{p_m\}$ be a sequence of mutually different boundary points of the unit disk Δ and let $\tau \in Z(\Delta)$. Then for any given $\epsilon > 0$, there is an asymptotically extremal Beltrami differential $\mu \in \tau$ such that $\|\mu\|_\infty < \|\tau\| + \epsilon$ and $b^*_{p_m}(\mu) = b_{p_m}(\tau)$ for all m.*

Proof. We use an inductive procedure. Let $m \ge 1$. For any given $\epsilon > 0$, by Corollary 3.1 (actually by Theorem 1), there is an asymptotically extremal Beltrami differential $\mu_m \in \tau$ such that

$$\|\mu_m\|_\infty < \|\tau\| + \sum_{j=1}^{m} \frac{\epsilon}{2^j} \tag{3.5}$$

and $b^*_{p_j}(\mu_m) = b_{p_j}(\tau)$ for all $1 \le j \le m$.

Choose a small neighborhood of p_{m+1} in Δ, say

$$B_{m+1} := \{z \in \Delta : |z - p_{m+1}| < \rho_{m+1}\},$$

where ρ_{m+1} is sufficiently small such that p_{m+1} is the only point of $\{p_n\}$ that is contained in $\overline{B}_{m+1}$.

Restrict μ_m on B_{m+1}. By Theorem 1, there is an asymptotically extremal Beltrami differential $\widetilde{\mu}_m \in [\mu_m|_{B_{m+1}}]_Z$ such that

$$\|\widetilde{\mu}_m\|_\infty < \|[\mu_m|_{B_{m+1}}]_Z\| + \frac{\epsilon}{2^{m+1}} \tag{3.6}$$

and $b^*_{p_{m+1}}(\widetilde{\mu}_m) = b_{p_{m+1}}([\mu_m|_{B_{m+1}}]_Z)$.

Combining (3.5) and (3.6), we have

$$\|\widetilde{\mu}_m\|_\infty < \|\tau\| + \sum_{j=1}^{m+1} \frac{\epsilon}{2^j}. \tag{3.7}$$

Put

$$\mu_{m+1}(z) := \begin{cases} \mu_m(z), & z \in \Delta \backslash B_{m+1}, \\ \widetilde{\mu}_m(z), & z \in B_{m+1}. \end{cases}$$

It is easy to check that

$$\|\mu_{m+1}\|_\infty < \|\tau\| + \sum_{j=1}^{m+1} \frac{\epsilon}{2^j}$$

and $b^*_{p_j}(\mu_{m+1}) = b_{p_j}(\tau)$ for all $1 \le j \le m+1$.

Thus, we can obtain two sequences $\{\mu_m\}$ and $\{B_m\}$ with the above conditions. Let

$$\mu(z) := \begin{cases} \mu_1(z), & z \in \Delta \setminus \bigcup_{j=2}^{\infty} B_j, \\ \mu_2(z), & z \in B_2, \\ \quad\vdots & \\ \mu_m(z), & z \in B_m, \\ \quad\vdots & \end{cases}$$

Then $\mu \in \tau$ is the desired asymptotically extremal Beltrami differential. □

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