

Induced representations arising from a character with finite orbit in a semidirect product

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ABSTRACT. Making use of a unified approach to certain classes of induced representations, we establish here a number of detailed spectral theoretic decomposition results. They apply to specific problems from noncommutative harmonic analysis, ergodic theory, and dynamical systems. Our analysis is in the setting of semidirect products, discrete subgroups, and solenoids. Our applications include analysis and ergodic theory of Bratteli diagrams and their compact duals; of wavelet sets, and wavelet representations.

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1. Introduction

The purpose of the present paper is to demonstrate how a certain induction (from representation theory) may be applied to a number of problems in dynamics, in spectral theory and harmonic analysis; yielding answers in the form of explicit invariants, and equivalences. In more detail, we show that a certain family of induced representations forms a unifying framework. The setting is unitary representations, infinite-dimensional semidirect products and crossed products. While the details in our paper involve a variant of

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Mackey's theory, our approach is extended, and it is constructive and algorithmic. Our starting point is a family of imprimitivity systems. By this we mean a pair: a unitary representation, and a positive operator-valued mapping, subject to a covariance formula for the two (called imprimitivity). One of its uses is a characterization of those unitary representations of some locally compact group G "which arise" as an induction from a subgroup; i.e., as induced from a representation of a specific subgroup of G . (Here, the notation "which arise" means "up to unitary equivalence.")

Below we give a summary of notation and terminologies, and we introduce a family of induced representations, see, e.g., [Jor88, Mac88, Ørs79].

The framework below is general; the context of locally compact (non-abelian) groups. But we shall state the preliminary results in the context of unimodular groups, although it is easy to modify the formulas and the results given to nonunimodular groups. One only needs to incorporate suitable factors on the respective modular functions, that of the ambient group and that of the subgroup. Readers are referred to [Ørs79] for additional details on this point. Another reason for our somewhat restricted setting is that our main applications below will be to the case of induction from suitable abelian subgroups. General terminology: In the context of locally compact abelian groups, say B , we shall refer to Pontryagin duality (see, e.g., [Rud62]); and so in particular, when the abelian group B is given, by "the dual" we mean the group of all continuous characters on B , i.e., the one-dimensional unitary representations of B . We shall further make use of Pontryagin's theorem to the effect that the double-dual of B is naturally isomorphic to B itself.

Our results are motivated in part by a number of noncommutative harmonic analysis issues involved in the analysis of wavelet representations, and wavelet sets; see especially [LiPT01]; and also [LaSST06, CM11, CM13, CMO14, MO14].

Definition 1.1. Let G be a locally compact group, $B \subset G$ a closed subgroup; both assumed unimodular. Given $V \in \text{Rep}(B, \mathcal{H}_V)$, a representation of B in the Hilbert space \mathcal{H}_V , let

$$(1.1) \quad U = \text{Ind}_B^G(V)$$

be the *induced representation*. U acts on the Hilbert space \mathcal{H}_U as follows:

\mathcal{H}_U consists of all measurable functions $f : G \rightarrow \mathcal{H}_V$ s.t.

$$(1.2) \quad f(bg) = V_b f(g), \quad \forall b \in B, \forall g \in G,$$

$$(1.3) \quad \|f\|_{\mathcal{H}_U}^2 = \int_{B \backslash G} \|f(g)\|^2 d\mu_{B \backslash G}(g) < \infty,$$

w.r.t. the measure on the homogeneous space $B \backslash G$, i.e., the Hilbert space carrying the induced representation. Set

$$(1.4) \quad (U_g f)(x) := f(xg), \quad x, g \in G.$$

Remark 1.2. If the respective groups B and G are nonunimodular, we select respective right-invariant Haar measures db, dg ; and corresponding modular functions Δ_B and Δ_G . In this case, the modification to the above is that Equation (1.2) will instead be

$$(1.5) \quad f(bg) = \left(\frac{\Delta_B(b)}{\Delta_G(b)} \right)^{1/2} V_b f(g); \quad b \in B, g \in G,$$

thus a modification in the definition of $\mathcal{H}(\text{Ind}_B^G(V))$.

We recall the following theorems of Mackey [Mac88, Ørs79].

Theorem 1.3 (Mackey). $U = \text{Ind}_B^G(V)$ is a unitary representation.

Theorem 1.4 (Imprimitivity Theorem). A representation $U \in \text{Rep}(G, \mathcal{H})$ is induced iff \exists a positive operator-valued mapping

$$(1.6) \quad \pi : C_c(B \backslash G) \longrightarrow \mathcal{B}(\mathcal{H}) \text{ s.t.}$$

$$(1.7) \quad U_g \pi(\varphi) U_g^* = \pi(R_g \varphi)$$

$\forall g \in G, \forall \varphi \in C_c(B \backslash G)$; where π is nondegenerate, and R_g in (1.7) denotes the right regular action.

Proof. See [Ørs79, Jor88]. □

Theorem 1.5. Given $U \in \text{Rep}(G, \mathcal{H})$, the following are equivalent:

- (1) $\exists \pi$ s.t. (1.7) holds.
- (2) $\exists V \in \text{Rep}(B, \mathcal{H}_V)$ s.t. $U \cong \text{Ind}_B^G(V)$.
- (3) $L_G(U, \pi) \cong L_B(V)$.
- (4) If $V_i \in \text{Rep}(B, \mathcal{H}_{V_i}), i = 1, 2$ and $U_i = \text{Ind}_B^G(V_i), i = 1, 2$, then

$$L_G((U_1, \pi_1); (U_2, \pi_2)) \cong L_B(V_1, V_2);$$

i.e., all intertwining operators $V_1 \rightarrow V_2$ lift to the pair $(U_i, \pi_i), i = 1, 2$.

(Here, “ \cong ” denotes unitary equivalence.) Specifically,

$$(1.8) \quad L_G((U_1, \pi_1), (U_2, \pi_2)) = \left\{ W : \mathcal{H}_{U_1} \rightarrow \mathcal{H}_{U_2} \mid WU_1(g) = U_2(g)W, \forall g \in G, \text{ and } U\pi_1(\varphi) = \pi_2(\varphi)W \right\}$$

and

$$(1.9) \quad L_B(V_1, V_2) = \{ w : \mathcal{H}_{V_1} \rightarrow \mathcal{H}_{V_2} \mid wV_1(b) = V_2(b)w, \forall b \in B \}.$$

Proof. See [Ørs79, Jor88, Mac88]. □

We also recall the following result:

Lemma 1.6. *Let B be a lattice in \mathbb{R}^d , let α be an action of \mathbb{Z} on B by automorphisms, and let $(K = \widehat{B}, \widehat{\alpha})$ represent the dual action of \mathbb{Z} on the compact abelian group K . Fix a $d \times d$ matrix A preserving the lattice B with $\text{spec}(A) \subset \{\lambda : |\lambda| > 1\}$.*

If $\alpha = \alpha_A \in \text{Aut}(B)$, then $\widehat{\alpha} \in \text{Aut}(K)$ is ergodic, i.e., if ν is the normalized Haar measure on K and if $E \subset K$ is measurable s.t. $\widehat{\alpha}E = E$, then $\nu(E)(1 - \nu(E)) = 0$.

Proof. By Halmos–Rohlin’s theorem (see [BreJ91]), we must show that if $b \in B$ and $A^n b = b$ for some $n \in \mathbb{N}$, then $b = 0$. This then follows from the assumption on the spectrum of A . \square

2. The representation $\text{Ind}_B^G(\chi)$

We now specify our notation and from this point on will restrict our study to the following setting:

- (i) B : a discrete abelian group (written additively);
- (ii) $\alpha \in \text{Aut}(B)$;
- (iii) $G_\alpha := B \rtimes_\alpha \mathbb{Z}$;
- (iv) $K := \widehat{B}$, the compact dual group;
- (v) $\widehat{\alpha} \in \text{Aut}(\widehat{B})$, dual action;
- (vi) $L_\alpha := K \rtimes_{\widehat{\alpha}} \mathbb{Z}$, the C^* -algebra crossed product [BreJ91, p.299]; also written as $C^*(K) \rtimes_{\widehat{\alpha}} \mathbb{Z}$.

More generally, let K be a compact Hausdorff space, and $\beta : K \rightarrow K$ a homeomorphism; then we study the C^* -crossed product $C^*(K) \rtimes_\beta \mathbb{Z}$ (see [BreJ91]).

We define the *induced representation*

$$(2.1) \quad U^\chi := \text{Ind}_B^G(\chi)$$

where $\chi \in K$, i.e., a character on B , and

$$(2.2) \quad G := B \rtimes_\alpha \mathbb{Z}$$

as a semi-direct product. Note that U^χ in (2.1) is induced from a one-dimensional representation.

Below, $\{\delta_k\}_{k \in \mathbb{Z}}$ denotes the canonical basis in $l^2(\mathbb{Z})$, i.e.,

$$\delta_k = (\dots, 0, 0, 1, 0, 0, \dots)$$

with “1” at the k^{th} place.

Lemma 2.1. *The representation $\text{Ind}_B^G(\chi)$ is unitarily equivalent to*

$$(2.3) \quad U^\chi : G \rightarrow \mathcal{B}(l^2(\mathbb{Z})), \text{ where}$$

$$(2.4) \quad U_{(j,b)}^\chi \delta_k = \chi(\alpha_{k-j}(b)) \delta_{k-j}, \quad (j, b) \in G.$$

Proof. First note that both groups B and $G = B \rtimes_{\alpha} \mathbb{Z}$ are discrete and unimodular, and the respective Haar measures are the counting measure. Since $B \backslash G \simeq \mathbb{Z}$, so $\mathcal{H}^{\chi} = L^2(B \backslash G) \simeq l^2(\mathbb{Z})$. Recall the multiplication rule in G :

$$(2.5) \quad (k, c)(j, b) = (k + j, \alpha_k(b) + c)$$

where $j, k \in \mathbb{Z}$, and $b, c \in B$.

The representation $\text{Ind}_B^G(\chi)$ acts on functions $f : G \rightarrow \mathbb{C}$ s.t.

$$(2.6) \quad f((j, b)) = \chi(b) f(j, 0)$$

and

$$(2.7) \quad \|f\|_{\mathcal{H}^{\chi}}^2 = \sum_{j \in \mathbb{Z}} |f(j, 0)|^2.$$

Moreover, the mapping

$$(2.8) \quad l^2 \ni \xi \longmapsto W\xi \in \mathcal{H}^{\chi}$$

given by

$$(2.9) \quad (W\xi)(j, b) = \chi(b) \xi_j$$

is a unitary intertwining operator, i.e.,

$$(2.10) \quad \text{Ind}_B^G(\chi) W\xi = WU^{\chi}\xi$$

where

$$(2.11) \quad \left(U_{(j,b)}^{\chi} \xi \right)_k = \chi(\alpha_k(b)) \xi_{k+j}$$

$\forall j, k \in \mathbb{Z}, \forall b \in B, \forall \xi \in l^2(\mathbb{Z})$. Note that (2.11) is equivalent to (2.4).

To verify (2.10), we have

$$\begin{aligned} \left(\text{Ind}_B^G(\chi)_{(j,b)} W\xi \right)(k, c) &= W\xi((k, c)(j, b)) \\ &\stackrel{(2.5)}{=} W\xi(k + j, \alpha_k(b) + c) \\ &\stackrel{(2.6)}{=} \chi(\alpha_k(b) + c) W\xi(k + j, 0) \\ &\stackrel{(2.9)}{=} \chi(\alpha_k(b) + c) \xi_{k+j} \end{aligned}$$

and

$$\begin{aligned} \left(WU_{(j,b)}^{\chi} \xi \right)(k, c) &\stackrel{(2.9)}{=} \chi(c) \left(U_{(j,b)}^{\chi} \xi \right)_k \\ &\stackrel{(2.11)}{=} \chi(c) \chi(\alpha_k(b)) \xi_{k+j} \\ &= \chi(\alpha_k(b) + c) \xi_{k+j}. \end{aligned}$$

In the last step we use that $\chi(\alpha_k(b) + c) = \chi(\alpha_k(b)) \chi(c)$, which is just the representation property of $\chi : B \rightarrow \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$,

$$(2.12) \quad \chi(b_1 + b_2) = \chi(b_1) \chi(b_2), \quad \forall b_1, b_2 \in B.$$

□

Remark 2.2. In summary, the representation $\text{Ind}_B^G(\chi)$ has three equivalent forms:

(1) On $l^2(\mathbb{Z})$,

$$(2.13) \quad (\text{Ind}_B^G(\chi)\xi)_k = \chi(\alpha_k(b))\xi_{k+j}, \quad (\xi_k) \in l^2(\mathbb{Z}).$$

(2) In the ONB $\{\delta_k \mid k \in \mathbb{Z}\}$,

$$(2.14) \quad \text{Ind}_B^G(\chi)_{(j,k)}\delta_k = \chi(\alpha_{k-j}(b))\delta_{k-j}, \quad k \in \mathbb{Z}.$$

(3) On \mathcal{H}^χ , consisting of functions $f : G \rightarrow \mathbb{C}$ s.t.

$$f(j, b) = \chi(b)f(j, 0), \quad \|f\|_{\mathcal{H}^\chi}^2 = \sum_{j \in \mathbb{Z}} |f(j, 0)|^2 < \infty$$

where

$$\begin{aligned} (\text{Ind}_B^G(\chi)_{(j,b)}f)(k, c) &= f((k, c)(j, b)) \\ &= f((k+j, \alpha_k(b)+c)) \\ &= \chi(\alpha_k(b)+c)f(k+j, 0), \end{aligned}$$

with $(j, b), (k, c) \in G = B \rtimes_\alpha \mathbb{Z}$, i.e., $j, k \in \mathbb{Z}, b, c \in B$.

3. Irreducibility

Let $G = B \rtimes_\alpha \mathbb{Z}$, and $\text{Ind}_B^G(\chi)$ be the induced representation. Our main results are summarized as follows:

(1) If $\chi \in K = \widehat{B}$ has infinite order, then $\text{Ind}_B^G(\chi) \in \text{Rep}_{\text{irr}}(G, l^2(\mathbb{Z}))$, i.e., $\text{Ind}_B^G(\chi)$ is irreducible. We have

$$(3.1) \quad \text{Ind}_B^G(\chi)_{(j,b)} = D_\chi(b)T_j$$

where $D_\chi(b)$ is diagonal, and $T_j : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$, $(T_j\xi)_k = \xi_{k+j}$, for all $\xi \in l^2(\mathbb{Z})$.

(2) Suppose χ has finite order p , i.e., $\exists p$ s.t. $\widehat{\alpha}^p\chi = \chi$, but $\widehat{\alpha}^k\chi \neq \chi$, $\forall 1 \leq k < p$. Set $U_p^\chi = \text{Ind}_B^G(\chi)$, then

$$(3.2) \quad U_p^\chi(j, b) = D_\chi(b)P^j$$

where $T_j\delta_k = \delta_{k-j}$ and P = the $p \times p$ permutation matrix. See (3.9).

Details below.

Lemma 3.1. Set $(T_j\xi)_k = \xi_{k+j}$, for $k, j \in \mathbb{Z}, \xi \in l^2$; equivalently, $T_j\delta_k = \delta_{k-1}$. Then

(1) The following identity holds:

$$(3.3) \quad T_n U^\chi = U^{\widehat{\alpha}^n \chi} T_n, \quad \forall n \in \mathbb{Z}, \forall \chi \in K.$$

(2) For $\omega = (\omega_i) \in l^\infty(\mathbb{Z})$, set

$$(3.4) \quad \begin{aligned} (\pi(\omega)\xi)_k &= \omega_k \xi_k, \quad \xi \in l^2(\mathbb{Z}); \text{ or} \\ \pi(\omega)\delta_k &= \omega_k \delta_k. \end{aligned}$$

Then

$$(3.5) \quad U_{(j,b)}^\chi \pi(\omega) (U_{(j,b)}^\chi)^* = \pi(T_j \omega)$$

$\forall (j, b) \in G, \forall \chi \in K$.

(3) Restriction to the representation:

$$(3.6) \quad U^\chi \Big|_B = \sum_{j \in \mathbb{Z}}^{\oplus} (\widehat{\alpha}^j \chi).$$

Recall that $G := B \rtimes_\alpha \mathbb{Z}$, and so the subgroup B corresponds to $j = 0$ in $\{(j, b) \mid j \in \mathbb{Z}, b \in B\}$. Note that $\widehat{\alpha}^j \chi$ is a one-dimensional representation of B .

Proof. One checks that

$$\begin{aligned} (T_n U_{(j,b)}^\chi \xi)_k &= (U_{(j,b)}^\chi \xi)_{k+n} = \chi(\alpha_{k+n}(b)) \xi_{k+n+j}, \text{ and} \\ (U_{(j,b)}^{\widehat{\alpha}^n \chi} T_n \xi)_k &= (\widehat{\alpha}^n \chi)(\alpha_k(b)) (T_n \xi)_{k+j} = \chi(\alpha_{k+n}(b)) \xi_{k+n+j}. \end{aligned}$$

The other assertions are immediate. □

Corollary 3.2. If $\widehat{\alpha}^p \chi = \chi$, then T_p commutes with U^χ . In this case,

$$(3.7) \quad U_{(j,b)}^\chi \delta_k = \widehat{\alpha}^{k-j}(\chi)(b) \delta_{k-j}$$

induces an action on $l^2(\mathbb{Z}/p\mathbb{Z})$. If m is fixed, the same representation is repeated, where the same action occurs in the sub-band

$$(3.8) \quad l^2(\{\delta_{j+mp} \mid 0 \leq j < p\}).$$

Proof. Immediate from Lemma 3.1. □

The p -dimensional representation in each band $\{\delta_{j+mp} \mid 0 \leq j < p\}$ may be given in matrix form:

Lemma 3.3. Let $\chi \in K$, and assume the orbit of χ under the action of $\widehat{\alpha}$ has finite order p , i.e., $\widehat{\alpha}^p \chi = \chi$ and $\widehat{\alpha}^k \chi \neq \chi, 1 \leq k < p$. For $b \in B$, set

$$D_\chi(b) = \begin{pmatrix} \chi(b) & & & \mathbf{0} \\ & \chi(\alpha(b)) & & \\ & & \ddots & \\ \mathbf{0} & & & \chi(\alpha^{p-1}(b)) \end{pmatrix} = \text{diag}(\chi(\alpha^i(b)))_0^{p-1},$$

a $p \times p$ diagonal matrix. If $\chi = (\chi_1, \dots, \chi_n)$, set $D_\chi(b) = \text{diag}(\chi_i(b))_{n \times n}$.

For the permutation matrix P , we shall use its usual $p \times p$ matrix representation

$$(3.9) \quad P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}_{p \times p}.$$

We have that:

(1) The following identity holds:

$$(3.10) \quad PD_\chi(b) = D_\chi(\alpha(b))P.$$

(2) For all $(j, b) \in G = B \rtimes_\alpha \mathbb{Z}$, let

$$(3.11) \quad U_p^{(\chi)}(j, b) := D_\chi(b)P^j.$$

Then $U_p^\chi \in \text{Rep}(G, \mathbb{C}^p)$, i.e.,

$$U_p^\chi(j, b)U_p^\chi(j', b') = U_p^\chi(j + j', \alpha_j(b') + b)$$

for all $(j, b), (j', b')$ in G .

Proof. It suffices to verify (3.10), and the rest of the lemma is straightforward. Set $\chi_k := \chi(\alpha^k(b))$, $0 \leq k < p$, and

$$(3.12) \quad D_\chi(b) = \text{diag}(\chi_0, \chi_1, \dots, \chi_{p-1}).$$

The assertion in (3.10) follows from a direct calculation. We illustrate this with $p = 3$: see Example 3.4 below. \square

Example 3.4. For $p = 3$, (3.10) reads:

$$PD_\chi(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 & 0 & 0 \\ 0 & \chi_1 & 0 \\ 0 & 0 & \chi_2 \end{pmatrix} = \begin{pmatrix} 0 & \chi_1 & 0 \\ 0 & 0 & \chi_2 \\ \chi_0 & 0 & 0 \end{pmatrix}.$$

$$D_\chi(\alpha(b))P = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \chi_1 & 0 \\ 0 & 0 & \chi_2 \\ \chi_0 & 0 & 0 \end{pmatrix}.$$

Lemma 3.5. Let $\chi \in K$, assume the orbit of χ under the action of $\hat{\alpha}$ has finite order p . Let U_p^χ be the corresponding representation. Then $U_p^\chi \in \text{Rep}_{\text{irr}}(G, l^2(\mathbb{Z}/p\mathbb{Z}))$, i.e., U_p^χ is irreducible.

Proof. Recall $U_p^\chi(j, b) = D_\chi(b)P^j$, $\forall (j, b) \in G$; see (3.11).

Let $A : l^2(\mathbb{Z}_p) \rightarrow l^2(\mathbb{Z}_p)$ be in the commutant of U_p^χ , so A commutes with P . It follows that A is a Toeplitz matrix

$$(3.13) \quad A = \begin{pmatrix} A_0 & A_2 & \cdots & \cdots & A_{p-1} \\ A_{-1} & A_0 & A_2 & \cdots & \cdots \\ \cdots & A_{-1} & A_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{-p+1} & \cdots & \cdots & \cdots & A_0 \end{pmatrix}$$

relative to the ONB $\{\delta_0, \delta_1, \dots, \delta_{p-1}\}$, where $A_{i,j} := \langle \delta_i, A\delta_j \rangle_{l^2}$. Note that

$$\begin{aligned} A_{i,j} &= \langle \delta_i, A\delta_j \rangle = \langle \delta_i, AP\delta_{j+1} \rangle = \langle \delta_i, PA\delta_{j+1} \rangle \\ &= \langle P^*\delta_i, A\delta_{j+1} \rangle = \langle \delta_{i+1}, A\delta_{j+1} \rangle = A_{i+1,j+1} \end{aligned}$$

where $i + 1, j + 1$ are the additions in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, i.e., $+ \pmod p$.

If A also commutes with D , then

$$(3.14) \quad A_k \left(\chi(b) - \chi(\alpha^k(b)) \right) = 0$$

for all $k = 1, 2, \dots, p - 1$, and all $b \in B$. But since $\chi, \widehat{\alpha}\chi, \dots, \widehat{\alpha}^{p-1}\chi$ are distinct, $A_k = 0, \forall k \neq 0$, and so $A = A_0 I_{p \times p}, A_0 \in \mathbb{C}$. \square

Proposition 3.6. *The orbit of $\chi \in K$ under the action of $\widehat{\alpha}$ has finite order p if and only if*

$$\text{Ind}_B^G(\chi) = \sum^\oplus (p\text{-dimensional irreducibles}).$$

Proof. This follows from an application of the general theory; more specifically from an application of Mackey’s imprimitivity theorem, in the form given to it in [Ors79]; see also Theorem 1.4 above. \square

Lemma 3.7. *$\text{Ind}_B^G(\chi)$ is irreducible iff the orbit of χ under the action of $\widehat{\alpha}$ has finite order p , i.e., iff the set $\{\widehat{\alpha}^k\chi \mid k \in \mathbb{Z}\}$ consists of distinct points.*

Proof. We showed that if χ has finite order p , then the translation T_p operator commutes with $U^\chi := \text{Ind}_B^G(\chi)$, where $T_p\delta_k = \delta_{k-p}, k \in \mathbb{Z}$. Hence assume χ has infinite order.

Consider the cases $U^\chi(j, b)$ of

- (1) $b = 0. U^\chi(j, 0) = T_j, j \in \mathbb{Z};$
- (2) $j = 0. U^\chi(0, b) = D_\chi(b), \delta_k \mapsto \chi(\alpha_k(b))\delta_k, \text{ where } \chi(\alpha_k(b)) = (\widehat{\alpha}^k\chi)(b), \text{ and } D_\chi = (\widehat{\alpha}^k\chi) = \text{diagonal matrix.}$

Thus if $A \in \mathcal{B}(l^2(\mathbb{Z}))$ is in the commutant of U^χ , then

$$(A\xi)_k = \sum_{j \in \mathbb{Z}} \eta_{k-j}\xi_j, \quad \eta \in l^\infty.$$

But

$$(3.15) \quad (\chi(\alpha_s(b)) - \chi(\alpha_k(b))) \eta_{k-s} = 0$$

$\forall b, \forall k, s$, so $\eta_t = 0$ if $t \in \mathbb{Z} \setminus \{0\}$ and $A = \eta_0 I$.

Note from (3.15) that if $k - s \neq 0$ then $\exists b$ s.t. $\chi(\alpha_s(b)) - \chi(\alpha_k(b)) \neq 0$, since $\widehat{\alpha}^s \chi \neq \widehat{\alpha}^k \chi$. (Compare with (3.14) in Lemma 3.5.) \square

Theorem 3.8. *Let $\text{Ind}_B^G(\chi)$ be the induced representation in (2.1)–(2.2).*

- (1) $\text{Ind}_B^G(\chi)$ is irreducible iff χ has no finite periods.
- (2) Suppose the orbit of χ under the action of $\widehat{\alpha}$ has finite order p , i.e., $\widehat{\alpha}^p \chi = \chi$, and $\widehat{\alpha}^k \chi \neq \chi$, $1 \leq k < p$. Then the commutant is as follows:

$$(3.16) \quad M_p := \{\text{Ind}_B^G(\chi)\}' \cong \{f(z^p) \mid f \in L^\infty(\mathbb{T})\}$$

where \cong in (3.16) denotes unitary equivalence.

Theorem 3.9. *Let $G = B \rtimes_\alpha \mathbb{Z}$, $K = \widehat{B}$, $\widehat{\alpha} \in \text{Aut}(K)$. Assume the orbit of χ under the action of $\widehat{\alpha}$ has finite order p , i.e., $\widehat{\alpha}^p \chi = \chi$, $\widehat{\alpha}^k \chi \neq \chi$, for all $1 \leq k < p$. Then the representation $\text{Ind}_B^G(\chi)$ has abelian commutant*

$$M_p = \{\text{Ind}_B^G(\chi)\}'$$

and M_p does not contain minimal projections.

Proof. Follows from the fact that M_p is L^∞ (Lebesgue) as a von Neumann algebra, and this implies the conclusion.

Details: Recall that $l^2(\mathbb{Z}) \simeq L^2(\mathbb{T})$, where

$$(3.17) \quad l^2 \ni \xi \longmapsto f_\xi(z) = \sum_{j \in \mathbb{Z}} \xi_j z^j \in L^2(\mathbb{T})$$

$$(3.18) \quad T\xi \longmapsto z^{-j} f_\xi(z), \quad z \in \mathbb{T}.$$

Note that $\text{Ind}_B^G(\chi)$ may be realized on $l^2(\mathbb{Z})$ or equivalently on $L^2(\mathbb{T})$ via (3.17), where

$$\|f_\xi\|_{L^2}^2 = \int_{\mathbb{T}} |f_\xi|^2 = \sum_{k \in \mathbb{Z}} |\xi_k|^2.$$

On $l^2(\mathbb{Z})$, we have

$$\text{Ind}_B^G(\chi)_{(j,b)} = D_\chi(b) T_j;$$

See (3.1), and Lemma 3.3.

And on $L^2(\mathbb{T})$, we have

$$\text{Ind}_B^G(\chi)_{(j,b)} = \widehat{D}_\chi(b) \widehat{T}_j, \text{ where}$$

$\widehat{D}_\chi(b)$ denotes rotation on \mathbb{T} , extended to the solenoid; and $\widehat{T}_j =$ multiplication by z^{-j} acting on $L^2(\mathbb{T})$.

By (3.16), M_p is abelian and has no minimal projections. Note projections in L^∞ are given by $P_E =$ multiplication by $\chi_E(z^p)$, where E is measurable in \mathbb{T} . \square

4. Super-representations

By “super-representation” we will refer here to a realization of noncommutative relations “inside” certain unitary representations of suitable groups acting in enlargement Hilbert spaces; this is in the sense of “dilation” theory [FK02], but now in a wider context than is traditional. Our present use of “super-representations” is closer to that of [BiDP05, DJ08, DJP09, DLS11].

Definition 4.1. Let G be a locally compact group, B a given subgroup of G , and let \mathcal{H}_0 be a Hilbert space. Let $U_0 : G \rightarrow \mathcal{B}(\mathcal{H}_0)$ be a positive definite operator-valued mapping, i.e., for all finite systems $\{c_j\}_{j=1}^n \subset \mathbb{C}$, $\{g_j\}_{j=1}^n \subset G$, we have

$$\sum_j \sum_k \bar{c}_j c_k U_0(g_j^{-1} g_k) \geq 0$$

in the usual ordering of Hermitian operators.

If there is a Hilbert space \mathcal{H} , an isometry $V : \mathcal{H}_0 \rightarrow \mathcal{H}$, and a unitary representation $U : G \rightarrow$ (unitary operators on \mathcal{H}) such that

- (1) U is induced from a unitary representation of B , and
- (2) $U_0(g) = V^* U(g) V, g \in G$,

then we say that U is a *super-representation*.

Let B be discrete, abelian as before, $K = \widehat{B}$, $\alpha \in \text{Aut}(B)$, $\widehat{\alpha} \in \text{Aut}(K)$, and $G = B \rtimes_{\alpha} \mathbb{Z}$. Set:

- $\text{Rep}(G)$: unitary representations of G ;
- $\text{Rep}_{\text{irr}}(G)$: irreducible representations in $\text{Rep}(G)$.

For $\chi \in K$, let $O(\chi)$ be the orbit of χ , i.e.,

$$(4.1) \quad O(\chi) = \{ \widehat{\alpha}^j(\chi) \mid j \in \mathbb{Z} \}.$$

Given $U \in \text{Rep}(G)$, let $\text{Class}(U) =$ the equivalence class of all unitary representations equivalent to U ; i.e.,

$$(4.2) \quad \begin{aligned} \text{Class}(U) &= \{ V \in \text{Rep}(G) \mid V \simeq U \} \\ &= \{ V \in \text{Rep}(G) \mid \exists W, \text{ unitary s.t. } WV_g = U_g W, g \in G \}. \end{aligned}$$

For $U_1, U_2 \in \text{Rep}(G)$, set

$$(4.3) \quad L(U^{(1)}, U^{(2)}) = \left\{ W : \mathcal{H}(U_1) \rightarrow \mathcal{H}(U_2) \mid W \text{ bounded s.t. } \right. \\ \left. WU_g^{(1)} = U_g^{(2)}W, g \in G \right\}.$$

Theorem 4.2. *The mapping {set of all orbits $O(\chi)$ } \rightarrow $\text{Class}(\text{Rep}(G))$*

$$K \ni \chi \mapsto U^\chi := \text{Ind}_B^G(\chi) \in \text{Rep}(G)$$

passes to

$$(4.4) \quad O(\chi) \mapsto \text{Class}(U^\chi).$$

Proof. (Sketch) By Lemma 3.1 Equation (3.3), if $\chi \in K$, $j \in \mathbb{Z}$, then $T_j U^\chi T_j^* = U^{\widehat{\alpha}^j \chi}$, and so $\text{Class}(U^\chi)$ depends only on $O(\chi)$ and not on the chosen point in $O(\chi)$.

Using (3.4)–(3.6), we can show that U^χ is irreducible, but if $O(\chi)$ is finite then we must pass to the quotient $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ and realize U^χ on $l^2(\mathbb{Z}_p)$, as a finite-dimensional representation. \square

Theorem 4.3. *There is a natural isomorphism:*

$$\begin{aligned} L_G(U^{\chi_1}, U^{\chi_2}) &\cong L_B(\chi_2, U^{\chi_1}|_B) \\ &= \sum_{j \in \mathbb{Z}} L_B(\chi_2 | \widehat{\alpha}^j(\chi_1)) = \#\{j \mid \chi_2 = \widehat{\alpha}^j(\chi_1)\}. \end{aligned}$$

Proof. Follows from Frobenius reciprocity. \square

Another application of Frobenius reciprocity:

Theorem 4.4. *Let $\chi \in K$, and assume the orbit of χ under the action of $\widehat{\alpha}$ has finite order p , i.e., $\widehat{\alpha}^p \chi = \chi$, and $\widehat{\alpha}^k \chi \neq \chi$ if $1 \leq k < p$. Let $U_p^\chi \in \text{Rep}(G, l^2(\mathbb{Z}_p))$. (Recall that U_p^χ is irreducible, $\dim U_p^\chi = p$.) Then*

$$L_G(U_p^\chi, \text{Ind}_B^G(\chi)) \simeq L_B(\chi, U_p^\chi|_B)$$

and

$$(4.5) \quad \dim L_B(\chi, U_p^\chi|_B) = 1.$$

Proof. Since $U_p^{(\chi)}(j, b) := D_\chi(b) P^j$ (see (3.11)), where P = the permutation matrix on \mathbb{Z}_p , we get

$$U_p^\chi|_B = \sum_{0 \leq k < p}^{\oplus} \widehat{\alpha}^k \chi;$$

see (3.6). But notice that all the p characters $\chi, \widehat{\alpha}(\chi), \dots, \widehat{\alpha}^{p-1}(\chi)$ are distinct, so (4.5) holds since $L_B(\chi, \widehat{\alpha}^k(\chi)) = 0$ if $k \not\equiv 0 \pmod{p}$. \square

We have also proved the following:

Theorem 4.5. *Let $\chi \in K$, and assume the orbit of χ under the action of $\widehat{\alpha}$ has finite order p . Assume G is compact. Let $U_p^\chi \in \text{Rep}_{\text{irr}}(G, l^2(\mathbb{Z}_p))$. Then U_p^χ is contained in $\text{Ind}_B^G(\chi)$ precisely once, i.e.,*

$$\dim L_G(U_p^\chi, \text{Ind}_B^G(\chi)) = 1.$$

But this form of Frobenius reciprocity only holds for certain groups G , e.g., when G is compact. Now for $G = B \rtimes_\alpha \mathbb{Z}$, the formal Frobenius reciprocity breaks down, and in fact:

Theorem 4.6. *$L_G(U_p^\chi, \text{Ind}_B^G(\chi)) = 0$ if $\chi \in K$ is an element of finite order p .*

Proof. We sketch the details for $p = 3$ to simplify notation. For $p = 3$,

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D_\chi(b) = \begin{pmatrix} \chi(b) & 0 & 0 \\ 0 & \chi(\alpha(b)) & 0 \\ 0 & 0 & \chi(\alpha_2(b)) \end{pmatrix}$$

and

$$(4.6) \quad U_p^\chi(j, b) = D_\chi(b) P^j$$

while

$$\left(\text{Ind}_B^G(\chi)_{(j,b)} \xi \right)_k = \chi(\alpha_k(b)) \xi_{k+j}$$

$\forall (j, b) \in G, \forall k \in \mathbb{Z}, \forall \xi \in l^2(\mathbb{Z}_p)$.

Let $W \in L_G(U_p^\chi, \text{Ind}_B^G(\chi))$, and u_0, u_1, u_2 be the canonical basis in $\mathcal{H}(U_p^\chi) = \mathbb{C}^3$, where

$$U_p^\chi(j, b) u_k = \chi(\alpha_{k+2j}(b)) u_{k+2j} \pmod 3$$

$\forall (j, b) \in G, k \in \{0, 1, 2\} \simeq \mathbb{Z}/3\mathbb{Z}$.

Set $Wu_k = \xi^{(k)} = (\xi_s^{(k)})_{s \in \mathbb{Z}} \in l^2(\mathbb{Z})$, where $\|\xi^{(k)}\|^2 = \sum_{s \in \mathbb{Z}} |\xi_s^{(k)}|^2 < \infty$.

It follows that

$$WU_p^\chi(j, b) u_k = \text{Ind}_B^G(\chi)_{(j,b)} Wu_k$$

$\forall (j, b) \in G, k \in \{0, 1, 2\}$. Thus

$$\chi(\alpha_{k+2j}(b)) \xi_s^{(k+2j)_3} = \chi(\alpha_s(b)) \xi_{s+j}^{(k)}, \quad \forall s, j \in \mathbb{Z}.$$

Now set $j = 3t \in 3\mathbb{Z}$, and we get

$$\chi(\alpha_k(b)) \xi_s^{(k)} = \chi(\alpha_s(b)) \xi_{s+3t}^{(k)}$$

and

$$(4.7) \quad |\xi_s^{(k)}| = |\xi_{s+3t}^{(k)}|, \quad \forall s, t \in \mathbb{Z}.$$

Since $\xi^{(k)} \in l^2(\mathbb{Z})$, $\lim_{t \rightarrow \infty} \xi_{s+3t}^{(k)} = 0$. We conclude from (4.7) that $\xi^{(k)} = 0$ in $l^2(\mathbb{Z})$. □

Remark 4.7. The decomposition of $\text{Ind}_B^G(\chi) : G \rightarrow B(l^2(\mathbb{Z}))$ is still a bit mysterious. Recall this representation commutes with $T_3 : (\xi_k) \mapsto (\xi_{k+3})$; or equivalently via $f(z) \mapsto z^3 f(z)$. So it is not irreducible.

The reason for $L_G(U_p^\chi, \text{Ind}_B^G(\chi)) = 0$, e.g., in the case of $p = 3$, is really that there is no isometric version of the 3×3 permutation matrix P in T , where $T\delta_k = \delta_{k-1}$.

If $W : \mathbb{C}^3 \rightarrow l^2(\mathbb{Z})$ is bounded, $WP = TW$, then applying the polar decomposition

$$W = (W^*W)^{1/2} V$$

with $V : \mathbb{C}^3 \rightarrow l^2$ isometric, and $VP = TV$, or

$$P = V^*TV, \quad P^j = V^*T_jV.$$

So T_j has the form

$$T_j = \begin{pmatrix} P^j & * \\ 0 & * \end{pmatrix}.$$

Pick $u \in \mathbb{C}^3$; then $\langle u, T_j u \rangle = \langle u, P^j u \rangle$. However, $\langle u, T_j u \rangle \rightarrow 0$ by Riemann–Lebesgue; while $\langle u, P^j u \rangle \not\rightarrow 0$ since $P^3 = I$.

Definition 4.8. A group L acts on a set S if there is a mapping

$$L \times S \longrightarrow S, \quad (\lambda, s) \longmapsto \lambda[s], \quad \lambda\lambda'[s] = \lambda[\lambda'[s]]$$

and λ^{-1} is the inverse of $S \ni s \longmapsto \lambda[s] \in S$, a bijection. Often S will have the structure of a topological space or will be equipped with a σ -algebra of measurable sets.

Remark 4.9. While we have stressed discrete decompositions of induced representations, the traditional literature has stressed direct integral decompositions, see, e.g., [Mac88]. A more recent use of continuous parameters in decompositions is a construction by J. Packer et al. [LiPT01] where “wavelet sets” arise as sets of support for direct integral measures. In more detail, starting with a wavelet representation of a certain discrete wavelet group (an induced representation of a semidirect product), the authors in [LiPT01] establish a direct integral where the resulting measure is a subset of \mathbb{R} called “wavelet set.” These wavelet sets had been studied earlier, but independently of representation theory. In the dyadic case, a wavelet set is a subset of \mathbb{R} which tiles \mathbb{R} itself by a combination of \mathbb{Z} -translations, and scaling by powers of 2. In the Packer et al. case, $\mathbb{R} \hookrightarrow K$, and

$$E \text{ wavelet set} \iff \int_E^{\oplus} \text{Ind}_B^G(\chi_t) dt \text{ is the wavelet representation in } L^2(\mathbb{R}).$$

Theorem 4.10. *The group $L := K \rtimes_{\hat{\alpha}} \mathbb{Z}$ acts on the set $\text{Rep}(G)$ by the following assignment:*

Arrange it so that all the representations $U^\chi := \text{Ind}_B^G(\chi)$, $\chi \in K$, acts on the same l^2 -space. Then $\text{Rep}(G) \sim \text{Rep}(G, l^2)$.

5. Induction and Bratteli diagrams

A Bratteli diagram is a group G with vertices V , and edges

$$E \subset V \times V \setminus \{\text{diagonal}\}.$$

It is assumed that

$$V = \bigcup_{n=0}^{\infty} V_n,$$

as a disjoint union in such a way that the edges E in G can be arranged in a sequence of lines $V_n \rightarrow V_{n+1}$, so no edge links pairs of vertices at the same level V_n . With a system of inductions and restrictions one then creates these diagrams.

Let G be created as follows: For a given V_n , let the vertices in V_n represent irreducible representations of some group G_n , and assume G_n is a subgroup in a bigger group G_{n+1} . For the vertices in V_{n+1} , i.e., in the next level in the Bratteli diagram G , we take the irreducible representations occurring in the decomposition of each of the restrictions:

$$\text{Ind}_{G_n}^{G_{n+1}}(L) \Big|_{G_n},$$

so the decomposition of the restriction of the induced representation; see Figure 5.1.

Multiple lines in a Bratteli diagram count the occurrence of irreducible representations with multiplicity; these are called *multiplicity lines*.

Counting multiplicity lines at each level $V_n \rightarrow V_{n+1}$ we get a so called *incidence matrix*. For more details on the use of Bratteli diagrams in representation theory, see [BraJKR00, BraJKR01, BraJKR02, BraJO04, BeKwY14, BeKa14, BeJ15].

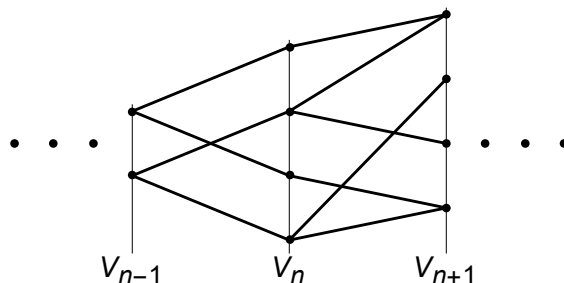


FIGURE 5.1. Part of a Bratteli diagram.

While there is a host of examples from harmonic analysis and dynamical systems; see the discussion above, we shall concentrate below on a certain family of examples when the discrete group B in the construction (from Sections 2 and 3) arises as a discrete Bratteli diagram, and, as a result, its Pontryagin dual compact group K is a solenoid; also often called a compact Bratteli diagram.

Example 5.2 below illustrates this in the special case of constant incidence diagrams, but much of this discussion applies more generally.

Example 5.1. Let $B = \mathbb{Z}[\frac{1}{2}]$, dyadic rationals in \mathbb{R} . Let $\alpha_2 =$ multiplication by 2, so that $\alpha_2 \in \text{Aut}(B)$, and set $G_2 := B \rtimes_{\alpha_2} \mathbb{Z}$.

Recall the Baumslag–Solitar group G_2 with two generators $\{u, t\}$, satisfying $utu^{-1} = t^2$. With the correspondence,

$$u \longleftrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \longleftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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