On solvability of some boundary value problems for a fractional analogue of the Helmholtz equation

B. Kh. Turmetov and B. T. Torebek

ABSTRACT. In this paper we study some boundary value problems for fractional analogue of Helmholtz equation in a rectangular and in a half-band. Theorems about existence and uniqueness of a solution of the considered problems are proved by spectral method.

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1. Introduction and problem statement

For an arbitrary positive α an operator of fractional integration in the sense of Riemann–Liouville of the α order is the following expression [10]:

$$I^\alpha\left[f\right](t) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_0^t (t-s)^{\alpha-1} f\left(s\right) ds.$$

given on the functions $f\left(t\right)$, defined on the interval $(0,\ell), \ell < \infty$. Since

$$I^\alpha[f](t) \to f(t)$$

almost everywhere as $\alpha \to 0$, then by definition we suppose that

$$I^0[f](t) = f(t).$$

Let $m - 1 < \alpha \leq m$, $m = 1, 2, \ldots$. Then the following expression

$$_{RL}D^\alpha [f] (t) = \frac{d^m}{dt^m} I^{m-\alpha} [f] (t)$$

is called a differentiation operator of the α order in the sense of Riemann–Liouville, and

$$_C D^\alpha [f] (t) = _{RL}D^\alpha \left[f(t) - f(0) - \frac{f'(0)}{1!} t - \cdots - \frac{f^{(m-1)}(0)}{(m-1)!} t^{m-1} \right]$$

is a differentiation operator of the α order in the sense of Caputo [10]. If $f(t) \in C^m[0, l]$, then the operator $_C D^\alpha$ can be rewritten as:

$$_C D^\alpha [f] (t) = I^{m-\alpha} \left[f^{(m)} \right] (t).$$

Furthermore, we will use another kind of fractional order derivative. Namely, the sequential derivative of the $k\alpha$, $k = 1, 2, \ldots$ order is:

$$D^\alpha = _C D^\alpha, \quad 0 < \alpha \leq 1, \quad D^{k\alpha} = D^\alpha D^{(k-1)\alpha}, \quad k = 2, 3, \ldots.$$

Note that the concept of sequential derivative was introduced in [14]. Various properties of these operators were studied in [10], [14], [18].

Correct formulation of the initial-boundary value problems for differential equations of fractional order have been studied in many papers [19], [1], [12], [6], [3], [5], [7]. Some questions of solvability of boundary value problems with fractional analogues of the Laplace operator were studied in [8], [21], [13], [4], [2], [11], [20].

As we know, if $0 < \alpha \leq 2$ and ∂_x^α is one of the operators $_{RL}D_x^\alpha$ or $_C D_x^\alpha$, for a fractional order differential equation of the form

$$\partial_x^\alpha u(x, y) - u_{yy}(x, y) = 0, \quad (x, y) \in \Omega$$

correct formulation of boundary value problems depends on the parameter α. So for values $\alpha \in (0, 1]$ as in the case of the conventional parabolic equation for the correctness of mixed problem together with boundary conditions it is enough to give one initial condition, and in the case $\alpha \in (1, 2]$ as for hyperbolic equations it is enough to give two initial conditions.

In the case of differential equation of the form:

$$D_x^{2\alpha} u(x, y) - u_{yy}(x, y) = 0, \quad (x, y) \in \Omega, \quad 0 < \alpha < 1,$$

for any $\alpha \in (0, 2]$ correct problem is a mixed problem with the Cauchy conditions (see e.g., [6]):

$$u(x, 0) = f(x), \quad D_x^\alpha u(x, 0) = g(x).$$

In this paper, we study boundary value problems for fractional analogue of elliptic equations. Denote

$$\Omega_\infty = \left\{ (x, y) \in R^2 : 0 < x < 1, 0 < y < \infty \right\},$$
$$\Omega_1 = \left\{ (x, y) \in R^2 : 0 < x < 1, 0 < y < 1 \right\},$$
$$\bar{\Omega}_\infty = \left\{ (x, y) : 0 \leq x \leq 1, y \geq 0 \right\},$$

$$\bar{\Omega}_1 = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}.$$

Furthermore, Ω will mean one of the domains Ω_∞ or Ω_1. Let $0 < \alpha \le 1$. Consider in the domain Ω the following equation:

(1.1) $\qquad D_x^{2\alpha} u(x, y) + u_{yy}(x, y) - c^2 u(x, y) = 0, \quad (x, y) \in \Omega,$

where c is a real number, $D_x^{2\alpha}$ means $D_x^{2\alpha} = D_x^\alpha D_x^\alpha$ and the operator D_x^α acts by the variable x.

Regular solution of Equation (1.1) is a function $u(x, y) \in C(\bar{\Omega})$, such that $D_x^\alpha u(x, y), D_x^{2\alpha} u(x, y), u_{yy}(x, y) \in C(\Omega)$.

Since for $\alpha = 1 : {}_C D^1 = D^1 = \frac{d}{dy}$, then

$$D_x^2 + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta,$$

i.e., in this case Equation (1.1) coincides with the Helmholtz equation.

For Equation (1.1) we consider the following problems:

Problem 1. Find in the domain Ω_1 a regular solution of Equation (1.1), satisfying the following boundary value conditions:

(1.2) $\qquad u(0, y) = f(y), \quad u(1, y) = g(y), \quad 0 \le y \le 1,$

(1.3) $\qquad u(x, 0) = 0, \quad u(x, 1) = 0, \quad 0 \le x \le 1,$

Problem 2. Find in the domain Ω_∞ a regular solution of Equation (1.1), satisfying the following boundary value conditions:

(1.4) $\qquad u(x, 0) = f(x), \quad 0 \le x \le 1,$

(1.5) $\qquad u(0, y) = 0, \quad u(1, y) = 0, \quad 0 \le y,$

and the condition:

(1.6) $\qquad \lim_{y \to \infty} |u(x, y)| \to 0.$

Problem 3. Find in the domain Ω_∞ a regular solution of Equation (1.1), such that $u_x(x, y) \in C(\bar{\Omega})$, and satisfying condition (1.4),

(1.7) $\qquad u_x(0, y) = 0, \quad u_x(1, y) = 0, \quad 0 \le y,$

and the condition:

(1.8) $\qquad \lim_{y \to \infty} |u(x, y)| \to 0,$

or the condition:

(1.9) $\qquad \lim_{y \to \infty} |u(x, y)| \le C, \quad C = \text{Const}.$

Note that Dirichlet type problem for fractional analogue of the Laplace equation:

(1.10) $\qquad {}_C D_x^\alpha u(x, y) + u_{yy}(x, y) = 0, \quad (x, y) \in \Omega, \quad 1 < \alpha < 2.$

was studied in [13]. Since for the operator ${}_C D^\alpha$, in general, the inequality

$$_C D^\alpha {}_C D^\beta \ne {}_C D^{\alpha+\beta}, \quad 0 < \alpha, \beta \notin N$$

holds (see [18]), then our problem 1 is different from the Dirichlet problems for Equation (1.10).

Need to study boundary value problems for Equation (1.1) is determined by using fractal Laplace equation to describe the production processes in mathematical modeling of socio-economic systems [17]. Note also that in [17] attention was drawn to the fact that the problem of finding a generalized two-factor Cobb-Douglas function is reduced to the Dirichlet problem for a generalized Laplace equation of fractional order.

2. Solution of one dimensional equation with fractional derivative

Let $0 < \alpha \le 1$, μ is a positive real number. For further research, we need to give some information about the solutions of differential equations of the form:

$$(2.1) \qquad D^{2\alpha}[y](t) - \mu^2 y(t) = 0, \quad t > 0.$$

Denote

$$S_1 = \{t : 0 < t < 1\},$$
$$S_\infty = \{t : 0 < t < \infty\},$$
$$\bar{S}_1 = \{t : 0 \le t \le 1\},$$
$$\bar{S}_\infty = \{t : 0 \le t < \infty\}.$$

We will looking for a solution of Equation (2.1) from the class $y(t)$, $D^\alpha y(t) \in C(\bar{S})$, $D^{2\alpha} y(t) \in C(S)$, where S is one of the domains S_1 or S_∞. Since $\mu > 0$, then Equation (2.1) is equivalent to the equation of the form:

$$(2.2) \qquad (D^\alpha - \mu)(D^\alpha + \mu) y(t) = 0.$$

It is well known (see [10]), that partial solution of the equation

$$(D^\alpha + \mu) y(t) = 0$$

is a function

$$y(t) = E_{\alpha,1}(-\mu t^\alpha),$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

is a Mittag-Leffler type function [10].

Then functions

$$(2.3) \qquad \{E_{\alpha,1}(\mu t^\alpha), E_{\alpha,1}(-\mu t^\alpha)\},$$

are solutions of Equation (2.2).

It is easy to show that the functions $E_{\alpha,1}\left(\mu t^{\alpha}\right)$ and $E_{\alpha,1}\left(-\mu t^{\alpha}\right)$ are linear independent. Hence, the system of functions (2.3) are the fundamental solutions of Equation (2.1), and therefore the general solution of this equation has the form:

$$(2.4) \qquad y\left(t\right) = D_1 E_{\alpha,1}\left(-\mu t^{\alpha}\right) + D_2 E_{\alpha,1}\left(\mu t^{\alpha}\right),$$

where D_1, D_2 are arbitrary constants.

Note that in the case $\alpha = \frac{1}{2}$:

$$E_{\alpha,1}\left(\lambda t^{\alpha}\right) = E_{\frac{1}{2},1}\left(\lambda \sqrt{t}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}+1\right)}$$

$$= e^{\lambda^2 t}\left[1 + \frac{2}{\sqrt{\pi}}\int_0^{\lambda\sqrt{t}} e^{-s^2} ds\right] = e^{\lambda^2 t}\,\mathrm{erfc}\left(-\lambda\sqrt{t}\right),$$

where $\mathrm{erfc}(z)$ is the error function.

Furthermore, for the functions $E_{\alpha,\beta}\left(z\right)$ as $|z| \to \infty$ the following asymptotic estimation holds [18]:

$$(2.5) \qquad E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma\left(\beta - \alpha k\right)} + O\left(\frac{1}{|z|^{p+1}}\right),$$

where $|\arg z| \le \rho_1 \pi, \rho_1 \in \left(\frac{\alpha}{2}, \min\{1,\alpha\}\right), \alpha \in (0,2)$. And if $\arg z = \pi$, then

$$(2.6) \qquad E_{\alpha,\beta}(z) = \frac{1}{1+|z|}, |z| \to \infty.$$

In particular, for the functions $E_{\alpha,1}\left(\mu t^{\alpha}\right)$ as $0 < \alpha \le 1$ we get the following estimation:

$$(2.7) \qquad E_{\alpha,1}\left(\mu t^{\alpha}\right) \to \infty, t \to \infty.$$

3. Studying Problem 1

Application of the Fourier method for solving Problem 1 leads to a spectral problem:

$$(3.1) \qquad \begin{cases} Y''\left(y\right) + \lambda Y\left(y\right) = 0, \ 0 < y < 1, \\ Y\left(0\right) = 0, Y\left(1\right) = 0. \end{cases}$$

Eigenvalues of Problem (3.1) have the form: $\lambda_k = (\pi k)^2, \ k = 1, 2, \ldots,$ and corresponding eigen functions

$$Y_k\left(y\right) = \sqrt{2}\sin k\pi y$$

form orthonormal basis of the space $L_2(0,1)$. Consequently, any regular solution of the Problem 1 can be represented at all y as a series:

$$(3.2) \qquad u\left(x,y\right) = \sum_{k=1}^{\infty} u_k\left(x\right) Y_k\left(y\right).$$

It is well known that if $f(y), g(y)$ are smooth enough in $[0, 1]$ and satisfy conditions (1.3), then they can be uniquely represented in the form of a uniformly and absolutely convergent Fourier series by the system $Y_k(y)$:

$$f(y) = \sum_{k=1}^{\infty} f_k Y_k(y),$$

$$g(y) = \sum_{k=1}^{\infty} g_k Y_k(y),$$

where

$$f_k = \int_0^1 f(y) Y_k(y) \, dx,$$

$$g_k = \int_0^1 g(y) Y_k(y).$$

Putting (3.2) into Equation (1.1) and boundary value conditions (1.2), for finding unknown functions $u_k(x)$ we obtain the following problem:

(3.3) $$D_x^{2\alpha} u_k(x) - \mu_k^2 u_k(x) = 0, 0 < x < 1,$$

(3.4) $$u_k(0) = f_k, u_k(1) = g_k.$$

where $\mu_k^2 = (k\pi)^2 + c^2$. Due to the equality (2.4) a general solution of Equation (3.3) has the form:

(3.5) $$u_k(x) = C_1 E_{\alpha,1}(-\mu_k x^\alpha) + C_2 E_{\alpha,1}(\mu_k x^\alpha).$$

Putting function (3.5) into the boundary condition (3.4), we get

$$u_k(x) = C_k(x) f_k + S_k(x) g_k,$$

where

(3.6) $$C_k(x) = \frac{E_{\alpha,1}(\mu_k) E_{\alpha,1}(-\mu_k x^\alpha) - E_{\alpha,1}(\mu_k x^\alpha) E_{\alpha,1}(-\mu_k)}{2\mu_k E_{2\alpha,\alpha+1}(\mu_k^2)},$$

(3.7) $$S_k(x) = \frac{x^\alpha E_{2\alpha,\alpha+1}(\mu_k^2 x^{2\alpha})}{E_{2\alpha,\alpha+1}(\mu_k^2)}.$$

It is easy to see that the function $E_{\alpha,1}(\mu_k^2 x^\alpha)$ satisfies the equation:

(3.8) $$y''(x) - \mu_k^2{}_{RL} D^{2-\alpha} y(x) = 0, 0 < x < 1.$$

It is also known (see [16]) that a regular solution of Equation (3.8) is not a constant (function $y(x)$ from the class $C[0, 1] \cap C^2(0, 1)$) can not attain to its positive maximum (negative minimum) within the segment. It is easy to show that functions $C_k(x)$ and $S_k(x)$ are solutions of Equation (3.8) and

$$C_k(0) = 1, \quad C_k(1) = 0,$$
$$S_k(0) = 0, \quad S_k(1) = 1.$$

Consequently, $0 \leq S_k(x), C_k(x) \leq 1$, for all $x \in [0,1]$.

Further, if the function $\varphi(x)$ belongs to the class $C^{m+\varepsilon}[0,l]$, $m \in \mathbb{Z}_+$ and $\varepsilon < 1$, then for Fourier coefficients of this function the following estimation holds (see [9]):

$$|\varphi_k| = O\left(\frac{1}{k^{m+\varepsilon}}\right), k \to \infty.$$

If $f''(y) \in C^\varepsilon[0,1]$, $g'(y) \in C^\varepsilon[0,1]$ and conditions $f(0) = f(1) = g(0) = g(1) = 0$ hold, then

$$|f_k| \leq \frac{C}{k^{2+\varepsilon}}, |g_k| \leq \frac{C}{k^{1+\varepsilon}}, C = \text{Const}.$$

For such functions, we obtain

$$|u_k(x)| \leq C\left(\frac{1}{k^{2+\varepsilon}} + \frac{1}{k^{1+\varepsilon}}\right).$$

Then the series (3.2) converges uniformly in the domain $\bar{\Omega}_1$, and therefore its sum $u(x,y) \in C(\bar{\Omega}_1)$.

Further, using estimations (2.5) and (2.6), we get

$$2x^\alpha \mu_k E_{2\alpha,\alpha+1}\left(\mu_k^2 x^{2\alpha}\right) = \frac{1}{\alpha} e^{\mu_k^{\frac{1}{\alpha}} x} + O\left(\frac{1}{\mu_k^2}\right),$$

$$E_{\alpha,1}(\mu_k x^\alpha) = \frac{1}{\alpha} e^{\mu_k^{\frac{1}{\alpha}} x} + O\left(\frac{1}{\mu_k}\right),$$

$$E_{\alpha,1}(-\mu_k x^\alpha) = O\left(\frac{1}{\mu_k}\right), x \geq x_0.$$

Then

$$S_k(x) = O\left(e^{\mu_k^{\frac{1}{\alpha}}(x-1)}\right),$$

$$C_k(x) = O\left(\frac{1}{\mu_k}\right).$$

Taking derivative term by term from the series (3.2) twice by y, we have:

$$u_{yy}(x,y) = -\sum_{k=1}^\infty \lambda_k u_k(x) Y_k(y).$$

Then for all $x \geq x_0 > 0$, $0 \leq y \leq 1$ we get

$$|u_{yy}(x,y)| \leq \sum_{k=1}^\infty \left|\mu_k^2\right| |u_k(x)| \leq C \sum_{k=1}^\infty \left[\frac{e^{-\mu_k(1-x)}}{k^\varepsilon} + \frac{1}{k^\varepsilon}\right] < \infty.$$

Similarly, estimate the series

$$D^{2\alpha} u(x,y) = \sum_{k=1}^\infty \mu_k^2 u_k(x) Y_k(y).$$

Then $u_{yy}(x,y), D^{2\alpha} u(x,y) \in C(\Omega_1)$.

Uniqueness of the solution of Problem 1 follows from uniqueness of the solution of problems (3.3) and (3.4). Thus, we have proved the following:

Theorem 1. *Let* $0 < \alpha \leq 1$, $f(y) \in C^{2+\varepsilon}[0,1]$, $g(y) \in C^{1+\varepsilon}[0,1]$ *and conditions* $f(0) = f(1) = 0$, $g(0) = g(1) = 0$ *hold. Then solution of Problem 1 exists, is unique and can be represented as:*

$$u(x,y) = \sum_{k=1}^{\infty} [f_k C_k(x) + g_k S_k(x)] \sin k\pi y,$$

where f_k, g_k *- Fourier coefficients of the functions* $f(y), g(y)$, *and* $C_k(x)$ *and* $S_k(x)$ *are defined by the equalities (3.6) and (3.7), respectively.*

4. Studying Problem 2

We formulate the main proposition concerning Problem 2.

Theorem 2. *Let* $f(x) \in C^{1+\varepsilon}[0,1]$, $f(0) = f(1) = 0$. *Then solution of Problem 2 exists, is unique and can be represented as:*

$$(4.1) \qquad u(x,y) = \sum_{k=1}^{\infty} f_k E_{\alpha,1}(-\mu_k y^\alpha) \sin k\pi x,$$

where

$$f_k = 2 \int_0^1 f(x) \sin k\pi x dx, k = 1, 2, \ldots.$$

Proof. Applying the Fourier method to solve Problem 2, we lead it to a spectral problem (3.1). As we have already noticed the eigen values of this problem have the form $\lambda_k = (\pi k)^2$, $k = 1, 2, \ldots$, and the corresponding eigen functions

$$Y_k(x) = \sqrt{2} \sin k\pi x.$$

System $Y_k(x)$ forms a orthonormal basis in the space $L_2(0,1)$. Consequently, any regular solution of the problem 2 at all $y > 0$ can be represented as a series:

$$(4.2) \qquad u(x,y) = \sum_{k=1}^{\infty} u_k(y) \sin k\pi x.$$

Expand the function $f(x)$ in Fourier series by the system $Y_k(x)$:

$$(4.3) \qquad f(x) = \sum_{k=1}^{\infty} f_k \sin k\pi x,$$

where $f_k = (f, Y_k)$.

Using methods of [15], we consider functions:

$$(4.4) \qquad u_k(y) = 2 \int_0^1 u(x,y) \sin k\pi x dx, \quad k = 1, 2, \ldots,$$

Applying the operator $D_y^{2\alpha} - c^2$ to functions (4.4) and taking account Equation (1.1), we obtain

$$\left(D_y^{2\alpha} - c^2\right)[u_k](y) = 2\int_0^1 \left(D_y^{2\alpha} - c^2\right)[u](x,y)\sin k\pi x\, dx$$

$$= -2\int_{-\pi}^{\pi} u_{xx}(x,y)\sin k\pi x\, dx.$$

Twice integrating by parts the last integral and using conditions (1.5) and (1.6), we receive:

$$(4.5) \qquad\qquad D_y^{2\alpha}u_k(y) - \mu_k^2 u_k(y) = 0,$$

$$(4.6) \qquad\qquad u_k(0) = f_k, \quad k = 1,2,\ldots,$$

$$(4.7) \qquad\qquad \lim_{y\to\infty} |u_k(y)| \to 0.$$

The general solution of Equation (4.5) has the form:

$$u_k(y) = D_1 E_{\alpha,1}(\mu_k y^\alpha) + D_2 E_{\alpha,1}(-\mu_k y^\alpha), \quad \mu_k = \sqrt{c^2 + (k\pi)^2}.$$

Due to the estimations (2.7)

$$E_{\alpha,1}(\mu_k y^\alpha) \to \infty$$

as $y \to \infty$. Thus for the condition (4.7) we need to choose $D_1 = 0$. Then

$$u_k(y) = D_2 E_{\alpha,1}(-\mu_k y^\alpha)$$

and by the condition (4.6) we get

$$u_k(y) = f_k E_{\alpha,1}(-\mu_k y^\alpha).$$

Further, equality (4.4) directly implies uniqueness of the solution of Problem 2, since if $f(x) = 0$ on $[0,1]$, then $u_k(y) = 0, k = 1,2,\ldots$ on $(0,\infty)$. Consequently, due to completeness of the system $\{Y_k(x)\}_{k=1}^{\infty}$, the function $u(x,y) = 0$ for all $(x,y) \in \bar{\Omega}_\infty$.

Therefore, the formal solution of Problem 2 can be represented as (4.1). If the function $f(x)$ satisfies conditions of Theorem 2, then Fourier coefficients we get inequality:

$$|f_k| \le \frac{C}{k^{1+\varepsilon}}.$$

Then for all $x \in [0,1], 0 \le y \le l, l < \infty$ we obtain

$$|u(x,y)| \le \sum_{k=1}^{\infty} \frac{C}{k^{1+\varepsilon}} < \infty,$$

i.e., series (4.1) converges uniformly in a domain $[0,1] \times [0,l]$, and therefore

$$u(x,y) \in C\left(\bar{\Omega}_\infty\right).$$

Further,
$$D_y^{2\alpha} u(x, y) = \sum_{k=1}^{\infty} \mu_k^2 f_k E_{\alpha,1}(-\mu_k y^\alpha) \sin k\pi x.$$

To estimate the last series we use the following properties of the Mittag-Leffler function:

$$-\lambda E_{\alpha,1}(-\lambda y^\alpha) = \sum_{j=0}^{\infty} (-\lambda)^{j+1} \frac{y^{\alpha(j+1)}}{\Gamma(\alpha j + 1)} \frac{1}{y^\alpha}$$

$$= \frac{1}{y^\alpha} \sum_{i=1}^{\infty} (-\lambda)^i \frac{y^{\alpha i}}{\Gamma(\alpha i + 1 - \alpha)}.$$

Then,

$$\left| \mu_k^2 E_{\alpha,1}(-\mu_k y^\alpha) \right| \leq \frac{C(k+1)}{y^\alpha} |E_{\alpha,1-\alpha}(-\mu_k y^\alpha)|, \alpha < 1.$$

Using estimation (2.6), for all $0 < x < 1, 0 < y_0 \leq y < \infty$ we get:

$$\left| D_y^{2\alpha} u(x, y) \right| = \left| \sum_{k=1}^{\infty} \mu_k^2 f_k E_{\alpha,1}(-\mu_k y^\alpha) \sin k\pi x \right| \leq \frac{C}{y_0^\alpha} \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}} < \infty.$$

Consequently, $D_y^{2\alpha} u(x, y) \in C(\Omega_\infty)$.

Similarly, show that $u_{xx}(x, y) \in C(\Omega_\infty)$. Theorem 2 is proved. □

5. Studying Problem 3

For Problem 3, we obtain the following:

Theorem 3. *Let $f(x) \in C^{2+\varepsilon}[0,1]$, $f'(0) = f'(1) = 0$. Then:*

(1) *If $u(x, y)$ is bounded at infinity, then the solution of Problem 3 exists, is unique and can be represented as*

(5.1) $$u(x, y) = f_0 E_{\alpha,1}(-|c|y^\alpha) + \sum_{k=1}^{\infty} f_k E_{\alpha,1}(-\mu_k y^\alpha) \cos k\pi x.$$

(2) *If the function $u(x, y)$ tends to zero at infinity, then:*
   (a) *If $c \neq 0$ the solution of Problem 3 exists, is unique and can be represented as (5.1).*
   (b) *If $c = 0$, Problem 3 is solvable if and only if the following condition holds:*

(5.2) $$\int_0^1 f(x)\, dx = 0,$$

and if a solution exists, it is unique and can be represented as

(5.3) $$u(x, y) = \sum_{k=1}^{\infty} f_k E_{\alpha,1}(-\mu_k y^\alpha) \cos k\pi x,$$

where

$$f_0 = \int_0^1 f(x)\, dx,$$

$$f_k = 2 \int_0^1 f(x) \cos k\pi x dx, \quad k = 1, 2, \ldots.$$

Proof. In the case of Problem 3 the corresponding spectral problem is represented as:

$$\begin{cases} -Y''(y) = \lambda Y(y), & 0 < y < 1, \\ Y'(0) = Y'(1) = 0. \end{cases}$$

Eigen values of the problem have the form: $\lambda_k = (2k\pi)^2, k = 0, 1, \ldots,$ and eigen functions

$$Y_0(y) = 1, Y_k(y) = \sqrt{2} \cos k\pi y, \quad k = 1, 2, \ldots,$$

form orthonormal basis in the space $L_2(0,1)$. Consequently, any regular solution of Problem 3 can be for all $y > 0$ represented as the series:

$$(5.4) \qquad u(x, y) = v_0(y) + \sum_{k=1}^{\infty} v_k(y) \cos k\pi x.$$

Since system of the functions $\{Y_k(x)\}_{k=0}^{\infty}$ is orthonormal, then the function $f(x)$ can be represented as:

$$(5.5) \qquad f(x) = f_0 + \sum_{k=1}^{\infty} f_k \cos k\pi x,$$

where $f_0 = (f, Y_0), f_k = (f, Y_k).$

Consider the function:

$$(5.6) \qquad v_0(y) = \int_0^1 u(x, y)\, dx,$$

$$(5.7) \qquad v_k(y) = 2 \int_0^1 u(x, y) \cos k\pi x dx, \quad k \geq 1.$$

Using the operator $D_y^{2\alpha} - c^2$ to the function (5.6), taking account Equation (1.1), we obtain

$$\left(D_y^{2\alpha} - c^2\right)[v_0](y) = \int_0^1 \left(D_y^{2\alpha} - c^2\right) u(x, y)\, dx = -\int_{-\pi}^{\pi} u_{xx}(x, y)\, dx.$$

Integrating by parts the last integral and using the boundary value condition (1.7), we conclude that $v_0(y)$ satisfies the equation:

(5.8) $$\left(D_y^{2\alpha} - c^2\right)[v_0](y) = 0,$$

and the boundary value condition:

(5.9) $$v_0(0) = f_0, \quad \lim_{y\to\infty} |v_0(y)| < \infty, \quad \left(\lim_{y\to\infty} |v_0(y)| = 0\right).$$

The general solution of Equation (5.8) has the form:

$$v_0(y) = D_1 E(|c|y^\alpha) + D_2 E(-|c|y^\alpha).$$

If $c \neq 0$, then due to the condition (2.7) as $y \to \infty$ function $E(|c|y^\alpha) \to \infty$, and therefore for the conditions (1.8) or (1.9) we assume that $D_1 = 0$. Then the problem (5.8)–(5.9) as

$$\lim_{y\to\infty} |v_0(y)| < \infty$$

or

$$\lim_{y\to\infty} |v_0(y)| = 0$$

and $c \neq 0$ has a unique solution in the form:

(5.10) $$v_0(y) = f_0 E_{\alpha,1}(-|c|y^\alpha).$$

If

$$\lim_{y\to\infty} |v_0(y)| = 0$$

and $c = 0$, then the problem (5.8)–(5.9) has a solution if and only if

$$f_0 = \int_0^1 f(x)\,dx = 0,$$

and consequently, has the form: $v_0(y) = 0$. Necessity of the condition (5.2) is proved. Let us show, that the condition (5.2) is sufficiency for solvability of Equations (5.8)–(5.9) if $c = 0$. Indeed, let the condition (5.2) hold. Then the problem (5.8)–(5.9) has only the trivial solution.

Analogously, for the function $v_k(y)$ we get the problem:

(5.11) $$D_y^{2\alpha}[v_k](y) - \mu_k^2 v_k(y) = 0,$$

with conditions:

(5.12) $$v_k(0) = f_k, \quad \lim_{y\to\infty} |v_k(y)| < \infty, \quad \left(\lim_{y\to\infty} |v_k(y)| = 0\right).$$

The general solution of the problem (5.11) is represented in the form (2.4). Estimation (2.7) yields that $D_1 = 0$. Then using the condition (5.12), we find a solution of Equation (5.11)–(5.12) in the form:

(5.13) $$v_k(y) = f_k E_{\alpha,1}(-\mu_k y^\alpha).$$

Formulas (5.10) and (5.13) directly imply uniqueness of the solution of the problem (5.11)–(5.12), since if $f(x) = 0$ on $[0, 1]$, then $u_k(y) = 0, k =$

$0, 1, \ldots$ on $(0, \infty)$. Consequently according completeness of the cosine system of function $u(x, y) = 0$ for all $(x, y) \in \bar{\Omega}_\infty$. Uniqueness is proved.

Due to the formulas (5.10) and (5.13), solution of Problem 3 can be rewritten in the form (5.1) and (5.3). If the function $f(x)$ satisfies conditions of Theorem 3, then for Fourier coefficients estimation

$$|f_k| \leq \frac{C}{k^{2+\varepsilon}}$$

holds.

Then for all $x \in [0, 1], 0 \leq y \leq l, l < \infty$ we get the estimation:

$$|u(x, y)| \leq \sum_{k=1}^{\infty} \frac{C}{k^{2+\varepsilon}} < \infty,$$

$$|u_x(x, y)| \leq \sum_{k=1}^{\infty} \frac{C}{k^{1+\varepsilon}} < \infty,$$

and therefore

$$u(x, y), u_x(x, y) \in C\left(\bar{\Omega}_\infty\right).$$

Analogously, as in Theorem 2, we prove that $D_y^{2\alpha} u, u_{xx} \in C(\Omega)$. Theorem 3 is proved. $\qquad \square$

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(B. Kh. Turmetov) B.Sattarkhanov street, 29, 161200, Department of Mathematics, Akhmet Yasawi International Kazakh-Turkish University, Kazakhstan, Turkistan
turmetovbh@mail.ru

(B. T. Torebek) A.Pushkin street, 125, 050010, Institute of Mathematics and Mathematical Modelling MES RK, Kazakhstan, Almaty
turebekb85@mail.ru