New York Journal of Mathematics

New York J. Math. **20** (2014) 831–843.

Discrete Duhamel product, restriction of weighted shift operators and related problems

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Abstract. By applying the discrete Duhamel product method we calculate the spectral multiplicity of the direct sum of some operators. In particular, we prove that $\mu(T|X_i \oplus A) = 1 + \mu(A)$ and $\mu(S \oplus A) = 2$ for the restriction of the weighted shift operator $T|X_i$, shift operator S and some appropriate operators A on the Banach spaces.

Contents

1. Introduction and preliminaries

Recall that the classical Duhamel product of two analytic functions $f(z) = \sum\limits_{n\geq 0} \widehat{f}(n)z^n$ and $g(z) = \sum\limits_{n\geq 0} \widehat{g}(n)z^n$ in $\mathrm{Hol}(\mathbb{D})$ is defined by

$$(f \circledast g)(z) = \frac{d}{dz}\int\limits_0^z f(z-t)g(t)dt = \int\limits_0^z f'(z-t)\,g(t)dt + f(0)g(z) \tag{1}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{n!m!}{(n+m)!}\widehat{f}(n)\,\widehat{g}(m)\,z^{n+m},$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc of the complex plane $\mathbb{C}$ (see for instance, Wigley [11, 12]). Beginning from these pioneering works of Wigley, subsequently many interesting and important problems of analysis, operator

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Received March 6, 2014.

2010 *Mathematics Subject Classification.* 47B38, 46B15.

Key words and phrases. Discrete Duhamel product, Weighted shift operator, Borel transform, Banach algebra, Spectral multiplicity.

This work is supported by Suleyman Demirel University with Project 3465-YL1-13. This paper was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

ISSN 1076-9803/2014

theory and Banach algebras theory were investigated namely by applying the method of Duhamel products, see for example, [3, 6, 7, 10, 11, 12].

Here we will consider a generalization of the usual Duhamel product (1), named as *discrete* Duhamel product, and use it to the study of spectral multiplicity of direct sums of some operators.

Recall that a subspace $E \subset X$ is called a cyclic subspace of an operator $A \in \mathcal{L}(X)$ (Banach algebra of all bounded linear operators on X) if

$$\operatorname{span}\{A^n E : n = 0, 1, 2, \dots\} = X,$$

where span stands for the closed linear hull. A vector $x \in X$ is called cyclic ($x \in \operatorname{Cyc}(A)$) if

$$\operatorname{span}\{A^n x : n = 0, 1, 2, \dots\} = X.$$

The spectral multiplicity $\mu(A)$ of the operator A is

$$\mu(A) := \inf\{\dim E : \operatorname{span}\{A^n E : n \geq 0\} = X\},$$

a nonnegative integer or ∞. A is a cyclic operator (i.e., there exists a vector $x \in X$ such that $x \in \operatorname{Cyc}(A)$) if and only if $\mu(A) = 1$. For example, it follows from the Weierstrass approximation theorem that $\mu(V) = \mu(M_x) = 1$ for the classical Volterra integration operator V and multiplication operator M_x defined in the space $C[0,1]$ by

$$Vf(x) = \int_0^x f(t)dt,$$
$$M_x f(x) = xf(x),$$

respectively. Also, by the classical Beurling theorem [1] in the Hardy space $H^2 = H^2(\mathbb{D})$ over the unit disc $\mathbb{D}$, $\mu(S) = \mu(S^*) = 1$, where $S : H^2 \to H^2$ is the classical unilateral shift operator in H^2 defined by $Sf(z) = zf(z)$, and S^* is the backward shift operator on H^2 defined by

$$S^* f(z) := \frac{f(z) - f(0)}{z}.$$

(S^* is the simple co-analytic Toeplitz operator $T_{\bar{z}}$ defined by

$$T_{\bar{z}} f(z) = P_+ \bar{z} f(z),$$

where $P_+ : L^2(\partial\mathbb{D}) \to H^2$ is the classical Riesz orthogonal projector.) However, it is well-known that $\mu(S^n) = n$ for any finite integer n, and $\mu(S_E) = \infty$ for the shift operator S_E acting in the vector valued Hardy space $H^2(E)$ with infinity dimensional Hilbert space E.

Note that as the norm, spectral radius, numerical radius, spectrum and numerical range of operator, the spectral multiplicity is also an important invariant of an operator. For this it is sufficient to remember, for example, the spectral theorem for normal operators on the Hilbert space (see, for example, Rudin [8]). Of course, the concept of cyclic subspace is very important in relation with the outstanding problem of existence of a nontrivial invariant subspace. Namely, it is easy to see that an operator $A : X \to X$ has no nontrivial invariant subspace if and only if $x \in \operatorname{Cyc}(A)$ for any nonzero vector $x \in X$. Note that a subspace $E \subset X$ is an invariant subspace for an operator A, if $AE \subset E$, that is, $Ax \in E$ for every $x \in E$.

Let $A \oplus B$ denote the direct sum of operators $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$, $(A \oplus B)(x \oplus y) = Ax \oplus By$, $x \oplus y \in X \oplus Y$. It is known that (see, for example, Halmos [2])

$$\mu(A \oplus B) \leq \mu(A) + \mu(B)$$

for any operator $A \oplus B \in \mathcal{L}(X \oplus Y)$. Here we will investigate the equality $\mu(A \oplus B) = \mu(A) + \mu(B)$ for some operators A and B (For basic facts on the spectral multiplicity of direct sums of operators we recommend the papers [4], [5] and references therein).

Let X be a Banach space with Schauder basis $(e_n)_{n\geq 0}$. Let $(\lambda_n)_{n\geq 0} \subset \mathbb{C}$ be a bounded sequence of nonzero numbers λ_n. We set $w_n := \lambda_0\lambda_1 \dots \lambda_{n-1}$, $w_0 := 1$,

$$X_i := \operatorname{span}(e_k : k = i, i+1, \dots), i = 0, 1, 2, \dots,$$

for any two vectors $x = \sum_{n=i}^{\infty} x_n e_n$ and $y = \sum_{n=i}^{\infty} y_n e_n$ in X_i $(i = 0, 1, 2, \dots)$. Then *discrete* Duhamel product (sometimes it is also called generalized Duhamel product, see for instance [3] and references therein, and also Karaev and Gürdal [6]) is defined by,

$$x \underset{i}{\tilde{\circledast}} y := \sum_{n=i}^{\infty} \sum_{m=i}^{\infty} \frac{w_{n+m-i}}{w_n w_m} x_n y_n e_{n+m-i}. \tag{2}$$

It is easy to see that the classical Duhamel product $\circledast$ (see formula (1)) corresponds to $i = 0$ and $\lambda_n := \frac{1}{n+1}$, $n \geq 0$, in (2). It is also easy to verify that the product $\underset{i}{\tilde{\circledast}}$ is commutative and associative.

Let T be the weighted shift operator acting in X by the formula

$$Te_n = \lambda_n e_{n+1},\ n = 0, 1, 2, \dots.$$

It can be easily shown that all subspaces X_i $(i \geq 0)$ are closed T-invariant subspace (i.e., $TX_i \subset X_i,\ i \geq 0$). Therefore the restricted weighted shift operators $T|X_i$, $i \geq 0$, are well-defined operators on the subspace X_i, $i \geq 0$.

In this paper, we will develop a method of the paper [9] (see the proof of Theorem 1 there) and investigate the spectral multiplicity of the operators $T|X_i \oplus A$, $i \geq 0$; here $\oplus$ stands for the direct sum of operators on the direct sum of Banach spaces. Our results also improve some results in [9] and [3]. Before giving the results of the paper, let us give some necessary definitions, notations and preliminaries.

The generalized Borel transform B_w from X onto the space of formal series over the field of complex numbers $\mathbb{C}$ is defined as follows

$$B_w\left(\sum_{n=0}^{\infty} x_n e_n\right) := \sum_{n=0}^{\infty} \frac{1}{w_n} x_n e_n,$$

where $w_n = \lambda_0 \lambda_1 \ldots \lambda_{n-1}$, $n \geq 0$. The inverse generalized Borel transform is defined by

$$B_w^{-1}\left(\sum_{n=0}^{\infty} x_n e_n\right) := \sum_{n=0}^{\infty} w_n x_n e_n.$$

Clearly, the classical Borel transform from $\operatorname{Hol}(\mathbb{D})$ (the space of all analytic functions on the unit disc $\mathbb{D}$) onto the space of formal power series $\mathbb{C}[[Z]]$ over the field of complex numbers $\mathbb{C}$ corresponds to the case $\lambda_n = \frac{1}{n+1}$, $n \geq 0$.

Recall that the class $\ell_A^p(w_n)$, $w_n \geq 0$, $n \leq 0$, $p \geq 1$ is defined by

$$\ell_A^p(w_n) := \left\{ f \in \operatorname{Hol}(D) : \|f\|_{\ell_A^p(w_n)} := \left(\sum_{n=0}^{\infty} \left|\widehat{f}(n)\right|^p w_n^p\right)^{1/p} < +\infty \right\},$$

where $\widehat{f}(n) := \frac{f^{(n)}(0)}{n!}$ is the n^{th} Taylor coefficient of the analytic function $f(z) = \sum\limits_{n \geq 0}^{\infty} \widehat{f}(n) z^n$ on $\mathbb{D}$. Note that:

(a) Every bounded linear operator C on a Banach space X admits the functional calculus from the class $\ell_A^1(\|C^n\|)$. Indeed, we can put $f(C) \overset{\text{def}}{=} \sum\limits_{n \geq 0}^{\infty} \widehat{f}(n) C^n$ for every function $f \in \ell_A^1(\|C^n\|)$, because in this case

$$\|f(C)\| = \left\|\sum_{n=0}^{\infty} \widehat{f}(n) C^n\right\| \leq \sum_{n=0}^{\infty} \left|\widehat{f}(n)\right| \|C^n\| = \|f\|_{\ell_A^1(\|C^n\|)},$$

for every $f \in \ell_A^1(\|C^n\|)$.

(b) Every operator $C \in \mathcal{L}(X)$ (the Banach algebra of all bounded linear operators on X), satisfying the condition $\sum\limits_{n=0}^{\infty} \|C^n\|^q < +\infty$, admits the functional calculus from the class

$$\ell_A^p := \ell_A^p(D) = \left\{ f \in \operatorname{Hol}(D) : \|f\|_{k_A^p}^p = \sum_{n=0}^{\infty} \left|\widehat{f}(n)\right|^p < +\infty \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq 1$. Indeed,

$$\|f(C)\|_{\mathcal{L}(X)} = \left\|\sum_{n=0}^{\infty} \widehat{f}(n) C^n\right\|_{\mathcal{L}(X)} \leq \sum_{n=0}^{\infty} \left|\widehat{f}(n)\right| \|C^n\|_{\mathcal{L}(X)}$$

$$\leq \left(\sum_{n=0}^{\infty}\left|\widehat{f}\,(n)\right|^{p}\right)^{1/p}\left(\sum_{n=0}^{\infty}\|C^{n}\|_{\mathcal{L}(X)}^{q}\right)^{1/q}$$
$$= \|f\|_{\ell_A^p}\, M\,(C,q)\,,$$

where $M\,(C,q) > 0$ is a constant.

Let us define also the following (closed) subspace of the space $\ell_A^p\,(\mathbb{D})$:

$$\ell_{A,i}^{p} := \left\{\sum_{n=0}^{\infty}\widehat{f}\,(n)\,z^{n}\in\ell_A^p : \widehat{f}\,(k)=0,\ k=0,1,2,\dots,i-1\right\},\ i=1,2,\dots\ .$$

2. The results

The following two lemmas can be proved by similar arguments used in the proofs of Theorem 1 in [9], Theorem 2 of [6] and Theorem 16 in [5], and therefore we omit their proofs.

Lemma 1. *Let X be a Banach space with a Schauder basis $(e_n)_{n\geq 0}$, x,y be two elements in $X_i = \operatorname{span}\{e_n : n=i,i+1,\dots\}$, $i\geq 0$. Let $Te_n=\lambda_n e_{n+1}$, $\lambda_n\neq 0$, $n\geq 0$, be the weighted shift operator with bounded weights sequence $(\lambda_n)_{n\geq 0}$ continuously acting in X. Then we have:*

$$x\underset{i}{\tilde{\circledast}}y = \sum_{n,m\geq i} x_n y_m \frac{w_{n+m-i}}{w_n w_m} e_{n+m-i} \tag{3}$$
$$= (B_w x)\,(T|X_i)\,y = (B_w g)\,(T|X_i)\,x,$$

where $T|X_i$ is a restricted weighted shift operator and

$$(B_w x)\,(T|X_i)\,y \overset{\text{def}}{=} \sum_{n=i}^{\infty}\frac{1}{w_n}x_n\,(T|X_i)^n\,y. \tag{4}$$

Lemma 2. *Let X be a Banach space with a Schauder basis $(e_n)_{n\geq 0}$ continuously embedded in ℓ^p for some $p\geq 1$. Let $Te_n=\lambda_n e_{n+1}$, $n\geq 0$, be the weighted shift operator continuously acting in X. We put $w_n = \lambda_0\lambda_1\dots\lambda_{n-1}$, $w_0 := 1$. Suppose that for any integer $i\geq 1$ there exists an integer $N\geq i$ such that*

$$\sum_{n,m\geq N}\left|\frac{w_{n+m-i}}{w_n w_m}\right|^q < +\infty \text{ for } p>1$$

and

$$\sum_{n,m\geq N}\left|\frac{w_{n+m-i}}{w_n w_m}\right| < +\infty \text{ for } p=1,$$

where $\frac{1}{p}+\frac{1}{q}=1$. Suppose also that $\|e_{n+m-i}\|\leq M_i\,\|e_n\|\,\|e_m\|$ for all $n,m\geq i$ and some $M_i>0$. Then we have:

(a)

$$\left\| x \underset{i}{\tilde{\circledast}} y \right\|_{X_i} \leq C_i \|x\|_{X_i} \|y\|_{X_i} \tag{5}$$

for all $x, y \in X_i$ *and some constant* $C_i > 0$, *i.e.,* $\left(X_i, \underset{i}{\tilde{\circledast}}\right)$ *is a unital Banach algebra with the unit element* $w_i e_i$.

(b) *An element* $x \in X_i$ *is* $\underset{i}{\tilde{\circledast}}$*-invertible if and only if* $x_i \neq 0$.

The main result of the present paper is the following.

Theorem 1. *Let* X *be a Banach space with a Schauder basis* $(e_n)_{n\geq 0}$ *which is embedded in* ℓ^p *for some integer* $p \geq 1$. *Let* T, $Te_n = \lambda_n e_{n+1}$, $n \geq 0$, *be the weighted shift operator with bounded weight sequence* $(\lambda_n)_{n\geq 0}$, *such that* T *is continuous in* X. *We put* $w_n := \lambda_0 \lambda_{n1} \ldots \lambda_{n-1}$, $w_0 := 1$. *Suppose that for any integer* $i \geq 1$ *there exists an integer* $N \geq i$ *such that*

$$\sum_{n,m\geq N} \left| \frac{w_{n+m-i}}{w_n w_m} \right|^q < +\infty \text{ for } p > 1 \tag{6}$$

and

$$\sum_{n,m\geq N} \left| \frac{w_{n+m-i}}{w_n w_m} \right| < +\infty \text{ for } p = 1, \tag{7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose also that* $\|e_{n+m-i}\| \leq M_i \|e_n\| \|e_m\|$ *for all* $n, m \geq i$ *and some* $M_i > 0$. *Let* $Q : Y \to Y$ *be an operator on a Banach space* Y *such that* $\sum_{k=1}^{\infty} \left(\frac{\|Q^k\|}{|w_k|} \right)^q < +\infty$. *Then*

$$\mu(T|X_i \oplus Q) \leq \mu(T|X_i) + \mu(Q) = 1 + \mu(Q).$$

Proof. First, note that the restricted operator $T|X_i$ is cyclic for every $i \geq 1$, and therefore $\mu(T|X_i) = 1$ for every $i \geq 1$. On the other hand, since

$$1 + \mu(Q) \geq \mu(T|X_i \oplus Q) \geq \max\{1, \mu(Q)\},$$

it is clear that if $\mu(Q) = +\infty$, then $\mu(T|X_i \oplus Q) = 1 + \mu(Q)$. So, we will assume that $\mu(Q) = n < +\infty$. For the proof, suppose in contrary that $\mu(T|X_i \oplus Q) \neq n+1$, that is $\mu(T|X_i \oplus Q) < n+1$ or $\mu(T|X_i \oplus Q) = \mu(Q) = n$. Then by the definition of the spectral multiplicity there exists n-dimensional cyclic subspace for the operator $T|X_i \oplus Q$. Let

$$\left\{x^{(1)} \oplus y^{(1)}, x^{(2)} \oplus y^{(2)}, \ldots, x^{(n)} \oplus y^{(n)}\right\}$$

be a cyclic tuple of vectors for the operator $T|X_i \oplus Q$. Then

$$\left\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right\}$$

is a cyclic tuple for the operator $T|X_i$.

By considering Lemma 2, it follows from the conditions (6), (7) that $\left(X_i, \underset{i}{\tilde{\circledast}}\right)$ is a Banach algebra. Consequently, for every $x \in X_i$ the "discrete Duhamel operator" $\mathcal{D}_x$, $\mathcal{D}_x y := x \underset{i}{\tilde{\circledast}} y$, $y \in X_i$, is continuous in X_i and , $\|\mathcal{D}_x\| \le C_i \|x\|_{X_i}$ (see inequality (5)). On the other hand, formula (2) shows that

$$(T|X_i)^m y = w_{i+m} e_{i+m} \underset{i}{\tilde{\circledast}} y, \ m \ge 0, \tag{8}$$

and therefore

$$\operatorname{span}\left\{(T|X_i)^m y : m \ge 0\right\} = clos\mathcal{D}_y \operatorname{span}\left\{w_{i+m} e_{i+m} : m \ge 0\right\},$$

which implies that $y \in \operatorname{Cyc}(T|X_i)$ if and only if $\overline{\mathcal{D}_y X_i} = X_i$. It is not difficult to prove that $\overline{\mathcal{D}_y X_i} = X_i$ if and only if $y_i \ne 0$, that is y is an invertible element of the Banach algebra $\left(X_i, \underset{i}{\tilde{\circledast}}\right)$, which is equivalent to the invertibility of the corresponding discrete Duhamel operator $\mathcal{D}_y$. Thus, the cyclicity of the tuple $\left\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right\}$ implies that there exists a number $i_0 \in \{1, \ldots, n\}$ such that $x_i^{(i_0)} \ne 0$. We assume without loss of generality that $i_0 = 1$, that is $x_i^{(1)} \ne 0$. Under this condition, as already mentioned above, $x^{(1)}$ is invertible in $\left(X_i, \underset{i}{\tilde{\circledast}}\right)$ (see the assertion (b) in Lemma 2). Therefore, there exists a unique element $z^{(1)} \in \left(X_i, \underset{i}{\tilde{\circledast}}\right)$ such that $z^{(1)} \underset{i}{\tilde{\circledast}} x^{(1)} = w_i e_i$. Since $\left(z^{(1)} \underset{i}{\tilde{\circledast}} x^{(1)}\right)_i = z_i^{(1)} x_i^{(1)}$, it follows that $z_i^{(1)} \ne 0$. Let us consider the following matrix:

$$M := \begin{pmatrix} z^{(1)} & 0 & 0 & \ldots & 0 \\ -x^{(2)} \underset{i}{\tilde{\circledast}} z^{(1)} & w_i e_i & 0 & \ldots & 0 \\ -x^{(3)} \underset{i}{\tilde{\circledast}} z^{(1)} & 0 & w_i e_i & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x^{(n)} \underset{i}{\tilde{\circledast}} z^{(1)} & 0 & 0 & \ldots & w_i e_i \end{pmatrix}.$$

Then, by using formulas (3) and (4) in Lemma 1 and formula (8), we have:

$$(B_w M)(T|X_i) \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ \vdots \\ x^{(n)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(B_w z^{(1)}\right)(T|X_i) & 0 & 0 & \dots & 0 \\ \left(B_w\left(-x^{(2)}\underset{i}{\tilde{\circledast}}z^{(1)}\right)\right)(T|X_i) & I_{X_i} & 0 & \dots & 0 \\ \left(B_w\left(-x^{(3)}\underset{i}{\tilde{\circledast}}z^{(1)}\right)\right)(T|X_i) & 0 & I_{X_i} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(B_w\left(-x^{(n)}\underset{i}{\tilde{\circledast}}z^{(1)}\right)\right)(T|X_i) & 0 & 0 & \dots & I_{X_i} \end{pmatrix} \cdot \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ \vdots \\ x^{(n)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(B_w z^{(1)}\right)(T|X_i)\,x^{(1)} \\ \left(B_w\left(-x^{(2)}\underset{i}{\tilde{\circledast}}z^{(1)}\right)\right)(T|X_i)\,x^{(1)} + x^{(2)} \\ \left(B_w\left(-x^{(3)}\underset{i}{\tilde{\circledast}}z^{(1)}\right)\right)(T|X_i)\,x^{(1)} + x^{(3)} \\ \vdots \\ \left(B_w\left(-x^{(n)}\underset{i}{\tilde{\circledast}}z^{(1)}\right)\right)(T|X_i)\,x^{(1)} + x^{(n)} \end{pmatrix}$$

$$= \begin{pmatrix} x^{(1)}\underset{i}{\tilde{\circledast}}z^{(1)} \\ x^{(1)}\underset{i}{\tilde{\circledast}}\left(-x^{(2)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) + x^{(2)} \\ x^{(1)}\underset{i}{\tilde{\circledast}}\left(-x^{(3)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) + x^{(3)} \\ \vdots \\ x^{(1)}\underset{i}{\tilde{\circledast}}\left(-x^{(n)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) + x^{(n)} \end{pmatrix}$$

$$= \begin{pmatrix} w_i e_i \\ -x^{(2)}\underset{i}{\tilde{\circledast}}\left(x^{(1)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) + x^{(2)} \\ -x^{(3)}\underset{i}{\tilde{\circledast}}\left(x^{(1)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) + x^{(3)} \\ \vdots \\ -x^{(n)}\underset{i}{\tilde{\circledast}}\left(x^{(1)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) + x^{(n)} \end{pmatrix}$$

$$= \begin{pmatrix} w_i e_i \\ -x^{(2)}\underset{i}{\tilde{\circledast}}w_i e_i + x^{(2)} \\ -x^{(3)}\underset{i}{\tilde{\circledast}}w_i e_i + x^{(3)} \\ \vdots \\ -x^{(n)}\underset{i}{\tilde{\circledast}}w_i e_i + x^{(n)} \end{pmatrix} = \begin{pmatrix} w_i e_i \\ -x^{(2)} + x^{(2)} \\ -x^{(3)} + x^{(3)} \\ \vdots \\ -x^{(n)} + x^{(n)} \end{pmatrix} = \begin{pmatrix} w_i e_i \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So

$$(B_wM)\,(T|X_i)\begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ \vdots \\ x^{(n)} \end{pmatrix} = \begin{pmatrix} w_ie_i \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since

$$B_wM = \begin{pmatrix} B_wz^{(1)} & 0 & 0 & \dots & 0 \\ B_w\left(-x^{(2)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) & e_i & 0 & \dots & 0 \\ B_w\left(-x^{(3)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) & 0 & e_i & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_w\left(-x^{(n)}\underset{i}{\tilde{\circledast}}z^{(1)}\right) & 0 & 0 & \dots & e_i \end{pmatrix},$$

we have

$$\underset{i}{\tilde{\circledast}} - \det(B_wM) = B_wz^{(1)}\underset{i}{\tilde{\circledast}}e_i\underset{i}{\tilde{\circledast}}\dots\underset{i}{\tilde{\circledast}}e_i = \frac{1}{w_i^{n-1}}B_wz^{(1)},$$

and hence

$$\left(\underset{i}{\tilde{\circledast}} - \det(B_wM)\right)_i = \frac{1}{w_i^{n-1}}\frac{1}{w_i}z_i^{(1)} = \frac{1}{w_i^{n}}z^{(1)} \neq 0$$

(because $z^{(1)}$ is $\underset{i}{\tilde{\circledast}}$-invertible element; see above). This implies that the operators $(B_wM)\,(T|X_i)$ and $(B_wM)\,(Q)$ are invertible in the spaces

$$X^n := \underbrace{X * X * \dots * X}_{n} \quad \text{and}$$
$$Y^n := \underbrace{Y * Y * \dots * Y}_{n},$$

respectively. Therefore, by Karaev's result (about this result see in reference of [5]), we have that

$$\left\{((B_wM)\,(T|X_i)\,x)_j \oplus ((B_wM)\,(Q)\,y)_j : j = 1, 2, \dots, n\right\}$$

is a cyclic tuple for the operator $T|X_i \oplus Q$. So, we have a new cyclic tuple of the form $\{w_ie_i \oplus \bar{y}_1, 0 \oplus \bar{y}_2, \dots, 0 \oplus \bar{y}_n\}$, and therefore for every $y \in Y$ there exists a family of vector-polynomials $\{P_{m,j}\}_{j=1}^{n}$ such that

$$\lim_{m\to\infty} P_{m,1}\,(T|X_i)\,w_ie_i = 0 \text{ in } X_i \tag{9}$$

and

$$\lim_{m\to\infty}\sum_{j=i}^{n} P_{m,j}\,(Q)\,\bar{y}_j = y \text{ in } Y. \tag{10}$$

We set

$$q_{m,1} := B_w^{-1} P_{m,1} = \sum_{k\geq 0} w_k \left(P_{m,1}\right)_k e_k.$$

It is clear that $P_{m,1}\left(T|X_i\right) w_i e_i = \sum\limits_{k\geq 0} w_k \left(P_{m,1}\right)_k e_k = q_{m,1}$, and by considering this in (9) we obtain that $\lim\limits_{m\to\infty} q_{m,1} = 0$ in X. Now by considering the condition

$$\sum_{k=i}^{\infty} \left(\frac{\left\|Q^k\right\|}{|w_k|} \right)^q =: C < +\infty,$$

we have

$$\begin{aligned} \left\|P_{m,1}(Q)\right\|_{\mathcal{L}(Y)} &= \left\| \sum_{k\geq 0} \left(P_{m,1}\right)_k Q^k \right\|_{\mathcal{L}(Y)} \\ &\leq \sum_{k\geq 0} \left|\left(P_{m,1}\right)_k\right| \left\|Q^k\right\|_{\mathcal{L}(Y)} = \sum_{k\geq 0} |w_k| \left|\left(P_{m,1}\right)_k\right| \frac{1}{|w_k|} \left\|Q^k\right\|_{\mathcal{L}(Y)} \\ &\leq \left(\sum_{k\geq 0} \left| w_k \left(P_{m,1}\right)_k \right|^p \right)^{\frac{1}{p}} \left(\sum_{k\geq 0} \left(\frac{\left\|Q^k\right\|_{\mathcal{L}(Y)}}{|w_k|} \right)^q \right)^{\frac{1}{q}} \\ &= C \left\|q_{m,1}\right\|_{\ell^p}. \end{aligned}$$

Since $X \subset \ell^p$, we have $\left\|q_{m,1}\right\|_{\ell^p} \leq \tilde{C} \left\|q_{m,1}\right\|_X$ and thus

$$\left\|q_{m,1}(Q)\right\|_{\mathcal{L}(Y)} \leq C\tilde{C} \left\|q_{m,1}\right\|_X.$$

On the other hand, since $q_{m,1} \to 0$ $(m \to \infty)$ in X, we obtain from this inequality that $q_{m,1}(Q) \rightrightarrows 0$, and therefore $\lim\limits_{m\to\infty} q_{m,1}(Q)\, \bar{y}_1 = 0$. Hence, it follows from (10) that

$$\lim_{m\to\infty} \sum_{j=2} P_{m,j}(Q)\, \bar{y}_j = y.$$

Since $y \in Y$ is arbitrary, this equality means that $\{\bar{y}_2, \bar{y}_3, \ldots, \bar{y}_n\} \in \operatorname{Cyc}(Q)$, which implies that $\mu(Q) \leq n-1$ (because $card\,\{\bar{y}_2, \bar{y}_3, \ldots, \bar{y}_n\} = n-1$). But, this contradicts to $\mu(Q) = n$. The theorem is proven. $\square$

The following is an immediate corollary of Theorem 1.

Corollary 1. *Let V, $Vf(z) = \int\limits_0^z f(t)\,dt$, be a Volterra integration operator on the space $\ell_A^p(\mathbb{D})$, $1 \leq p < \infty$, and let Q be a bounded operator on a separable Banach space Y satisfying*

$$\sum_{k=1}^{\infty} \left(k! \left\|Q^k\right\| \right)^q < +\infty, \textit{ for } p > 1,$$

and

$$\left\|Q^k\right\| = O\left(\frac{1}{k!}\right), \text{ for } p = 1;$$

here $\frac{1}{p}+\frac{1}{q}=1$. *Then* $\mu(V \oplus Q) = 1 + \mu(Q)$.

For the proof, it is sufficient to put in Theorem 1, $X = \ell_A^p$, $\lambda_n = \frac{1}{n+1}$ $(n \geq 0)$, $e_n = z^n$ $(n \geq 0)$, $i = 0$, and to modify the proof of the theorem in case $p = 1$.

Let S be the shift operator on $\ell_A^p = \ell_A^p(\mathbb{D})$ defined by $Sf(z) = zf(z)$. Our next result calculates $\mu(S + C)$ for some $C \in \mathcal{L}(X)$.

Theorem 2. *Let* S *be a shift operator on* ℓ_A^p, $1 < p < \infty$, *and* C *be a cyclic bounded linear operator on a separable Banach space* X *such that* $\sum_{k=0}^{\infty} \left\|C^k\right\|^q < +\infty$, *where* $\frac{1}{p}+\frac{1}{q}=1$. *Then* $\mu(S \oplus C) = 2$.

Proof. As we have proved in Section 1,

$$\|f(C)\|_{\mathcal{L}(X)} \leq M(C,q)\,\|f\|_{\ell_A^p} \tag{11}$$

for all $f \in \ell_A^p$, where $M(C,q) := \left(\sum_{k=0}^{\infty} \left\|C^k\right\|^q\right)^{\frac{1}{q}}$, which means that C admits the functional calculus in the class ℓ_A^p. Since $\mu(S) = 1$ and $\mu(C) = 1$, it is clear that $1 \leq \mu(S \oplus C) \leq \mu(S) + \mu(C) = 2$. Suppose in contrary that $\mu(S \oplus C) = 1$. Then the operator $S \oplus C$ has a cyclic vector $f \oplus x \in \ell_A^p \oplus X$, which implies that there exists polynomials p_n such that

$$\lim_{n\to\infty} p_n(S \oplus C)(f \oplus x) = 1 + 0 \text{ in } \ell_A^p \oplus X,$$

or

$$\lim_{n\to\infty} (p_n(S) \oplus p_n(C))(f \oplus x) = 1 \oplus 0.$$

Whence

$$\lim_{n\to\infty} p_n(z) f(z) = 1 \text{ in } \ell_A^p \tag{12}$$

and

$$\lim_{n\to\infty} p_n(C)\, x = 0 \text{ in } X. \tag{13}$$

It is obvious that $p_n f - 1 \in \ell_A^p$. Let $p_n(z) f(z) - 1 = \sum_{k=0}^{\infty} a_{n,k} z^k$. Then by considering inequality (11) and (12), we have

$$\|(p_n f - 1)(C)\|_{\mathcal{L}(X)} \leq M(C,q)\, \|p_n f - 1\|_{\ell_A^p} \to 0 \ \text{ (as } n \to \infty),$$

which means that $p_n(C) f(C) \rightrightarrows I_X$ (as $n \to \infty$). In particular,

$$\lim_{n\to\infty} p_n(C) f(C)\, x = x \text{ in } X.$$

On the other hand, it follows from (13) that

$$0 = \lim_{n\to\infty} f(C)\, p_n(C)\, x = \lim_{n\to\infty} p_n(C) f(C)\, x = x,$$

or $x = 0$, which is impossible, because x is a cyclic vector of the operator C. This proves the theorem. □

Let us denote

$$\begin{aligned}\ell_A^{p,n} &:= \left\{ f \in \ell_A^p(\mathbb{D}) : f^{(n)} \in \ell_A^p \right\} \\ &= \Bigg\{ f \in \ell_A^p(\mathbb{D}) : f(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k \text{ and} \\ &\qquad \sum_{k=n}^{\infty} \left(k(k-1)\dots(k-n+1) \left| \widehat{f}(k) \right| \right)^p < +\infty \Bigg\}.\end{aligned}$$

Theorem 3. *Let S be a shift operator on the space $\ell_A^{p,n}$, where $p \geq 1$, and C be a cyclic operator on the separable Banach space X satisfying*

$$\sum_{k=n}^{\infty} \left(\frac{\left\| C^k \right\|}{k(k-1)\dots(k-n+1)} \right)^q < +\infty, \tag{14}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then $\mu(S \oplus C) = 2$.

Proof. By using (14), we have for every $f \in \ell_A^{p,n}$ that

$$\begin{aligned}\| f(C) \| &= \left\| \sum_{k=0}^{\infty} \widehat{f}(k) C^k \right\| \leq \sum_{k=0}^{\infty} \left| \widehat{f}(k) \right| \left\| C^k \right\| \\ &= \left| \widehat{f}(0) \right| + \left| \widehat{f}(1) \right| \|C\| + \dots + \left| \widehat{f}(n-1) \right| \left\| C^{n-1} \right\| + \\ &+ \sum_{k=n}^{\infty} k(k-1)\dots(k-n+1) \left| \widehat{f}(k) \right| \frac{\left\| C^k \right\|}{k(k-1)\dots(k-n+1)} \\ &\leq M_1(C,n) \|f\|_{\ell_A^{p,n}} + \left(\sum_{k=n}^{\infty} \left(k(k-1)\dots(k-n+1) \left| \hat{f}(k) \right| \right)^p \right)^{\frac{1}{p}} \cdot \\ &\cdot \left(\sum_{k=n}^{\infty} \left(\frac{\left\| C^k \right\|}{k(k-1)\dots(k-n+1)} \right)^q \right)^{\frac{1}{q}} \\ &= M_2(C,n) \|f\|_{\ell_A^{p,n}},\end{aligned}$$

where $M_1(C,n)$, $M_2(C,n) > 0$ are some numbers. Thus

$$\| f(C) \| \leq M_2(C,n) \|f\|_{\ell_A^{p,n}} \tag{15}$$

for all $f \in \ell_A^{p,n}$. Note that $\mu(S) = 1$. Now in order to complete the proof of the theorem, it remains only to use inequality (15) and the arguments for the proof of Theorem 2. □

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This paper is available via `http://nyjm.albany.edu/j/2014/20-42.html`.