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A modular description of $ER(2)$

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Abstract. We give a description of the maximally unramified extension of completed second Real Johnson–Wilson theory using supersingular elliptic curves with $\Gamma_0(3)$-level structures.

Contents

1. Introduction

1.1. Real Johnson–Wilson theories. Complex conjugation acts on the complex K-theory spectrum KU and the homotopy fixed points of this action is KO. In fact the complex orientation $MU \to KU$ is equivariant with respect to this C_2-action. Localizing at the prime 2, there is a C_2-equivariant orientation

$$BP \to KU_{(2)}$$

with kernel $\langle v_i, i > 1\rangle$. Hu and Kriz ([7],[8]) have realized this as a map of honest C_2-equivariant spectra

$$BP\mathbb{R} \to K\mathbb{R}$$

where $K\mathbb{R}$ is Atiyah's *real* K-theory ([1]) localized at the prime 2. They have further generalized this to a map of equivariant spectra

$$BP\mathbb{R} \to E\mathbb{R}(n)$$

which is an equivariant refinement of the orientation $BP \to E(n)$ with kernel $\langle v_i, i > n\rangle$. Here $E(n)$ is the $2(2^n - 1)$-periodic Johnson–Wilson theory. It has coefficients $\mathbb{Z}_{(2)}[v_1, \dots, v_n^{\pm 1}]$ with $|v_i| = 2(2^i - 1)$.

The underlying nonequivariant spectrum of $E\mathbb{R}(n)$ is $E(n)$ and the homotopy fixed points with respect to the complex conjugation action is denoted by $ER(n)$. This is $2^{n+2}(2^n - 1)$ periodic. The spectrum $ER(1)$ is $KO_{(2)}$.

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Kitchloo and Wilson have done extensive computations with the spectrum $ER(2)$ ([11],[12]).

The spectrum $ER(n)$ is homotopy commutative, but its completion is an E_∞-ring spectrum. In fact Averett ([2]) has shown that after completion the fixed points inclusion map $ER(n) \to E(n)$ is a higher chromatic generalization of the C_2-Galois extension (in the sense of [16]) $KO \to KU$.

More elaborately, let E_n be the n-th Morava E-theory spectrum associated to the Lubin–Tate space of deformations of the Honda formal group Γ_n over $\mathbb{F}_{2^n}$. This Lubin–Tate space is noncanonically isomorphic to $\operatorname{Spf} \pi_0(E_n) = \operatorname{Spf} W(\mathbb{F}_{2^n})[[u_1, \ldots, u_{n-1}]]$ where $W(k)$ denotes the Witt ring of k. Hopkins–Miller theory gives a unique E_∞-ring structure on E_n, with coefficients

$$\pi_* E_n = \pi_0(E_n)[u^{\pm 1}]$$

where $|u| = 2$. The n-th Morava stabilizer group $\mathbb{S}_n = \mathcal{O}^\times_{D_{\frac{1}{n}}, \mathbb{Q}_2}$ (the group of units in the maximal order of the division algebra over $\mathbb{Q}_2$ with Hasse invariant $\frac{1}{n}$) and the Galois group $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ act on the Lubin–Tate space. By Hopkins–Miller theory this lifts to an action of the extended stabilizer group $\mathbb{S}_n \rtimes \mathrm{Gal}$ on the spectrum E_n through E_∞-ring maps, up to contractible choices.

There is a subgroup $H(n) = \mathbb{F}^\times_{2^n} \rtimes \mathrm{Gal}$ of the extended stabilizer and a central element $[-1]_{\Gamma_n} \in \mathbb{S}_n$ corresponding to the formal inverse of the Honda formal group law. Averett shows that the standard equivalence $L_{K(n)} E(n) \simeq E_n^{hH(n)}$ is equivariant with respect to the C_2 action coming from $[-1]_{\Gamma_n}$. In other words there is an equivalence

$$L_{K(n)} ER(n) \simeq E_n^{h(C_2 \times H(n))}. \tag{1}$$

1.2. Main result. In this paper we want to show that the Real Johnson–Wilson theory $ER(2)$ arises from certain modular curves in the same way the spectrum TMF of topological modular forms arises from the moduli stack of smooth elliptic curves [6].

The stabilizer group $\mathbb{S}_2$ has the maximal finite subgroup $\widehat{A}_4 = Q_8 \rtimes C_3$ where C_3 acts on Q_8 by conjugation, and $H(2) = C_3 \rtimes \mathrm{Gal} \subset \widehat{A}_4 \rtimes \mathrm{Gal}$.

Completion of TMF at the chromatic height 2 admits the description (for the primes 2 and 3)

$$L_{K(2)} TMF = \left(\prod_{x \in \mathcal{E}ll^{ss}(\overline{\mathbb{F}}_p)} E_2^{h\,\mathrm{Aut}(x)} \right)^{h\,\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}. \tag{2}$$

Here $\mathcal{E}ll^{ss}$ is the locus of supersingular elliptic curves at the prime p and E_2 is the Hopkins–Miller spectrum associated to the Lubin–Tate space of deformations of the formal group law of x over $\overline{\mathbb{F}}_p$. Since there is a unique isomorphism class of elliptic curves x at these primes and $\mathrm{Aut}(x)$ is the maximal finite subgroup of $\mathbb{S}_2$, the right hand side is the higher real K-theory $EO_2^{h\,\mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)}$.

We want to describe completed $ER(2)$ in terms of modular curves. This raises the question: *Given $p = 2$, does there exist an étale cover of $\mathcal{X}$ of Ell^{ss} so that there exists $x \in \mathcal{X}$ for which* $\mathrm{Aut}(x)$ *is the subgroup $C_2 \times C_3$ of $\mathbb{S}_2$?*

We state the main theorem here.

Theorem 1.1. *For a p-local commutative S-algebra A, let $A(\mu_{\infty,p})$ denote the S-algebra obtained by adjoining all the roots of unity of order prime to p. Then $L_{K(2)}ER(2)(\mu_{\infty,2})$ is an algebra over TMF.*

There is a $K(2)$-local C_2-Galois extension

$$L_{K(2)}TMF_0(3)(\mu_{\infty,2}) \to L_{K(2)}TMF_1(3)(\mu_{\infty,2})$$

which is isomorphic to the extension $L_{K(2)}ER(2)(\mu_{\infty,2}) \to L_{K(2)}E(2)(\mu_{\infty,2})$.

It is worth pointing out that Behrens and Hopkins ([4]) have answered in great detail the following question: Given a *maximal* finite subgroup G of $\mathbb{S}_n$ at the prime p, is the associated higher real K-theory EO_n a summand of the $K(n)$-localization of a TAF spectrum associated to a unitary similitude group of type $(1, n-1)$? This paper is concerned with a similar question where $p = 2$, $n = 2$ and G is not maximal.

It is evident from the calculations of Mahowld and Rezk ([14]) that the spectrum $TMF_1(3)$ is an example of a *generalized* $E(2)$ (see [3]). Presumably there is a notion of a generalized $E\mathbb{R}(n)$ so that $TMF_0(3)$ is an example of a generalized $ER(2)$.

In the next section we reformulate our main theorem (as Theorem 2.3) in algebraic-geometric language and review the various moduli spaces appearing in the formulation. The final section contains the proof of the main theorem. In the rest of the paper the prime p is 2.

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2. Setup

2.1. Stacks from ring spectra. Let BP denote the p-local Brown–Peterson spectrum. We can consider the associated flat Hopf algebroid:

$$V = BP_* \simeq \mathbb{Z}_{(p)}[v_1, \ldots]$$
$$VT = BP_*BP \simeq BP_*[t_1, \ldots].$$

Denote the stack associated to (V, VT) by $\mathcal{M}$. This is an algebraic stack in the *fpqc*-topology ([15]). Such stacks are representable by flat Hopf algebroids. This differs from the standard notion of an algebraic stack in algebraic geoemtry (see [13, 4.1]) since its diagonal is not necessarily of finite type.

The algebraic stack $\mathcal{M}$ is closely related to the moduli stack of one-dimensional commutative formal groups. A formal group over a scheme S is a commutative group object in the category of formal schemes over S that is *fpqc*-locally isomorphic to $(\widehat{\mathbb{A}}^1, 0)$ as schemes. Let's denote this stack by $X_{1,fg}$. Given a ring R, the groupoid $X_{1,fg}(R)$ comprises formal groups G/R over Spec R and their isomophisms.

The stack $X_{1,fg}$ carries a canonical line bundle ω. For every R we can construct the locally free rank one R-module $\omega_{G/R}$ of invariant 1-forms of G over Spec R, and its formation is compatible with base change and therefore defines a line bundle ω over X.

The stack $\mathcal{M} = \mathcal{M}_{(V,VT)}$ is a $\mathbb{G}_m$ torsor over $X_{1,fg} \otimes \mathbb{Z}_{(p)}$. Its points can be described as follows. For a $\mathbb{Z}_{(p)}$-algebra R, the groupoid $\mathcal{M}(R)$ consists of pairs $(G/R, \alpha : \omega_{G/R} \simeq R)$ of a formal group and a trivialization of the ω section, and isomorphisms of G that respect the trivializations.

Definition 2.1. Let X be a p-local homotopy commutative ring spectrum so that BP_*X is a commutative ring. Define $\mathcal{M}_X$ to be the stack associated to the Hopf algebroid

$$V(X) = BP_*X$$
$$VT(X) = VT \otimes_V BP_*X.$$

We now make clear the relation of $\mathcal{M}_X$ with formal groups. The unit $S \to X$ gives a map of stacks $\mathcal{M}_X \to \mathcal{M}$. There is an algebraic stack M_X over $X_{1,fg} \otimes \mathbb{Z}_{(p)}$ along with a line bundle $\omega_X : M_X \to B\mathbb{G}_m$, so that $\mathcal{M}_X$ is a $\mathbb{G}_m$-torsor over M_X. The R-points of $\mathcal{M}_X$ can then be identified with pairs $(P \in M_X(R), \alpha : \omega_X(P/R) \simeq R)$ of objects of M_X over Spec R and trivializations of their ω_X sections.

If X is a homotopy BP-algebra then $\mathcal{M}_X$ is an affine scheme and is Spec X_*. In this case the stack $\mathcal{M}_X$ is defined to be the one associated to the Hopf algebroid

$$V(X) \simeq VT \otimes_V X_*$$
$$VT(X) \simeq VT \otimes_V VT \otimes_V X_*$$

which is a representation for Spec $X_* \times_{\mathcal{M}} \mathcal{M}$.

Lemma 2.1. *If X and Y are p-local homotopy commutative ring spectra so that BP_*X and BP_*Y are commutative rings and $\mathcal{M}_Y \to \mathcal{M}$ is flat, then the stack associated to the smash product $X \wedge Y$ can be identified with the pullback.*

$$\mathcal{M}_{X\wedge Y} \simeq \mathcal{M}_X \times_{\mathcal{M}} \mathcal{M}_Y$$

Proof. Suppose E is a BP-algebra and the map $\mathcal{M}_E \to \mathcal{M}$ is flat. This means that E is a Landweber exact cohomology theory. Given an arbitrary X, the pullback Spec $E_* \times_{\mathcal{M}} \mathcal{M}_X$ is then the stack associated to the Hopf algebroid

$$E_* \otimes_V VT \otimes_V V(X) \simeq V(E \wedge X)$$
$$E_* \otimes_V VT \otimes_V VT \otimes_V V(X) \simeq VT(E \wedge X).$$

In other words the pullback is $\mathcal{M}_{E\wedge X}$.

Suppose $\mathcal{M}_Y$ is flat over $\mathcal{M}$, then $BP\wedge Y$ and $BP\wedge BP\wedge Y$ are Landweber exact. Therefore there are equivalences of stacks:

$$\text{Spec } V(Y)\times_{\mathcal{M}}\mathcal{M}_X \simeq \mathcal{M}_{BP\wedge Y\wedge X} \simeq \text{Spec } V(X\wedge Y),$$
$$\text{Spec } VT(Y)\times_{\mathcal{M}}\mathcal{M}_X \simeq \mathcal{M}_{BP\wedge BP\wedge Y\wedge X} \simeq \text{Spec } VT(X\wedge Y). \qquad \square$$

The notion of height of a formal group gives a filtration of the moduli stack $X_{1,fg}$ and this canonically lifts to a filtration of the $\mathbb{G}_m$-torsor $\mathcal{M}$. One can give an explicit construction of the filtration. Let $I_n = (p, v_1, \ldots, v_{n-1})$ denote the invariant prime ideals of V. The associated substacks correspond to formal groups of height at least n.

$$\mathcal{M}^{\geq n} = \text{Spec}\,(V/I_n, V/I_n \otimes_V VT \otimes_V V/I_n).$$

There is a filtration on $\mathcal{M}$ by closed substacks

$$\mathcal{M} = \mathcal{M}^{\geq 0} \supseteq \mathcal{M}^{\geq 1} \supseteq \ldots \supseteq \mathcal{M}^{\geq \infty}.$$

Let $\mathcal{U}^n = \mathcal{M} - \mathcal{M}^{\geq n+1}$ $(-1 \leq n \leq \infty)$ be the open substack of $\mathcal{M}$ complementary to $\mathcal{M}^{\geq n+1}$. There is an ascending chain on open immersions

$$\emptyset = \mathcal{U}^{-1} \subseteq \mathcal{U}^0 \subseteq \mathcal{U}^1 \subseteq \ldots \subseteq \mathcal{U}^\infty \subseteq \mathcal{M}.$$

Since for $0 \leq n < \infty$, I_n is finitely generated, the open immersion $\mathcal{U}^n \subseteq \mathcal{M}$ is quasi-compact and $\mathcal{U}^n$ is an algebraic stack. The R-points of $\mathcal{U}^n$ consist of formal groups of height at most n over Spec R along with trivializations of their corresponding cotangent bundles as before. The following corollary to [15, Theorem 26] gives an explicit atlas for $\mathcal{U}^n$.

Proposition 2.1. *Let $E(n)$ be the n-th Johnson–Wilson spectrum with $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n^{\pm 1}]$. Let*

$$(V_n, VT_n) := (E(n)_*, E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*)$$

be the Hopf algebroid induced from (BP_*, BP_*BP) by the Landweber exact map $BP_* \to E(n)_*$. Then the Hopf algebroid (V_n, VT_n) is flat and its associated algebraic stack is $\mathcal{U}^n$.

By Lemma 2.1 the pullback $\mathcal{U}^n \times_{\mathcal{M}} \mathcal{M}_X$ is the stack associated with the Hopf algebroid

$$(V_n \otimes_V V(X), VT_n \otimes_V V(X)).$$

Let $\mathcal{M}^n = \mathcal{M}^{\geq n}[v_n^{-1}]$ denote the height n-layer. The stack $\mathcal{M}^n$ is contained inside $\mathcal{U}^n$ as a closed substack. Define the formal neighborhood of $\mathcal{M}^n$ by taking the completion of $\mathcal{U}^n$ at $\mathcal{M}^n$.

Definition 2.2. $\widehat{\mathcal{M}}^n = (\mathcal{U}^n)^\wedge_{I_n}$

This is the stack associated to the Hopf algebroid

$$\left(V_n^\wedge, V_n^\wedge \otimes_{V_n} VT_n \otimes_{V_n} V_n^\wedge\right)$$

where $V_n^\wedge = (V_n)^\wedge_{I_n}$.

Let $\eta : \mathrm{Spec}\ \mathbb{F}_{p^n}[u^{\pm 1}] \to \mathcal{M}^n$ be a lift of the the Honda formal group $\Gamma_n : \mathrm{Spec}\ \mathbb{F}_{p^n} \to X_{1,fg}$ of height n. Then η is a presentation for $\mathcal{M}^n$ and a pro-étale $\mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$-torsor. Let $\mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n})$ be the Lubin–Tate space of deformations of Γ_n. Let $\mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n})[u^{\pm 1}] := \mathrm{Def}(\Gamma, \mathbb{F}_{p^n}) \times \mathrm{Spec}\ \mathbb{Z}[u^{\pm 1}]$. There is a map

$$\mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n})[u^{\pm 1}] \to \widehat{\mathcal{M}}^n$$

which is a presentation and a pro-étale $\mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$-torsor. Furthermore there is a pullback of stacks,

$$\begin{array}{ccc} \mathrm{Spec}\ \mathbb{F}_{p^n}[u^{\pm 1}] & \longrightarrow & \mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n})[u^{\pm 1}] \\ \downarrow & & \downarrow \\ \mathcal{M}^n & \longrightarrow & \widehat{\mathcal{M}}^n \end{array}$$

Let E_n denote the Hopkins–Miller spectrum associated to $\mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n})$. There is an isomorphism $\mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n})[u^{\pm 1}] \simeq \mathrm{Spf}\ \pi_* E_n$. The diagonal is

$$\begin{aligned} &\mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n}) \times_{\widehat{\mathcal{M}}^n} \mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n})[u^{\pm 1}] \\ &\simeq \mathrm{Def}(\Gamma_n, \mathbb{F}_{p^n})[u^{\pm 1}] \times (\mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)) \\ &\simeq \mathrm{Spf}\ \pi_* L_{K(n)}(E_n \wedge E_n). \end{aligned}$$

Therefore $\widehat{\mathcal{M}}^n$ can also be represented by the Hopf algebroid

$$(E_{n*}, \mathrm{map}\,(\mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p), E_{n*}))$$

where map means the set of continous maps.

Definition 2.3. $\widehat{\mathcal{M}}^n \times_{\mathcal{M}} \mathcal{M}_X = (\mathcal{U}^n \times_{\mathcal{M}} \mathcal{M}_X)^{\wedge}_{I_n}$

By Lemma 2.1 $\widehat{\mathcal{M}}^n \times_{\mathcal{M}} \mathcal{M}_X$ is the stack associated to the Hopf algebroid

$$(E_{n*} \otimes_V V(X), \mathrm{map}\,(\mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p), E_{n*} \otimes_V V(X)).$$

The remainder of this paper is concerned with the structure of

$$\widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{ER(2)}.$$

2.2. Elliptic curves. A Weierstrass curve over R is the closure in $\mathbb{P}^2_R$ of the affine curve

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \tag{3}$$

over R. The curve is smooth if and only if $\Delta = \Delta(a_1, \ldots, a_6)$ is invertible in R. A strict isomorphism of Weierstrass curves is given by the change of coordinates

$$x' = x + r,\ \ y' = y + sx + t.$$

The Weierstrass curves along with their coordinate changes form an algebraic stack $\mathcal{M}_{(A,\Gamma)}$ determined by the Hopf algebroid (A, Γ) where

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6],\ \ \Gamma = A[s, r, t].$$

The Hopf algebroid structure maps are implicit in the definitions (see [6, section 3]).

We now explain how this stack $\mathcal{M}_{(A,\Gamma)}$ is associated to elliptic curves. Let $\mathcal{E}ll$ denote the moduli stack of elliptic curves over Spec $\mathbb{Z}$. A morphism $\mathrm{Spec}(R) \to \mathcal{E}ll$ classifies an elliptic curve $C \to R$, which is a smooth proper morphism whose geometric fibers are elliptic curves. Let $\overline{\mathcal{E}ll}$ denote the compactified moduli stack classifying generalized elliptic curves. There exists a line bundle $\omega \to \overline{\mathcal{E}ll}$ associated to the cotangent space at the identity section of a generalized elliptic curve. Given a smooth elliptic curve $C : \mathrm{Spec}\ R \to \mathcal{E}ll$, the set of sections $\Gamma(\mathrm{Spec}\ R, \omega(C))$ is the set of invariant 1-forms on C.

Let Ell be the $\mathbb{G}_m$-torsor over $\mathcal{E}ll$ whose R-points are given by pairs $(C/R, \alpha : \omega(C) \simeq R)$. Here C/R is an elliptic curve over R and α is a choice of trivialization of the R-sections of ω.

Any generalized elliptic curve $C \to S$ admits a presentation in the Weierstrass normal form locally over S in the flat topology. The identity element of the elliptic curve is identified with the unique point at infinity of the Weierstrass curve. This gives a map of stacks $\overline{\mathsf{Ell}} \to \mathcal{M}_{(A,\Gamma)}$ which is an equivalence on the substack of smooth elliptic curves, $\mathsf{Ell} \simeq \mathcal{M}_{(A[\Delta^{-1}],\Gamma[\Delta^{-1}])}$.

There is a natural map $\mathcal{E}ll \to X_{1,fg}$ classifying the formal group associated to an elliptic curve. This map lifts canonically to a map of the $\mathbb{G}_m$-torsors.

$$\mathsf{Ell} \to \mathcal{M}$$

Consider the substack $\overline{\mathsf{Ell}}_p \to \overline{\mathsf{Ell}}$ which is the p-completion of $\overline{\mathsf{Ell}}$. Note that $\overline{\mathsf{Ell}}_p$ is a *formal* Deligne–Mumford stack. For any p-complete ring R the map $\mathrm{Spf}(R) \to \overline{\mathsf{Ell}}$ classifies an ind-system $C_m/\,\mathrm{Spec}(R/p^n)$ of compatible generalized elliptic curves.

Define $(\overline{\mathsf{Ell}})_{\mathbb{F}_p} = \overline{\mathsf{Ell}} \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{F}_p)$. Let $(\mathsf{Ell}^{ord})_{\mathbb{F}_p} \subset (\overline{\mathsf{Ell}})_{\mathbb{F}_p}$ denote the locus of ordinary generalized elliptic curves in characteristic p, and let

$$(\mathsf{Ell}^{ss})_{\mathbb{F}_p} = (\overline{\mathsf{Ell}})_{\mathbb{F}_p} - (\mathsf{Ell}^{ord})_{\mathbb{F}_p}$$

denote the locus of supersingular elliptic curves in characteristic p.

Consider the substack

$$\mathsf{Ell}_p^{ss} \subset \overline{\mathsf{Ell}}_p$$

where Ell_p^{ss} is the completion of $\overline{\mathsf{Ell}}$ at $(\mathsf{Ell}^{ss})_{\mathbb{F}_p}$.

Define the Hopf algebroid,

$$A' = A[\Delta^{-1}]^{\wedge}_{(p,a_1)},\ \Gamma' = A' \otimes_A \Gamma \otimes_A A'.$$

Then,

$$\mathsf{Ell}_p^{ss} = \mathcal{M}_{(A',\Gamma')}.$$

The following is a special case of the Serre–Tate theorem for abelian schemes [9, Theorem 1.2.1].

Theorem 2.1 (Serre, Tate)**.** *Let R be a Noetherian ring with p nilpotent and I a nilpotent ideal in R, $R_0 = R/I$. Let $\mathcal{E}ll_p^{ss}(R)$ denote the category of* *supersingular elliptic curves in characteristic* p *over* R *and* $D(R, R_0)$ *denote the category of triples*

$$(C_0, G, \epsilon)$$

consisting of a supersingular elliptic curve C_0 *over* R_0*, a formal group* G *over* R *and an isomorphism of formal groups over* R_0*,* $\epsilon : G_0 \simeq C_0^\wedge$*, where* G_0 *is reduction modulo* I *of* G*. Then the functor*

$$\mathcal{E}ll_p^{ss}(R) \to D(R, R_0)$$
$$C \mapsto (C_0, C^\wedge, \textit{natural } \epsilon)$$

is an equivalence of categories.

The following is implied by the Serre–Tate theorem.

Proposition 2.2. *There is an equivalence*

$$\mathsf{Ell}_p^{ss} \simeq \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \overline{\mathsf{Ell}}$$

of formal Deligne–Mumford stacks.

Let E be a supersingular elliptic curve over a field k of characteristic p classifying a point $\eta : \operatorname{Spec} k \to (\mathsf{Ell})_{\mathbb{F}_p}$. Serre–Tate theory gives an isomophism of the formal neighborhood of η with the universal deformation space for the formal group $\widehat{E}$. Therefore there is an equivalence of deformation spaces

$$\operatorname{Def}(E, k) \simeq \operatorname{Def}(\widehat{E}, k).$$

2.3. Level structures. In this section we review the modular curves. Let S be a scheme over $\mathbb{Z}[1/N]$ and C a smooth elliptic curve over S. Let $C[N]$ denote the N-torsion points of C. The group scheme $C[N]$ is étale locally over S isomorphic to the discrete group scheme $(\mathbb{Z}/N\mathbb{Z})^2$ over C. Let $\mathcal{E}ll(N)$ denote the moduli stack of pairs (C, ϕ) where C is a smooth elliptic curve and ϕ is a *full level-n structure*, a choice of isomorphism

$$\phi : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\cong} C[N] \ .$$

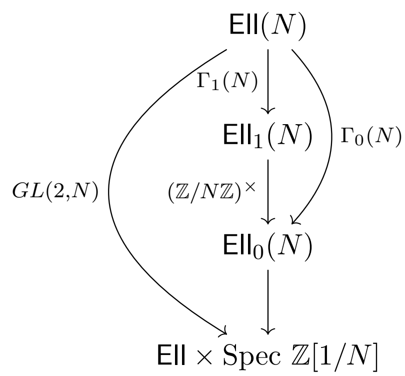
Equivalently, the points of $\mathcal{E}ll(N)$ are triples (C, P, Q) where P and Q are a pair of sections $S \to C[N]$ that are, locally over S, linearly independent.

Let $\mathcal{E}ll_1(N)$ denote the moduli stack of pairs (C, P) where P is a primitive N-torsion point over C. Finally, let $\mathcal{E}ll_0(N)$ denote the moduli stack of pairs (C, H) where H is a choice of a subgroup scheme $H \subset C[N]$ locally isomorphic to $\mathbb{Z}/N\mathbb{Z}$.

In this paper we'll work with stacks $\mathsf{Ell}(N)$, $\mathsf{Ell}_1(N)$ and $\mathsf{Ell}_0(N)$ which are $\mathbb{G}_m$-torsors over the modular curves $\mathcal{E}ll(N)$, $\mathcal{E}ll_1(N)$ and $\mathcal{E}ll_0(N)$. The R-points of $\mathsf{Ell}(N)$ consists of triples $(C, \eta, (P, Q))$, where C is an elliptic curve over R, η is a choice of a nowhere vanishing invariant 1-form of C and (P, Q) is a full level-N structure described as before.

Theorem 2.2 (Deligne, Rapoport ([10]))**.** *The moduli stack $\mathcal{E}ll(N)$ is a smooth affine scheme over* Spec $\mathbb{Z}[1/N]$ *for $N \geq 3$. For $N \geq 4$ the moduli stack $\mathcal{E}ll_1(N)$ is a smooth affine scheme over* Spec $\mathbb{Z}[1/N]$*.*

The maps forgetting level structures induce a diagram



where all the arrows are finite étale and the labeled ones are Galois.

The Galois groups are defined as follows:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(2,N) \right\},$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(2,N) \right\}.$$

Let $\mathsf{Ell}(N)_p$, $\mathsf{Ell}_1(N)_p$ and $\mathsf{Ell}_0(N)_p$ denote the completions at p. Let $\mathsf{Ell}(N)_p^{ss}$ denote the pullback

$$\begin{array}{ccc} \mathsf{Ell}(N)_p^{ss} & \longrightarrow & \mathsf{Ell}(N)_p \\ \downarrow & & \downarrow \\ \mathsf{Ell}_p^{ss} & \longrightarrow & \mathsf{Ell}_p. \end{array}$$

Since $\mathsf{Ell}(N)_p$ is formal affine (assuming $N \geq 3$) and the right vertical arrow is an étale $GL(2,N)$-torsor, Serre–Tate theory implies there is an equivalence

$$\mathsf{Ell}(N)_p^{ss} \simeq \coprod_i \operatorname{Spf} W(k_i)[[u_1]][u^{\pm 1}]$$

for a finite set of fields k_i (depending on N).

2.4. Restatement of the main theorem. Given a formal Deligne–Mumford stack $\mathcal{S}$ over $\mathbb{Z}_p$, define its maximal unramified cover $\mathcal{S}^{nr}$ to be the pullback $\mathcal{S} \times_{\mathbb{Z}_p} \operatorname{Spf} W(\overline{\mathbb{F}}_p)$. The map $\mathcal{S}^{nr} \to \mathcal{S}$ is an pro-étale cover with Galois group $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}}$. Our main Theorem 1.1 can be restated in the following way.

Theorem 2.3. *There is a map of stacks over* $\widehat{\mathcal{M}}^2$,

$$\widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{ER(2)}{}^{nr} \to \mathsf{Ell}_2^{ss}.$$

The map over $\widehat{\mathcal{M}}^2$ *induced by forgetting level structure*

$$(\mathsf{Ell}_1(3)_2^{ss})^{nr} \to (\mathsf{Ell}_0(3)_2^{ss})^{nr}$$

is equivalent to the map

$$\widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)}{}^{nr} \to \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{ER(2)}{}^{nr}$$

over $\widehat{\mathcal{M}}^2$ *induced by the inclusion of fixed points* $ER(2) \to E(2)$.

3. Real Johnson–Wilson theory from modular curves

3.1. Level 3-structures at the prime 2. Consider the supersingular elliptic curve

$$C : x^3 + y^2 + y = 0 \in \mathbb{P}^2_{\mathbb{F}_4}. \tag{4}$$

The automorphism group $G_{24} = \mathrm{Aut}_{\mathbb{F}_4}(C)$ is the group of units in the maximal order of a rational quarternion algebra $\mathbb{Q}\{i, j, k\}$. It's isomorphic to the binary tetrahedral group $\widehat{A}_4 = Q_8 \rtimes C_3$ of order 24, where C_3 acts on Q_8 by conjugation. It contains the quarternion group $Q_8 = \{\pm 1, \pm i, \pm j \pm k\}$ and 16 other elements $(\pm 1 \pm i \pm j \pm k)/2$.

Let $C^\wedge$ be the completion of C at the identity section. $C^\wedge$ is a formal group of height 2 over $\mathbb{F}_4$. The automorphism group $\mathrm{Aut}(C^\wedge) = \mathcal{O}^\times_{D_{1/2},\mathbb{Q}_2}$ is the group of units in the maximal order of the 2-adic quaternion algebra

$$D_{\frac{1}{2},\mathbb{Q}_2} = \mathbb{Q}_2\{i, j, k\}.$$

Abstractly this is the completion of the Hurwitz lattice

$$\mathbb{Z}(\pm 1, \pm i, \pm j, \pm k) \coprod (\pm 1 \pm i \pm j \pm k)/2$$

at the ideal (2). Notice that G_{24} is the maximal finite subgroup.

Proposition 3.1. *The map* $\mathrm{Spec}\, \mathbb{F}_4 \to (\mathcal{E}ll^{ss})_{\mathbb{F}_2}$ *classifying* C *is a presentation and an étale* $G_{24} \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$*-torsor.*

It follows that the map $\mathrm{Def}(C, \mathbb{F}_4) \to \mathcal{E}ll_2^{ss}$ classifying the universal deformation of C is an étale $G_{24} \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$-torsor. By Serre–Tate theory there is an isomorphism of deformation spaces

$$\mathrm{Def}(C, \mathbb{F}_4) \simeq \mathrm{Spf}\, W(\mathbb{F}_4)[[a_1]].$$

The curve C lifts to $W(\mathbb{F}_4)[[a_1]]$ as $\widetilde{C} : y^2 + a_1 xy + y = x^3$, which is the universal deformation curve over $\mathrm{Def}(C, \mathbb{F}_4)$. $\widetilde{C}$ lifts further to

$$y^2 + a_1 uxy + u^3 y = x^3 \tag{5}$$

over $\mathrm{Spf}\, W(F_4)[[a_1]][u^{\pm 1}]$. We shall call this $\widetilde{C}$ from now.

Proposition 3.2 ([6, Theorem 3.1])**.** *The map*

$$\operatorname{Spf} W(\mathbb{F}_4)[[a_1]][u^{\pm 1}] \to \mathsf{Ell}_2^{ss} \simeq \mathcal{M}_{(A',\Gamma')}$$

classifying the curve

$$\widetilde{C} : y^2 + a_1 uxy + u^3 y = x^3$$

is a presentation and an étale $G_{24} \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$*-torsor.*

This is a restatement of

$$L_{K(2)} TMF = EO_2.$$

The level 3-structures on elliptic curves and their associated moduli stacks are related by finite étale morphisms

$$\mathsf{Ell}(3) \xrightarrow{6} \mathsf{Ell}_1(3) \xrightarrow{2} \mathsf{Ell}_0(3) \xrightarrow{\quad\quad 4 \quad\quad} \mathsf{Ell} \times \operatorname{Spec} \mathbb{Z}[1/3]$$

of degrees 6, 2 and 4. The related modular groups are as follows:

$$\Gamma_1(3) = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(2,3) \right\} \simeq C_3 \rtimes C_2,$$
$$\Gamma_0(3) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(2,3) \right\} \simeq C_6 \rtimes C_2.$$

The point $(0,0)$ on the Weierstrass normal form of a smooth elliptic curve is a point of order 3 if it is of the form

$$C(a_1, a_3) : y^2 + a_1 xy + a_3 y = x^3$$

where a_3 is a unit. The curve $C(a_1, a_3)$ comes with the invariant 1-form

$$\eta = \frac{dx}{2y + a_1 x + a_3} = \frac{dy}{3x^2 - a_1 y}.$$

Furthermore, if (C', P, η') is a smooth elliptic curve with a point P of order 3 and an invariant 1-form η' then there exists a *unique* isomorphism of the triple (see [14, Prop. 3.2])

$$(C(a_1, a_3), (0,0), \eta) \simeq (C', P, \eta').$$

This gives us the following equivalence of stacks:

$$\mathsf{Ell}_1(3) = \operatorname{Spec} \mathbb{Z}[1/3][a_1, a_3^{\pm 1}, \Delta^{-1}],$$
$$\mathsf{Ell}_1(3)_2^{ss} = \operatorname{Spf} \mathbb{Z}_2[[a_1]][a_3^{\pm 1}],$$

where $\Delta = a_3^3(a_1^3 - 27a_3)$.

Proposition 3.3. *The map*

$$\operatorname{Spf} W(\mathbb{F}_4)[[a_1]][u^{\pm 1}] \to \mathsf{Ell}_1(3)_2^{ss}$$

classifying the point $(\widetilde{C}, (0,0), \frac{dx}{2y + a_1 ux + u^3})$ *is a presentation and an étale* $C_3 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$*-torsor. The embedding* $C_3 \subset \mathbb{S}_2$ *is induced by lifting the* 2*-adic Teichmüller character* $\mathbb{F}_4^\times \to W(\mathbb{F}_4)^\times$*.*

There is a C_2-étale map forgetting level structure,

$$\mathsf{Ell}_1(3) \to \mathsf{Ell}_0(3),$$
$$(C, P, \eta) \mapsto (C, \langle P \rangle, \eta).$$

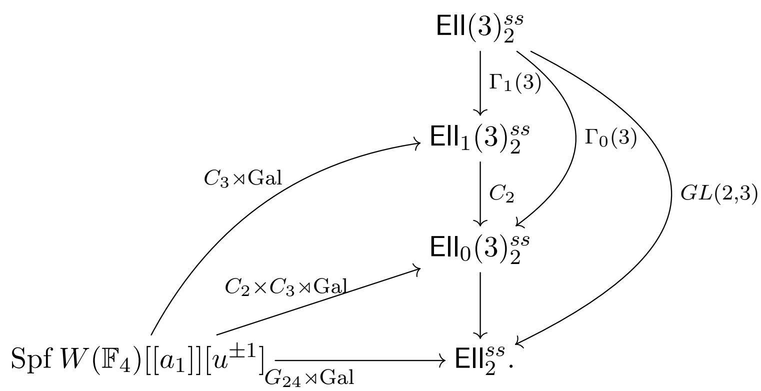
The group C_2 acts on $\mathsf{Ell}_1(3)$ by $(C, P, \eta) \mapsto (C, -P, \eta)$.

Proposition 3.4. *The map*

$$\operatorname{Spf} W(\mathbb{F}_4)[[a_1]][u^{\pm 1}] \to \mathsf{Ell}_0(3)_2^{ss}$$

classifying the point $(\widetilde{C}, \langle (0,0) \rangle, \frac{dx}{2y + a_1 u x + u^3})$ *is a presentation and an étale* $C_2 \times C_3 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$*-torsor. The group* C_2 *is the normal subgroup of* $Q_8 \rtimes C_3$ *generated by the formal inverse* $[-1]_{C^\wedge}$.

The following diagram shows how the various modular curves at the supersingular locus in characterstic 2 are related. All the arrows are finite étale and the labeled ones are Galois. Notation: $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$.



As a consequence there is a diagram of $K(2)$-local elliptic spectra.

$$\begin{array}{ccc} L_{K(2)}TMF_1(3) & \xrightarrow{\simeq} & E_2^{h(C_3 \rtimes \mathrm{Gal})} \\ \uparrow & & \uparrow \\ L_{K(2)}TMF_0(3) & \xrightarrow{\simeq} & E_2^{h(C_2 \times C_3 \rtimes \mathrm{Gal})} \\ \uparrow & & \uparrow \\ L_{K(2)}TMF & \xrightarrow{\simeq} & E_2^{h(\hat{A}_4 \rtimes \mathrm{Gal})}. \end{array}$$

Here E_2 is the Hopkins–Miller Morava E-theory spectrum associated to the deformation space $\mathrm{Def}(C, \mathbb{F}_4)$.

3.2. The structure of $\mathcal{M}_{E(2)}$ near the height 2 point. Let the point $f : \operatorname{Spec} \mathbb{F}_4 \to \mathcal{U}^2$ classify the formal group $C^\wedge$ associated to the supersingular elliptic curve $C : y^2 + y + x^3 = 0$. Let $\widetilde{C}$ denote the universal deformation of the curve C over $\operatorname{Spf} W(\mathbb{F}_4)[[a_1]][u^{\pm 1}]$ as in equation (5).

Since $\mathcal{U}^2$ denotes the moduli stack of formal groups of height at most 2 we know from Prop. 2.1 there is a faithfully flat presentation

$$\operatorname{Spec} \mathbb{Z}_{(2)}[v_1, v_2^{\pm 1}] \to \mathcal{U}^2.$$

As a consequence there exists an extension K of $\mathbb{F}_4$ such that

$$f : \operatorname{Spf} W(K)[[a_1]][u^{\pm 1}] \to \operatorname{Spf} W(\mathbb{F}_4)[[a_1]][u^{\pm 1}]$$

is an fpqc cover and the pullback $f^*(\widetilde{C}^\wedge)$ is isomorphic to a 2-typical formal group law $\psi : \operatorname{Spf} W(K)[[a_1]][u^{\pm 1}] \to \operatorname{Spec} \mathbb{Z}_{(2)}[v_1, v_2^{\pm 1}]$.

$$\alpha : f^*(\widetilde{C}^\wedge) \simeq \psi.$$

Consider the map

$$\operatorname{Spf} W(K)[[a_1]][u^{\pm 1}] \xrightarrow{(f^*(\widetilde{C}^\wedge), \alpha, \psi)} \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)}$$

classifying the formal group $f^*(\widetilde{C}^\wedge)$, ψ and the isomorphism α. This map is a surjection and an étale presentation.

The Teichmüller character map $\mathbb{F}_4^\times \to W(\mathbb{F}_4)^\times$ lifts to give an embedding $\mathbb{F}_4^\times \subset \operatorname{Aut}(f^*(C^\wedge))$. For a $\omega \in \mathbb{F}_4^\times$ the linear formal power series $g(t) = \omega t$ gives an element of $\mathbb{S}_2$. Since the action of $\operatorname{Aut}(f^*(C^\wedge))$ extends to an action on the deformation space $\operatorname{Spf} W(K)[[a_1]][u^{\pm 1}]$, we obtain an action of g on the ring $W(K)[[a_1]][u^{\pm 1}]$ by

$$g(u) = \omega u,$$
$$g(ua_1) = \omega^2 u a_1,$$

which leaves v_1 and v_2 invariant. Moreover these are the only elements of $\mathbb{S}_2$ which act invariantly on v_1 and v_2. Since

$$\widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \operatorname{Spec} \mathbb{Z}_{(2)}[v_1, v_2^{\pm 1}] \simeq \operatorname{Spf} \mathbb{Z}_2[[v_1]][v_2^{\pm 1}]$$

we can identify the pullback in the following diagram of stacks:

$$\begin{array}{ccc} \operatorname{Spf} W(K)[[a_1]][u^{\pm 1}] \times (C_3 \rtimes \operatorname{Gal}(K, \mathbb{F}_2)) & \longrightarrow & \operatorname{Spf} W(K)[[a_1]][u^{\pm 1}] \\ \downarrow & & \downarrow \\ \operatorname{Spf} W(K)[[a_1]][u^{\pm 1}] & \longrightarrow & \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)}. \end{array}$$

Therefore, the map $\operatorname{Spf} W(K)[[a_1]][u^{\pm 1}] \to \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)}$ is an étale $C_3 \rtimes \operatorname{Gal}(K, \mathbb{F}_2)$-torsor. The conjugation action of $\operatorname{Gal}(K/\mathbb{F}_2)$ on C_3 factors through the quotient $\operatorname{Gal}(K/\mathbb{F}_2) \to \operatorname{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ since every automorphism of $\widetilde{C}^\wedge$ is already defined over $\mathbb{F}_4$. In terms of $K(2)$-localization,

$$L_{K(2)} E(2) = E_2^{hH}$$

where $H = \mathbb{F}_4^\times \rtimes \operatorname{Gal}(K/\mathbb{F}_2)$ and E_2 is the Hopkins–Miller spectrum associated to $\operatorname{Def}(C^\wedge, K)$.

More generally, we can use the maximal unramified extension E_2^{nr}, the Hopkins–Miller spectrum associated to the deformation space $\mathrm{Def}(C,\overline{\mathbb{F}}_2)$. The extended Morava stabilizer group in this case is

$$\mathbb{S}_2 \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2) = \mathbb{S}_2 \rtimes \hat{\mathbb{Z}}.$$

Proposition 3.5. *There is a map of stacks* $\mathrm{Def}(C,\overline{\mathbb{F}}_2) \to \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)}$ *which is a surjection and an étale* $C_3 \rtimes \hat{\mathbb{Z}}$*-torsor.*

The maximal unramified cover of the deformation space $\mathrm{Def}(C,\mathbb{F}_4)$ can be identified with $\mathrm{Def}(C,\overline{\mathbb{F}}_2)$. There is a natural map of stacks (we use the notation $/\!\!/$ for quotient stacks)

$$\mathrm{Def}(C,\overline{\mathbb{F}}_2) \to \mathrm{Def}(C,\mathbb{F}_4)/\!\!/\,\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2), \tag{6}$$

which realizes as a $K(n)$-local $\hat{\mathbb{Z}}$-Galois extension of S-algebras

$$E_2^{\mathrm{Gal}} \to E_2^{nr}.$$

The $\hat{\mathbb{Z}}$-extension is obtained by adjoining all the roots of unity of order prime to p to the p-complete spectrum E_2^{Gal} and p-completing ([16, 5.4.6]).

The map (6) factors through the quotient

$$\mathrm{Def}(C,\overline{\mathbb{F}}_2)/\!\!/C_3 \to \mathrm{Def}(C,\mathbb{F}_4)/\!\!/\widehat{A}_4 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

In terms of $K(2)$-localization there is a map

$$L_{K(2)}TMF \simeq (E_2)^{\widehat{A}_4 \rtimes \mathrm{Gal}} \to (E_2^{nr})^{hC_3} \simeq L_{K(2)}E(2)(\mu_{\infty,2}). \tag{7}$$

Combining Proposition 3.3 and Proposition 3.5 proves the following.

Proposition 3.6. *There are equivalences of stacks over* $\widehat{\mathcal{M}}^2$*.*

$$\mathrm{Def}(C,\overline{\mathbb{F}}_2)/\!\!/C_3 \simeq \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)}{}^{nr} \simeq (\mathsf{Ell}_1(3)_2^{ss})^{nr}.$$

3.3. The structure of $\mathcal{M}_{ER(2)}$ near the height 2 point.

Proposition 3.7. *The map of stacks*

$$\widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)} \to \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{ER(2)}$$

is a surjection and an étale C_2*-torsor.*

Proof. Combining Averett's formula (1) and [16, Theorem 5.4.4] we see that the map of ring spectra $L_{K(2)}ER(2) \to L_{K(2)}E(2)$ is a $K(2)$-local C_2-Galois extension of 2-complete commutative S-algebras. □

Given the description of $\widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)}$ as the quotient stack

$$\mathrm{Def}(C,\overline{\mathbb{F}}_2)/\!\!/C_3 \rtimes \hat{\mathbb{Z}},$$

the action of C_2 must come from the inverse map $[-1]_C$ on the curve C.

The induced action on the deformation space is as follows:

$$\begin{gathered}[-1]_C : W(\mathbb{F}_4)[[a_1]][u^{\pm 1}] \longrightarrow W(\mathbb{F}_4)[[a_1]][u^{\pm 1}] \\ [-1]_C(u) = -u, \\ [-1]_C(ua_1) = ua_1.\end{gathered}$$

Proposition 3.8. *There is a map of stacks* $\mathrm{Def}(C, \overline{\mathbb{F}}_2) \to \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{ER(2)}$ *which is an étale surjection and a* $C_2 \times C_3 \rtimes \hat{\mathbb{Z}}$*-torsor. The* C_2*-action on* $\mathrm{Def}(C, \overline{\mathbb{F}}_2)$ *comes from the inverse* $[-1]_C$*.*

This produces a map

$$\mathrm{Def}(C, \overline{\mathbb{F}}_2) /\!\!/ C_2 \times C_3 \to \mathrm{Def}(C, \mathbb{F}_4) /\!\!/ \widehat{A}_4 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

In terms of $K(2)$-localization the map of spectra (7) factors through the real part:

(8)

$$L_{K(2)}TMF \to (E_2^{nr})^{h(C_2 \times C_3)} \simeq L_{K(2)}ER(2)(\mu_{\infty,2}) \to L_{K(2)}E(2)(\mu_{\infty,2}).$$

This proves the first part of Theorem 2.3. Combining Proposition 3.4 with Proposition 3.8 proves the following which together with Proposition 3.6 proves the rest of Theorem 2.3.

Proposition 3.9. *There are equivalences of stacks over* $\widehat{\mathcal{M}}^2$*,*

$$\mathrm{Def}(C, \overline{\mathbb{F}}_2) /\!\!/ C_2 \times C_3 \simeq \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{ER(2)}{}^{nr} \simeq (\mathsf{Ell}_0(3)_2^{ss})^{nr}.$$

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