

# An elementary approach to $C^*$ -algebras associated to topological graphs

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ABSTRACT. We develop notions of a representation of a topological graph  $E$  and of a covariant representation of a topological graph  $E$  which do not require the machinery of  $C^*$ -correspondences and Cuntz–Pimsner algebras. We show that the  $C^*$ -algebra generated by a universal representation of  $E$  is isomorphic to the Toeplitz algebra of Katsura’s topological-graph bimodule, and that the  $C^*$ -algebra generated by a universal covariant representation of  $E$  is isomorphic to Katsura’s topological graph  $C^*$ -algebra. We exhibit our results by constructing the isomorphism between the  $C^*$ -algebra of a row-finite directed graph  $E$  with no sources and the  $C^*$ -algebra of the topological graph arising from the shift map acting on the infinite-path space  $E^\infty$ .

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## 1. Introduction

Let  $E$  be a countable directed graph with vertex set  $E^0$ , edge set  $E^1$  and range and source maps  $r, s : E^1 \rightarrow E^0$ . The Toeplitz–Cuntz–Krieger algebra  $\mathcal{TC}^*(E)$  is the universal  $C^*$ -algebra generated by a family of mutually orthogonal projections  $\{p_v : v \in E^0\}$  and a family of partial isometries  $\{s_e : e \in E^1\}$  such that:

$$(1) \quad s_e^* s_e = p_{s(e)} \text{ for every } e \in E^1.$$

Received May 15, 2012; revised May 5, 2014.

2010 *Mathematics Subject Classification.* 46L05.

*Key words and phrases.*  $C^*$ -algebra; graph algebra;  $C^*$ -correspondence; topological graph; Cuntz–Pimsner algebra.

This research was supported by the Australian Research Council.

$$(2) \quad p_v \geq \sum_{e \in F} s_e s_e^* \text{ for every } v \in E^0 \text{ and finite } F \subset r^{-1}(v).$$

The graph algebra  $C^*(E)$  is universal for families as above satisfying the additional relation that  $p_v = \sum_{r(e)=v} s_e s_e^*$  whenever  $r^{-1}(v)$  is nonempty and finite.

These  $C^*$ -algebras have been studied very extensively over the last fifteen years, part of their appeal being precisely that they can be defined, as above, in the space of a paragraph. Combined with key structure theorems like the gauge-invariant uniqueness theorem [1] and the Cuntz–Krieger uniqueness theorem [11], or Fowler and Raeburn’s analogue of Coburn’s Theorem for the Toeplitz algebra of a directed graph [4], the above elementary presentation often makes it easy to verify that a given  $C^*$ -algebra is isomorphic to  $C^*(E)$  or  $\mathcal{T}C^*(E)$  as appropriate.

Some years after the introduction of graph algebras, Katsura introduced topological graphs and their  $C^*$ -algebras [5]. His construction is based on that of [6], which in turn was a modification of Pimsner’s construction of  $C^*$ -algebras associated to  $C^*$ -correspondences in [9]. Roughly speaking, a topological graph is like a directed graph except that  $E^0$  and  $E^1$  are locally compact Hausdorff spaces rather than countable discrete sets, and  $r$  and  $s$  are required to be topologically well-behaved. Building on Fowler and Raeburn’s construction of a  $C^*$ -correspondence from a directed graph given in [4], Katsura associated to each topological graph  $E$  a  $C^*$ -correspondence  $X(E)$ , and defined the  $C^*$ -algebra of the topological graph  $E$  to be the  $C^*$ -algebra  $\mathcal{O}_{X(E)}$  he had associated to the module  $X(E)$  in [6].

The drawback of this approach is that the relations defining the  $C^*$ -algebra are complicated and cannot be stated without first introducing at least the rudiments of  $C^*$ -correspondences, which is quite a bit of overhead — see Section 3. Clearly it would be handy to be able to circumvent all this technical overhead when defining and dealing with the  $C^*$ -algebras of topological graphs. This paper takes the first step in this direction. Our main theorems show that if  $E$  is a topological graph, then the Toeplitz algebra  $\mathcal{T}(E)$  and Katsura’s topological graph algebra  $\mathcal{O}(E)$  of [5] can be described as  $C^*$ -algebras universal for relatively elementary relations involving elements of  $C_0(E^0)$  and  $C_c(E^1)$ . In particular, our presentations can be stated without any reference to  $C^*$ -correspondences. We state our results so that they apply to arbitrary topological graphs, but we also show how our crucial covariance condition simplifies when the range map  $r : E^1 \rightarrow E^0$  is a local homeomorphism. Even this special situation is interesting: it includes, for example, Cantor minimal systems, and all crossed products of abelian  $C^*$ -algebras by  $\mathbb{Z}$ .

To emphasise how little background is needed to present our definition of a representation of a topological graph and our covariance condition, we begin by stating all our key definitions and main theorems in Section 2. In Section 3 we recall Katsura’s construction of a  $C^*$ -correspondence from a topological graph and his definitions of the associated  $C^*$ -algebras. In

Section 4, we prove our main results. We finish in Section 5 by considering the topological graph  $\widehat{E}$  obtained from a row-finite directed graph  $E$  with no sources by taking  $\widehat{E}^0 = \widehat{E}^1 = E^\infty$  and defining the range map to be the identity map, and the source map to be the left-shift map  $\sigma$ . We apply our results to provide a relatively elementary proof that  $\mathcal{O}(\widehat{E})$  is canonically isomorphic to  $C^*(E)$ . This result could be recovered from [2, 3], but working out the details provides a good example of the efficacy of our results.

## 2. Main results

**Definition 2.1** ([5, Definition 2.1]). A quadruple  $E = (E^0, E^1, r, s)$  is called a *topological graph* if  $E^0, E^1$  are locally compact Hausdorff spaces,  $r : E^1 \rightarrow E^0$  is a continuous map, and  $s : E^1 \rightarrow E^0$  is a local homeomorphism.

We think of  $E^0$  as a space of vertices, and we think of each  $e \in E^1$  as an arrow pointing from  $s(e)$  to  $r(e)$ . If  $E^0, E^1$  are both countable and discrete, then  $E$  is a directed graph in the sense of [8, 10].

Given  $x \in C_c(E^1)$  and  $f \in C_0(E^0)$ , we define  $x \cdot f$  and  $f \cdot x$  in  $C_c(E^1)$  by

$$(2.1) \quad (x \cdot f)(e) = x(e)f(s(e)) \quad \text{and} \quad (f \cdot x)(e) = f(r(e))x(e).$$

To define our notion of a representation of a topological graph, we need a couple of preliminary ideas. An *s-section* in a topological graph  $E$  is a subset  $U \subset E^1$  such that  $s|_U$  is a homeomorphism. An *r-section* is defined similarly, and a *bisection* is a set which is both an *s-section* and an *r-section*.

If  $x \in C_c(E^1)$  then  $\text{osupp}(x)$  denotes the precompact open set

$$\{e \in E^1 : x(e) \neq 0\},$$

and  $\text{supp}(x)$  is the closure of  $\text{osupp}(x)$ . If  $x \in C_c(E^1)$  and  $\text{osupp}(x)$  is an *s-section*, we define  $\widehat{x} : E^0 \rightarrow \mathbb{C}$  by

$$(2.2) \quad \widehat{x}(v) := \begin{cases} |x(e)|^2 & \text{if } v = s(e) \text{ for } e \in \text{osupp}(x), \\ 0 & \text{for } v \notin s(\text{osupp}(x)). \end{cases}$$

**Definition 2.2.** Let  $E$  be a topological graph. A *Toeplitz representation* of  $E$  in a  $C^*$ -algebra  $B$  is a pair  $(\psi, \pi)$  where  $\psi : C_c(E^1) \rightarrow B$  is a linear map,  $\pi : C_0(E^0) \rightarrow B$  is a homomorphism, and:

- (1)  $\psi(f \cdot x) = \pi(f)\psi(x)$ , for all  $x \in C_c(E^1)$ ,  $f \in C_0(E^0)$ ;
- (2) for  $x \in C_c(E^1)$  such that  $\text{supp}(x)$  is contained in an open *s-section*,  $\pi(\widehat{x}) = \psi(x)^*\psi(x)$ ;
- (3) for  $x, y \in C_c(E^1)$  such that  $\text{supp}(x)$  and  $\text{supp}(y)$  are contained in disjoint open *s-sections*,  $\psi(x)^*\psi(y) = 0$ .

We say that a Toeplitz representation  $(\psi, \pi)$  of  $E$  in  $B$  is *universal* if for any Toeplitz representation  $(\psi', \pi')$  of  $E$  in  $C$ , there exists a homomorphism  $h : B \rightarrow C$ , such that  $h \circ \psi = \psi'$  and  $h \circ \pi = \pi'$ .

**Remark 2.3.** Suppose that  $x \in C_c(E^1)$  and  $\text{supp}(x)$  is contained in an open  $s$ -section  $U$ . Since  $E^1$  is locally compact, we may cover  $\text{supp}(x)$  with precompact open sets  $\{V_i : i \in I\}$ , and since  $\text{supp}(x)$  is compact, we may pass to a finite subcover  $\{V_i : i \in F\}$ . Then each  $V_i \cap U$  is precompact and open, so  $\bigcup_{i \in F} (V_i \cap U)$  is a precompact open  $s$ -section containing  $\text{supp}(x)$ . Thus we may assume in Conditions (2) and (3) that the open  $s$ -sections containing  $\text{supp}(x)$  and  $\text{supp}(y)$  are precompact.

To state our first main theorem, recall that if  $E$  is a topological graph, then  $\mathcal{T}(E)$  denotes the Toeplitz algebra of Katsura's topological-graph bimodule (see Notation 3.6).

**Theorem 2.4.** *Let  $E$  be a topological graph. Then there is a universal Toeplitz representation  $(i_1, i_0)$  of  $E$  which generates  $\mathcal{T}(E)$ . Moreover the  $C^*$ -algebra generated by the image of any universal Toeplitz representation of  $E$  is isomorphic to  $\mathcal{T}(E)$ .*

**Remark 2.5.** Since the map  $\psi$  occurring in a Toeplitz representation of  $E$  is not a homomorphism, it will not usually be norm-decreasing with respect to the supremum norm. So it is not clear that one can just extend by continuity a linear map  $\psi_0$  defined on a dense subspace of  $C_c(E^1)$ . We show in Proposition 4.12 how to get around this difficulty: the map  $\psi$  is norm-decreasing with respect to the supremum norm when applied to functions supported on  $s$ -sections.

We now describe the covariance condition for a Toeplitz representation of a topological graph. The condition is somewhat technical, but we will indicate how it simplifies under additional hypotheses in Corollary 2.15.

We first need a little notation from [5].

**Definition 2.6** ([5, Definition 2.6]). Let  $E$  be a topological graph. We define:

- (1)  $E_{\text{sce}}^0 = E^0 \setminus \overline{r(E^1)}$ .
- (2)  $E_{\text{fin}}^0 = \{v \in E^0 : \text{there exists a neighbourhood } N \text{ of } v \text{ such that } r^{-1}(N) \text{ is compact}\}$ .
- (3)  $E_{\text{rg}}^0 = E_{\text{fin}}^0 \setminus \overline{E_{\text{sce}}^0}$ .

**Remark 2.7.** The set  $E_{\text{fin}}^0$  is open in  $E^0$ . To see this, fix  $v \in E_{\text{fin}}^0$ . Then there exists a neighbourhood  $U$  of  $v$  such that  $r^{-1}(U)$  is compact and there exists an open neighbourhood  $V$  of  $v$  contained in  $U$ . So  $V \subset E_{\text{fin}}^0$ , whence  $E_{\text{fin}}^0$  is open. It follows that  $E_{\text{sce}}^0, E_{\text{fin}}^0, E_{\text{rg}}^0$  are all open in  $E^0$ . Moreover  $E_{\text{rg}}^0$  is the intersection of  $E_{\text{fin}}^0$  with the interior of  $\overline{r(E^1)}$ . Finally, as proved by Katsura (see [5, Lemma 1.23]), for any compact subset  $K \subset E_{\text{fin}}^0$ , the set  $r^{-1}(K)$  is compact in  $E^1$ .

**Notation 2.8.** Let  $X$  be a locally compact Hausdorff space and  $U$  be an open subset of  $X$ . Then the standard embedding  $\iota_U$  of  $C_0(U)$  as an ideal of

$C_0(X)$  is given by

$$\iota_U(f)(x) := \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

We will usually suppress the map  $\iota_U$  and just identify each of  $C_0(U)$  and  $C_c(U)$  with their images in  $C_0(X)$  under  $\iota_U$ . That is, we think of  $C_0(U)$  as an ideal of  $C_0(X)$ , and we regard  $C_c(U)$  as an (algebraic) ideal of  $C_0(X)$ .

**Remark 2.9.** Let  $X$  be a Hausdorff space, and let  $K \subset X$  be compact. Fix a finite cover  $\mathcal{N}$  of  $K$  by open subsets of  $X$ . By [7, Chapter 5.W], there exists a partition of unity  $\{h_N : N \in \mathcal{N}\}$  on  $K$  subordinate to  $\{N \cap K : N \in \mathcal{N}\}$ , where  $h_N(x) \in [0, 1]$ , for all  $N \in \mathcal{N}$ ,  $x \in K$ . Since  $X$  may not be normal, we cannot necessarily extend this to a partition of unity on  $X$ . Nevertheless, if  $f \in C_c(X)$  with  $\text{supp}(f) \subset K$ , then the functions  $f_N : X \rightarrow \mathbb{C}$  given by

$$(2.3) \quad f_N(x) := \begin{cases} f(x)h_N(x) & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$$

belong to  $C_c(X)$ . Hence each  $\text{osupp}(f_N) \subset N \cap K$ , and  $\sum_{N \in \mathcal{N}} f_N = f$ .

**Definition 2.10.** Let  $E$  be a topological graph, and let  $(\psi, \pi)$  be a Toeplitz representation of  $E$  in a  $C^*$ -algebra  $B$ . We call  $(\psi, \pi)$  *covariant* if there exists a collection  $\mathcal{G} \subset C_c(E_{\text{rg}}^0)$  of nonnegative functions which generates  $C_0(E_{\text{rg}}^0)$  as a  $C^*$ -algebra, and for each  $f \in \mathcal{G}$  there exist a finite cover  $\mathcal{N}_f$  of  $r^{-1}(\text{supp}(f))$  by open  $s$ -sections, and a collection of nonnegative functions  $\{f_N : N \in \mathcal{N}_f\} \in C_c(E^1)$  such that:

- (1)  $\text{osupp}(f_N) \subset N \cap r^{-1}(\text{supp}(f))$  for all  $N \in \mathcal{N}_f$ .
- (2)  $\sum_{N \in \mathcal{N}_f} f_N = f \circ r$ .
- (3)  $\pi(f) = \sum_{N \in \mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*$ .

We say that a covariant Toeplitz representation  $(\psi, \pi)$  of  $E$  in  $B$  is *universal* if for any covariant Toeplitz representation  $(\psi', \pi')$  of  $E$  in  $C$ , there exists a homomorphism  $h : B \rightarrow C$ , such that  $h \circ \psi = \psi'$  and  $h \circ \pi = \pi'$ .

**Remark 2.11.** Definition 2.10 is formulated so as to make it easy to check that a given pair  $(\psi, \pi)$  is covariant. However, when *using* covariance of a pair  $(\psi, \pi)$  it is helpful to know that (3) of Definition 2.10 holds for every  $f \in C_c(E_{\text{rg}}^0)$ , every  $\mathcal{N}_f$ , and every  $\{f_N : N \in \mathcal{N}_f\}$  satisfying (1) and (2) of Definition 2.10. We prove this in Proposition 4.8.

**Remark 2.12.** In Definition 2.10, we made implicit use of Remark 2.9 because the preimage  $r^{-1}(\text{supp}(f))$  is compact by Remark 2.7. Observe that the covariance condition for a Toeplitz representation of  $E$  only involves functions in  $C_0(E_{\text{rg}}^0) \subset C_0(E^0)$ .

Recall that if  $E$  is a topological graph, then  $\mathcal{O}(E)$  denotes Katsura's topological graph  $C^*$ -algebra (see Notation 3.6).

**Theorem 2.13.** *Let  $E$  be a topological graph. Then there is a universal covariant Toeplitz representation  $(j_1, j_0)$  of  $E$  which generates  $\mathcal{O}(E)$ . Moreover the  $C^*$ -algebra generated by the image of any universal covariant Toeplitz representation of  $E$  is isomorphic to  $\mathcal{O}(E)$ .*

Although Definition 2.10 looks complicated, the hypotheses are relatively easy to check in specific instances. To give some intuition, we indicate how the definition simplifies if the range map  $r : E^1 \rightarrow E^0$  is a local homeomorphism. This situation still includes many interesting examples.

**Definition 2.14.** Let  $E$  be a topological graph. A pair  $(\mathcal{U}, V)$  is called a *local  $r$ -fibration* if  $V$  is a subset of  $E^0$ , and  $\mathcal{U}$  is a finite collection of mutually disjoint bisections such that  $r^{-1}(V) = \bigcup_{U \in \mathcal{U}} U$ , and  $r(U) = V$  for each  $U \in \mathcal{U}$ . A local  $r$ -fibration is *precompact* if each  $U \in \mathcal{U}$  is precompact and  $V$  is precompact. A local  $r$ -fibration is *open* if each  $U \in \mathcal{U}$  is open and  $V$  is open.

Suppose that  $(\mathcal{U}, V)$  is an open local  $r$ -fibration. Suppose that  $U \in \mathcal{U}$  and that  $f \in C_c(V)$ . We write  $r_U^* f \in C_c(U)$  for the function

$$r_U^* f : e \mapsto \begin{cases} f(r(e)) & \text{if } e \in U \\ 0 & \text{otherwise.} \end{cases}$$

**Corollary 2.15.** *Let  $E$  be a topological graph. Suppose that  $r$  is a local homeomorphism. Let  $(\psi, \pi)$  be a Toeplitz representation of  $E$ . Then  $(\psi, \pi)$  is covariant if and only if there exists a collection  $\mathcal{G} \subset C_c(E_{\text{rg}}^0)$  of nonnegative functions which generates  $C_0(E_{\text{rg}}^0)$  as a  $C^*$ -algebra, and for each  $f \in \mathcal{G}$  there exists an open local  $r$ -fibration  $(\mathcal{U}, V)$  such that  $\text{supp}(f) \subset V$ , and*

$$(2.4) \quad \pi(f) = \sum_{U \in \mathcal{U}} \psi(\sqrt{r_U^* f}) \psi(\sqrt{r_U^* f})^*.$$

*If  $(\psi, \pi)$  is covariant, then Equation (2.4) holds for every  $f \in C_c(E_{\text{rg}}^0)$  and open local  $r$ -fibration  $(\mathcal{U}, V)$  with  $\text{supp}(f) \subset V$ .*

**Remark 2.16.** Let  $E$  be a topological graph. Fix  $f \in C_c(E_{\text{rg}}^0)$  and suppose that  $(\mathcal{U}, V)$  is an open local  $r$ -fibration such that  $\text{supp}(f) \subset V$ . By Remark 2.7,  $r^{-1}(\text{supp}(f))$  is compact. Since  $\mathcal{U}$  is an open cover of  $r^{-1}(\text{supp}(f))$  by disjoint open bisections, Remark 2.9 gives functions  $\{f_U : U \in \mathcal{U}\}$  such that  $\text{osupp}(f_U) \subset U \cap r^{-1}(\text{supp}(f))$ , and  $\sum_{U \in \mathcal{U}} f_U = f \circ r$ , for all  $U \in \mathcal{U}$ . We then have  $r_U^* f = f_U$ , for all  $U \in \mathcal{U}$ .

### 3. $C^*$ -correspondences, $C^*$ -algebras and Katsura's construction

We recall some background on Hilbert  $C^*$ -modules. For more detail see [12].

**Definition 3.1.** Let  $A$  be a  $C^*$ -algebra. A *right Hilbert  $A$ -module* is a right  $A$ -module  $X$  equipped with a map  $\langle \cdot, \cdot \rangle_A : X \times X \rightarrow A$  such that for  $x, y \in X, a \in A$ :

- (1)  $\langle x, x \rangle_A \geq 0$  with equality only if  $x = 0$ .
- (2)  $\langle x, y + z \rangle_A = \langle x, y \rangle_A + \langle x, z \rangle_A$ .
- (3)  $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$ .
- (4)  $\langle x, y \rangle_A = \langle y, x \rangle_A^*$ .
- (5)  $X$  is complete in the norm<sup>1</sup> defined by  $\|x\|_A^2 = \|\langle x, x \rangle_A\|$ .

Recall from, for example, [10, Page 72], that a right Hilbert  $A$ -module  $X$  is a  $C^*$ -correspondence over  $A$  if there is a left action of  $A$  on  $X$  such that

$$(3.1) \quad \langle a \cdot y, x \rangle_A = \langle y, a^* \cdot x \rangle_A \quad \text{for all } a \in A, x, y \in X.$$

An operator  $T : X \rightarrow X$  is *adjointable* if there exists  $T^* : X \rightarrow X$  such that  $\langle T \cdot x, y \rangle_A = \langle y, T^* x \rangle_A$ , for all  $x, y \in X$ . The adjoint  $T^*$  is unique, and  $T$  is automatically bounded and linear. The set  $\mathcal{L}(X)$  of adjointable operators on  $X$  is a  $C^*$ -algebra. Equation (3.1) implies that there is a homomorphism  $\phi : A \rightarrow \mathcal{L}(X)$  such that  $\phi(a)x = a \cdot x$  for all  $a \in A$  and  $x \in X$ .

**Definition 3.2.** Let  $A$  be a  $C^*$ -algebra and let  $X$  be a right Hilbert  $A$ -module. Fix  $x, y \in X$ , we define  $\Theta_{x,y} : X \rightarrow X$  by  $\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_A$ , for all  $z \in X$ . We define  $\mathcal{K}(X) := \overline{\text{span}}\{\Theta_{x,y} : x, y \in X\}$ .

A calculation shows that  $T\Theta_{x,y} = \Theta_{Tx,y}$ , and  $\Theta_{x,y}^* = \Theta_{y,x}$ , for all  $x, y \in X, T \in \mathcal{L}(X)$ . Hence  $\mathcal{K}(X)$  is a closed two-sided ideal of  $\mathcal{L}(X)$ .

Toeplitz representations and Toeplitz algebras of  $C^*$ -correspondences were introduced and studied in [9]. Cuntz–Pimsner algebras were also introduced in [9], but the definition was later modified by Katsura in [6, Definition 3.5] so as to include graph algebras as a special case [10, Example 8.13]. We use Katsura’s definition in this paper.

**Definition 3.3** ([6, Definition 2.1]). Let  $A$  be a  $C^*$ -algebra and let  $X$  be a  $C^*$ -correspondence over  $A$ . A *Toeplitz representation* of  $X$  in a  $C^*$ -algebra  $B$  is a pair  $(\psi, \pi)$ , where  $\psi : X \rightarrow B$  is a linear map,  $\pi : A \rightarrow B$  is a homomorphism, and for any  $x, y \in X$  and  $a \in A$ :

- (1)  $\psi(a \cdot x) = \pi(a)\psi(x)$ .
- (2)  $\psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A)$ .

We say that a Toeplitz representation  $(\psi, \pi)$  of  $X$  in  $B$  is *universal* if for any Toeplitz representation  $(\psi', \pi')$  of  $X$  in  $C$ , there exists a homomorphism  $h : B \rightarrow C$ , such that  $h \circ \psi = \psi'$ , and  $h \circ \pi = \pi'$ .

**Remark 3.4** ([6, Page 370]). Let  $X$  be a  $C^*$ -correspondence over a  $C^*$ -algebra  $A$  and let  $(\psi, \pi)$  be a Toeplitz representation of  $X$ . For  $x \in X, a \in A$ , a calculation using Definition 3.3(2) and that  $\pi$  is a homomorphism

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<sup>1</sup>It is not supposed to be immediately obvious that this defines a norm, but it is true (see [12]).

shows that  $(\psi(xa) - \psi(x)\pi(a))^*(\psi(xa) - \psi(x)\pi(a)) = 0$ , and hence that  $\psi(x \cdot a) = \psi(x)\pi(a)$ . Definition 3.3(2) show that  $\psi$  is bounded with  $\|\psi\| \leq 1$ .

Proposition 1.3 of [4] implies that there exists a  $C^*$ -algebra  $\mathcal{T}_X$  generated by the image of a universal Toeplitz representation  $(i_X, i_A)$  of  $X$ . This  $C^*$ -algebra is unique up to canonical isomorphism and we call it the *Toeplitz algebra* of  $X$ . Given another Toeplitz representation  $(\psi, \pi)$  of  $X$  in a  $C^*$ -algebra  $B$ , we write  $h_{\psi, \pi}$  for the induced homomorphism from  $\mathcal{T}_X$  to  $B$ .

Recall from [9, Page 202] (see also [4, Proposition 1.6]) that if  $(\psi, \pi)$  is a Toeplitz representation of a  $C^*$ -correspondence  $X$  in a  $C^*$ -algebra  $B$ , then there is a unique homomorphism  $\psi^{(1)} : \mathcal{K}(X) \rightarrow B$  such that  $\psi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$  for all  $x, y \in X$ . Recall also that given a  $C^*$ -algebra  $A$  and a closed two-sided ideal  $J$  of  $A$ , we define  $J^\perp := \{a \in A : ab = 0 \text{ for all } b \in J\}$ . Then  $J^\perp$  is also a closed two-sided ideal of  $A$ .

**Definition 3.5** ([6, Definitions 3.4 and 3.5]). Let  $A$  be a  $C^*$ -algebra, let  $X$  be a  $C^*$ -correspondence over  $A$ , and write  $\phi : A \rightarrow \mathcal{L}(X)$  for the homomorphism implementing the left action. A Toeplitz representation  $(\psi, \pi)$  of  $X$  is *covariant* if  $\psi^{(1)}(\phi(a)) = \pi(a)$  for all  $a \in \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^\perp$ .

A covariant Toeplitz representation  $(\psi, \pi)$  of  $X$  in  $B$  is *universal* if for any covariant Toeplitz representation  $(\psi', \pi')$  of  $X$  in  $C$ , there exists a homomorphism  $h$  from  $B$  into  $C$ , such that  $h \circ \psi = \psi'$ , and  $h \circ \pi = \pi'$ .

Recall from [10, Page 75] that there exists a  $C^*$ -algebra  $\mathcal{O}_X$  generated by the image of a universal covariant Toeplitz representation  $(j_X, j_A)$  of  $X$ . This  $C^*$ -algebra is unique up to canonical isomorphism. Given another covariant Toeplitz representation  $(\psi, \pi)$  of  $X$  in a  $C^*$ -algebra  $B$ , we write  $\psi \times \pi$  for the induced homomorphism from  $\mathcal{O}_X$  to  $B$ .

Let  $E$  be a topological graph. As in [5], define actions of  $C_0(E^0)$  on  $C_c(E^1)$  by Equation (2.1). For  $x_1, x_2 \in C_c(E^1)$ , define  $\langle x_1, x_2 \rangle_{C_0(E^0)} : E^0 \rightarrow \mathbb{C}$  by

$$\langle x_1, x_2 \rangle_{C_0(E^0)}(v) = \sum_{s(e)=v} \overline{x_1(e)}x_2(e).$$

(By convention an empty sum is 0.) If  $\text{osupp}(x)$  is an  $s$ -section, then the function  $\widehat{x}$  of Equation (2.2) is equal to  $\langle x, x \rangle_{C_0(E^0)}$ . As in [5] (see also [10, Page 79]),  $\langle \cdot, \cdot \rangle_{C_0(E^0)}$  defines a  $C_0(E^0)$ -valued inner product on  $C_c(E^1)$ , and the completion  $X(E)$  of  $C_c(E^1)$  in the norm  $\|x\|_{C_0(E^0)}^2 = \|\langle x, x \rangle_{C_0(E^0)}\|$  is a  $C^*$ -correspondence over  $C_0(E^0)$ . We call  $X(E)$  the *graph correspondence* of  $E$ .

**Notation 3.6.** We denote by  $\mathcal{T}(E)$  [5, Definition 2.2] the Toeplitz algebra  $\mathcal{T}_{X(E)}$ , and we denote by  $\mathcal{O}(E)$  [5, Definition 2.10] the  $C^*$ -algebra  $\mathcal{O}_{X(E)}$ .

### 4. Proofs of the main results

To prove Theorem 2.4, we must show that  $\psi(x)^*\psi(y) = \pi(\langle x, y \rangle_{C_0(E^0)})$  for all  $x, y \in C_c(E^1)$ . We establish this formula for  $x$  and  $y$  supported on



a common  $s$ -section in Lemma 4.1, and then extend it to arbitrary  $x, y \in C_c(E^1)$  in Proposition 4.3.

Let  $U, V$  be complex vector spaces. Then any sesquilinear form  $\varphi : V \times V \rightarrow U$  which is conjugate linear in the first variable satisfies the polarisation identity

$$\varphi(v_1, v_2) = \frac{1}{4} \sum_{n=0}^3 (-i)^n \varphi(v_1 + i^n v_2, v_1 + i^n v_2).$$

(To prove this, just expand the sum.)

**Lemma 4.1.** *Let  $E$  be a topological graph and let  $(\psi, \pi)$  be a Toeplitz representation of  $E$ . Fix  $x_1, x_2 \in C_c(E^1)$ . Suppose that  $\text{supp}(x_1) \cup \text{supp}(x_2)$  is contained in an open  $s$ -section. Then  $\pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) = \psi(x_1)^* \psi(x_2)$ .*

**Proof.** The polarisation identity for  $\langle \cdot, \cdot \rangle_{C_0(E^0)}$  and Definition 2.2(2) give

$$\begin{aligned} \pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) &= \frac{1}{4} \sum_{n=0}^3 (-i)^n \pi(\langle x_1 + i^n x_2, x_1 + i^n x_2 \rangle_{C_0(E^0)}) \\ &= \frac{1}{4} \sum_{n=0}^3 (-i)^n \psi(x_1 + i^n x_2)^* \psi(x_1 + i^n x_2) \\ &= \psi(x_1)^* \psi(x_2). \end{aligned} \quad \square$$

In the following and throughout the rest of the paper we write  $U^c$  for the complement  $X \setminus U$  of a subset  $U$  of a set  $X$ .

**Remark 4.2.** Let  $X$  be locally compact Hausdorff space. Fix a compact subset  $K \subset X$  and an open neighbourhood  $U$  of  $K$ . Since  $U^c$  is closed and disjoint from  $K$ , there is a function  $f \in C(X, [0, 1])$  which is identically 1 on  $K$  and identically 0 on  $U^c$  (see for example [14, Theorem 37.A]). So  $V := f^{-1}((1/2, 3/2))$  is open and satisfies  $K \subset V \subset \overline{V} \subset U$ .

**Proposition 4.3.** *Let  $E$  be a topological graph and let  $(\psi, \pi)$  be a Toeplitz representation of  $E$ . Fix open  $s$ -sections  $U_1, U_2 \subset E^1$ , and  $x_1, x_2 \in C_c(E^1)$  with  $\text{supp}(x_i) \subset U_i$ , for  $i = 1, 2$ . Then  $\pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) = \psi(x_1)^* \psi(x_2)$ .*

**Proof.** Let  $K_i = \text{supp}(x_i)$ , for  $i = 1, 2$ , and  $K = K_1 \cup K_2$ . Since  $E^1$  is locally compact and Hausdorff, Remark 4.2 implies that there exist open sets  $U'_i \subset E^1$  such that  $K_i \subset U'_i \subset \overline{U'_i} \subset U_i$ , for  $i = 1, 2$ .

For each  $e \in K$  and  $W \in \{U_i, U'_i, \overline{U'_i}^c : i = 1, 2\}$  such that  $e \in W$ , fix an open neighbourhood  $V(e, W)$  of  $e$  such that  $\overline{V(e, W)} \subset W$ . Define

$$N_e = \left( \bigcap_{e \in U_i} V(e, U_i) \right) \cap \left( \bigcap_{e \in K_i} V(e, U'_i) \right) \cap \left( \bigcap_{e \in U_i^c} V(e, \overline{U'_i}^c) \right).$$

Then for  $i = 1, 2$ :

- (1) If  $e \in U_i$ , then  $\overline{N_e} \subset U_i$ .
- (2) If  $e \in K_i$ , then  $\overline{N_e} \subset U'_i$ .

(3) If  $e \notin U_i$ , then  $\overline{N_e} \cap \overline{U'_i} = \emptyset$ .

Since  $K$  is compact, there is a finite  $F \subset K$  such that  $\{N_e : e \in F\}$  covers  $K$ .

Fix  $e, f \in F$ . Suppose that  $\overline{N_e} \cap \overline{N_f} \neq \emptyset$ . We claim that

$$(4.1) \quad \text{either } \overline{N_e} \cup \overline{N_f} \subset U_1, \text{ or } \overline{N_e} \cup \overline{N_f} \subset U_2.$$

When  $e = f$ , this is trivial. Suppose  $e \neq f$ . Assume without loss of generality that  $e \in K_1$ . Then (2) forces  $\overline{N_e} \subset U'_1$ , so  $\overline{N_e} \cap \overline{N_f} \neq \emptyset$  forces  $\overline{N_f} \cap \overline{U'_1} \neq \emptyset$ . Condition (3) then forces  $f \in U_1$ , so (1) forces  $\overline{N_f} \subset U_1$  and hence  $\overline{N_e} \cup \overline{N_f} \subset U_1$  as required.

Since  $\{N_e : e \in F\}$  is a finite open cover of  $K$ , Remark 2.9 implies that there are finite collections of functions

$$\{x_{1,e} : e \in F\}, \{x_{2,e} : e \in F\} \subset C_c(E^1)$$

such that  $\text{osupp}(x_{i,e}) \subset N_e \cap K$  for all  $e \in F$ ,  $i = 1, 2$ , and  $\sum_{e \in F} x_{1,e} = x_1$ ,  $\sum_{e \in F} x_{2,e} = x_2$ . Linearity of  $\psi$  gives  $\psi(x_1)^* \psi(x_2) = \sum_{e,f \in F} \psi(x_{1,e})^* \psi(x_{2,f})$ . Fix  $e, f \in F$  such that  $\overline{N_e} \cap \overline{N_f} = \emptyset$ . By Remark 4.2 there are disjoint open  $s$ -sections  $O_e, O_f \subset E^1$ , such that  $\overline{N_e} \subset O_e$ , and  $\overline{N_f} \subset O_f$ . Thus condition (3) of Definition 2.2 gives  $\psi(x_{1,e})^* \psi(x_{2,f}) = 0$  since  $\text{supp}(x_{i,e}) \subset \overline{N_e}$  for all  $e \in F$ ,  $i = 1, 2$ . It follows that  $\psi(x_1)^* \psi(x_2) = \sum_{\overline{N_e} \cap \overline{N_f} \neq \emptyset} \psi(x_{1,e})^* \psi(x_{2,f})$ . Whenever  $\overline{N_e} \cap \overline{N_f} \neq \emptyset$ , Equation (4.1) implies that  $\overline{N_e} \cup \overline{N_f}$  is contained in an open  $s$ -section, so Lemma 4.1 gives

$$\psi(x_1)^* \psi(x_2) = \sum_{\overline{N_e} \cap \overline{N_f} \neq \emptyset} \pi(\langle x_{1,e}, x_{2,f} \rangle_{C_0(E^0)}) = \pi(\langle x_1, x_2 \rangle_{C_0(E^0)}). \quad \square$$

**Proposition 4.4.** *Let  $E$  be a topological graph and let  $(\psi, \pi)$  be a Toeplitz representation of  $E$  in  $B$ . Then*

$$\psi : (C_c(E^1), \|\cdot\|_{C_0(E^0)}) \rightarrow B$$

*is a bounded linear map. Let  $\tilde{\psi}$  be the unique extension of  $\psi$  to  $X(E)$ . Then the pair  $(\tilde{\psi}, \pi)$  is a Toeplitz representation of  $X(E)$ .*

**Proof.** Fix  $x_1, x_2 \in C_c(E^1)$ . Let  $K = \text{supp}(x_1) \cup \text{supp}(x_2)$ . For each  $e \in K$ , there exists an open  $s$ -section  $N_e$  containing  $e$ . Remark 4.2 yields an open neighbourhood  $N'_e$  of  $e$  such that  $\overline{N'_e} \subset N_e$ . Since  $K$  is compact, there is a finite subset  $F \subset K$ , such that  $\{N'_e : e \in F\}$  covers  $K$ . By Remark 2.9, there exist  $\{x_{1,e} : e \in F\}, \{x_{2,e} : e \in F\} \subset C_c(E^1)$  such that  $\text{osupp}(x_{i,e}) \subset N'_e \cap K$  for all  $e \in F$  and  $i = 1, 2$ , and such that  $\sum_{e \in F} x_{1,e} = x_1$  and  $\sum_{e \in F} x_{2,e} = x_2$ . Proposition 4.3 implies that

$$\begin{aligned} \pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) &= \sum_{e,f \in F} \pi(\langle x_{1,e}, x_{2,f} \rangle_{C_0(E^0)}) \\ &= \sum_{e,f \in F} \psi(x_{1,e})^* \psi(x_{2,f}) = \psi(x_1)^* \psi(x_2). \end{aligned}$$

Then

$$\|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\| = \|\pi(\langle x, x \rangle_{C_0(E^0)})\| \leq \|\langle x, x \rangle_{C_0(E^0)}\| = \|x\|_{C_0(E^0)}^2$$

for all  $x \in C_c(E^1)$ . So  $\psi$  is bounded, and hence extends uniquely to a bounded linear map  $\tilde{\psi}$  on  $X(E)$ . By continuity,  $(\tilde{\psi}, \pi)$  is a Toeplitz representation of  $X(E)$ .  $\square$

**Remark 4.5.** Let  $(\psi, \pi)$  be a Toeplitz representation of the graph correspondence  $X(E)$ . Then the pair  $(\psi|_{C_c(E^1)}, \pi)$  is a Toeplitz representation of  $E$ . So Proposition 4.4 implies that  $(\psi, \pi) \mapsto (\psi|_{C_c(E^1)}, \pi)$  is a bijection between Toeplitz representations of  $X(E)$  and Toeplitz representations of  $E$ , with inverse described by Proposition 4.4.

**Proof of Theorem 2.4.** Let  $(i_X, i_A)$  be the universal Toeplitz representation of  $X(E)$  in  $\mathcal{T}(E)$ . Then  $(i_1, i_0) := (i_X|_{C_c(E^1)}, i_A)$  is a Toeplitz representation of  $E$ . Fix another Toeplitz representation  $(\psi, \pi)$  of  $E$  in a C\*-algebra  $B$ . By Proposition 4.4,  $\psi$  extends to  $\tilde{\psi} : X(E) \rightarrow B$  such that  $(\tilde{\psi}, \pi)$  is a Toeplitz representation of  $X(E)$ . By the universal property of  $(i_X, i_A)$ , there exists a homomorphism  $h_{\tilde{\psi}, \pi} : \mathcal{T}(E) \rightarrow B$ , such that  $h_{\tilde{\psi}, \pi} \circ i_X = \tilde{\psi}$ , and  $h_{\tilde{\psi}, \pi} \circ i_A = \pi$ . In particular  $h_{\tilde{\psi}, \pi} \circ i_1 = \psi$ . Hence  $(i_1, i_0)$  is a universal Toeplitz representation of  $E$  which generates  $\mathcal{T}(E)$ . The second statement follows easily.  $\square$

Our next task is to prove Theorem 2.13. We first need some background results.

**Remark 4.6.** Let  $E$  be a topological graph. Fix  $v \in E_{\text{fin}}^0$ . There exists a neighbourhood  $U$  of  $v$  such that  $r^{-1}(U)$  is compact, and there exists an open neighbourhood  $V$  of  $v$  such that  $V \subset U$ . By Remark 4.2, there exists an open neighbourhood  $W$  of  $v$  such that  $\overline{W} \subset V$ . Since  $r^{-1}(\overline{W})$  is closed and is contained in  $r^{-1}(U)$ , it is compact. Hence  $v \in E_{\text{fin}}^0$  if and only if there exists an open neighbourhood  $N$  of  $v$  such that  $r^{-1}(\overline{N})$  is compact.

Let  $E$  be a topological graph. Recall from Definition 3.5 that Katsura’s covariance condition for a Toeplitz representation of the correspondence  $X(E)$  involves the ideal  $\phi^{-1}(\mathcal{K}(X(E))) \cap (\ker \phi)^\perp$ . Katsura computed this ideal in [5]. We quote his result and give a simple proof.

Observe that  $\ker \phi = \{f \in C_0(E^0) : f(\overline{r(E^1)}) \equiv 0\}$ . Hence

$$(\ker \phi)^\perp = \{f \in C_0(E^0) : f(\overline{E_{\text{sce}}^0}) \equiv 0\}.$$

**Proposition 4.7** ([5, Proposition 1.24]). *Let  $E$  be a topological graph. Then*

$$\phi^{-1}(\mathcal{K}(X(E))) = C_0(E_{\text{fin}}^0).$$

Moreover  $\phi^{-1}(\mathcal{K}(X(E))) \cap (\ker \phi)^\perp = C_0(E_{\text{rg}}^0)$ .

**Proof.** The final statement follows from the previous one and definition of  $E_{\text{rg}}^0$ . So we just need to show that  $\phi^{-1}(\mathcal{K}(X(E))) = C_0(E_{\text{fin}}^0)$ .

Fix  $f \in C_0(E^0) \setminus C_0(E_{\text{fin}}^0)$ . We must show that  $\phi(f) \notin \mathcal{K}(X(E))$ . Fix  $v_0 \in (E_{\text{fin}}^0)^c$ , such that  $f(v_0) \neq 0$ . Let  $m = |f(v_0)|$  and let

$$N_0 = \{v \in E^0 : |f(v)| > m/2\},$$

so  $N_0$  is an open neighbourhood of  $v_0$ . By Remark 4.6,  $r^{-1}(\overline{N_0})$  is not compact. Fix  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  in  $C_c(E^1)$ . Let

$$K = \bigcup_{i=1}^n \text{supp}(x_i) \cup \text{supp}(y_i).$$

Then  $K$  is compact, so  $r^{-1}(\overline{N_0})$  is not contained in  $K$ . So there exists  $e_0 \in r^{-1}(\overline{N_0}) \setminus K$ . By Remark 4.2 there exists  $x_0 \in C_c(E^1)$  such that  $x_0(e_0) = 1$ . Hence

$$\begin{aligned} & \left\| \phi(f) - \sum_{i=1}^n \Theta_{x_i, y_i} \right\| \\ & \geq \left\| \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i, y_i}(x_0) \right\|_{C_0(E^0)} \\ & \geq \left| \left\langle \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i, y_i}(x_0), \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i, y_i}(x_0) \right\rangle_{C_0(E^0)} \right|^{1/2} \\ & \geq m/2. \end{aligned}$$

Thus  $\|\phi(f) - a\| \geq m/2$  for all  $a \in \text{span}\{\Theta_{x,y} : x, y \in C_c(E^1)\}$ . Since  $\overline{\text{span}\{\Theta_{x,y} : x, y \in C_c(E^1)\}} = \mathcal{K}(X(E))$ , it follows that  $\phi(f) \notin \mathcal{K}(X(E))$ .

Now fix a nonnegative function  $f \in C_c(E_{\text{fin}}^0)$ . We must show that  $\phi(f) \in \mathcal{K}(X(E))$ . Let  $K' = \text{supp}(f)$ . For any  $e \in r^{-1}(K')$ , there exists an open  $s$ -section  $N_e$  containing  $e$ . Remark 2.7 shows that  $r^{-1}(K')$  is compact. Hence there exists a finite subset  $F \subset r^{-1}(K')$  such that  $\{N_e\}_{e \in F}$  covers  $r^{-1}(K')$ . Since  $\text{supp}(f \circ r) \subset r^{-1}(K')$ , Remark 2.9 yields a finite collection of functions  $\{f_e : e \in F\} \subset C_c(E^1)$  such that each  $\text{osupp}(f_e) \subset N_e \cap r^{-1}(K')$  and  $\sum_{e \in F} f_e = f \circ r$ . Since the  $f_e$  are supported on the  $s$ -sections  $N_e$ , we have  $\theta_{\sqrt{f_e}, \sqrt{f_e}}(x)(e') = f_e(e')x(e')$  for all  $e' \in E^1$ . Hence

$$(4.2) \quad \phi(f) = \sum_{e \in F} \Theta_{\sqrt{f_e}, \sqrt{f_e}} \in \mathcal{K}(X(E)). \quad \square$$

**Proposition 4.8.** *Let  $E$  be a topological graph and let  $(\psi, \pi)$  be a covariant Toeplitz representation of  $E$ . Then the Toeplitz representation  $(\tilde{\psi}, \pi)$  of  $X(E)$  from Proposition 4.4 is also covariant. For any nonnegative function  $f \in C_c(E_{\text{rg}}^0)$ , any finite cover  $\mathcal{N}$  of  $r^{-1}(\text{supp}(f))$  by open  $s$ -sections and*

any collection of nonnegative functions  $\{f_N : N \in \mathcal{N}\} \subset C_c(E^1)$ , such that  $\text{osupp}(f_N) \subset N \cap r^{-1}(\text{supp}(f))$ , and  $\sum_{N \in \mathcal{N}} f_N = f \circ r$ , we have

$$\pi(f) = \sum_{N \in \mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*.$$

**Proof.** Choose collections  $\mathcal{G}$ ,  $\{\mathcal{N}_f : f \in \mathcal{G}\}$ , and  $\{f_N : N \in \mathcal{N}_f\}$  as in Definition 2.10. By Corollary 4.7, to prove that  $(\tilde{\psi}, \pi)$  is a covariant Toeplitz representation of  $X(E)$ , must show that  $\tilde{\psi}^{(1)} \circ \phi(f) = \pi(f)$ , for all  $f \in \mathcal{G}$ . Fix  $f \in \mathcal{G}$ . Since  $\phi(f) = \sum_{N \in \mathcal{N}_f} \Theta_{\sqrt{f_N}, \sqrt{f_N}}$ , we have  $\tilde{\psi}^{(1)} \circ \phi(f) = \sum_{N \in \mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*$ . Hence  $\tilde{\psi}^{(1)} \circ \phi(f) = \pi(f)$  by Definition 2.10. Therefore  $(\tilde{\psi}, \pi)$  is a covariant Toeplitz representation of  $X(E)$ .

For the second statement observe that since  $(\tilde{\psi}, \pi)$  is a covariant Toeplitz representation of  $X(E)$ ,

$$\pi(f) = \tilde{\psi}^{(1)} \circ \phi(f) = \sum_{N \in \mathcal{N}_f} \tilde{\psi}^{(1)}(\Theta_{\sqrt{f_N}, \sqrt{f_N}}) = \sum_{N \in \mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*. \quad \square$$

**Proposition 4.9.** *Let  $E$  be a topological graph and let  $(\psi, \pi)$  be a covariant Toeplitz representation of  $X(E)$ . Then  $(\psi|_{C_c(E^1)}, \pi)$  is a covariant Toeplitz representation of  $E$ .*

**Proof.** Remark 4.5 implies that  $(\psi|_{C_c(E^1)}, \pi)$  is a Toeplitz representation of  $E$ . Let  $\mathcal{G}$  be the set of all nonnegative functions in  $C_c(E_{\text{rg}}^0)$ . Fix  $f \in \mathcal{G}$ . By Equation (4.2), there exists a finite cover  $\mathcal{N}$  of  $r^{-1}(\text{supp}(f))$  by open  $s$ -sections and a finite collection of functions  $\{f_N : N \in \mathcal{N}\} \subset C_c(E^1)$ , such that  $\text{osupp}(f_N) \subset N \cap r^{-1}(\text{supp}(f))$ ,  $\sum_{N \in \mathcal{N}} f_N = f \circ r$ , and  $\phi(f) = \sum_{N \in \mathcal{N}} \Theta_{\sqrt{f_N}, \sqrt{f_N}}$ . Since  $(\psi, \pi)$  is a covariant Toeplitz representation of  $X(E)$ , we have

$$\begin{aligned} \pi(f) &= \psi^{(1)} \circ \phi(f) = \sum_{N \in \mathcal{N}} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^* \\ &= \sum_{N \in \mathcal{N}} \psi|_{C_c(E^1)}(\sqrt{f_N})\psi|_{C_c(E^1)}(\sqrt{f_N})^*. \end{aligned}$$

Hence  $(\psi|_{C_c(E^1)}, \pi)$  is covariant. □

**Proof of Theorem 2.13.** Propositions 4.8 and 4.9 provide a bijective map from covariant Toeplitz representations of  $E$  to covariant Toeplitz representations of  $X(E)$ . The result now follows from the same argument as Theorem 2.4. □

We still need to prove Corollary 2.15. We must first show that under the hypotheses of the corollary, there are plenty of local  $r$ -fibrations (see Definition 2.14).

**Lemma 4.10.** *Let  $E$  be a topological graph. Suppose that  $r$  is a local homeomorphism. Then for any  $v \in E_{\text{rg}}^0$ , there exists a precompact open local  $r$ -fibration  $(\mathcal{U}, V)$ , such that  $v \in V \subset \overline{V} \subset E_{\text{rg}}^0$ .*

**Proof.** Fix  $v \in E_{\text{rg}}^0$ . Since  $E_{\text{rg}}^0 = E_{\text{fin}}^0 \setminus \overline{E_{\text{sce}}^0} \subset E_{\text{fin}}^0 \setminus E_{\text{sce}}^0$ , Lemma 1.22 of [5] shows that  $r^{-1}(v)$  is nonempty. Lemma 1.23 of [5] shows that it is also compact. Since  $r$  is a local homeomorphism,  $r^{-1}(v)$  is finite. Since  $E^1$  is locally compact Hausdorff, we can separate points in  $r^{-1}(v)$  by mutually disjoint precompact open bisections  $\{U_e : e \in r^{-1}(v)\}$ . We can assume, by shrinking if necessary, that the  $U_e$  have common range  $N$  with  $\overline{N} \subset E_{\text{rg}}^0$ . Now  $|r^{-1}(w)| \geq |r^{-1}(v)|$  for all  $w \in N$ .

We claim there exists an open neighbourhood  $V$  of  $v$  such that  $V \subset N$ , and  $|r^{-1}(w)| = |r^{-1}(v)|$  for all  $w \in V$ . Suppose for a contradiction there exists a convergent net  $(v_\alpha)_{\alpha \in \Lambda} \subset N$  with limit  $v$  satisfying  $|r^{-1}(v_\alpha)| > |r^{-1}(v)|$  for all  $\alpha \in \Lambda$ . Then for any  $\alpha \in \Lambda$ , there exists  $e_\alpha \notin \bigcup_{e \in r^{-1}(v)} U_e$  such that  $r(e_\alpha) = v_\alpha$ . Since  $r^{-1}(\overline{N})$  is compact, [13, Theorem IV.3] implies that there exists a convergent subnet  $(e_{h(a)})_{a \in A}$  of  $(e_\alpha)_{\alpha \in \Lambda}$  with limit  $e \notin \bigcup_{f \in r^{-1}(v)} U_f$ . By continuity of  $r$ , we have  $r(e) = v$ , which is a contradiction.

Hence there exists an open neighbourhood  $V$  of  $v$  satisfying  $V \subset N$ , such that  $|r^{-1}(w)| = |r^{-1}(v)|$  for all  $w \in V$ . So with

$$\mathcal{U} = \{U_e \cap r^{-1}(V) : e \in r^{-1}(v)\},$$

the pair  $(\mathcal{U}, V)$  is a precompact open local  $r$ -fibration.  $\square$

**Lemma 4.11.** *Let  $E$  be a topological graph. Suppose that  $r$  is a local homeomorphism. Let  $\mathcal{G}$  be the set of all nonnegative functions  $f$  in  $C_c(E_{\text{rg}}^0)$  such that  $\text{supp}(f) \subset V$  for some open local  $r$ -fibration  $(\mathcal{U}, V)$ . Then  $\mathcal{G}$  generates  $C_0(E_{\text{rg}}^0)$ .*

**Proof.** By Lemma 4.10, for all  $v \neq w \in E_{\text{rg}}^0$ , there is a local  $r$ -fibration  $(\mathcal{U}, V)$  such that  $v \in V$  and  $w \notin V$ . So there exists  $f \in \mathcal{G}$  such that  $f(v) = 1$  and  $f(w) = 0$ . Now the result follows from the Extended Stone–Weierstrass Theorem.  $\square$

**Proof of Corollary 2.15.** Let  $(\psi, \pi)$  be a covariant Toeplitz representation of  $E$ . Let  $\mathcal{G}$  be the set of all nonnegative functions  $f$  in  $C_c(E_{\text{rg}}^0)$  such that  $\text{supp}(f) \subset V$  for some open local  $r$ -fibration  $(\mathcal{U}, V)$ . Lemma 4.11 implies that  $\mathcal{G}$  generates  $C_0(E_{\text{rg}}^0)$ . Fix  $f \in \mathcal{G}$  and an open local  $r$ -fibration  $(\mathcal{U}, V)$  with  $\text{supp}(f) \subset V$ . Since  $\mathcal{U}$  is a finite cover of  $r^{-1}(\text{supp}(f))$  by open  $s$ -sections, each  $\text{osupp}(r_U^* f) \subset U \cap r^{-1}(\text{supp}(f))$ , and  $\sum_{U \in \mathcal{U}} r_U^* f = f \circ r$ . By Proposition 4.8, we have  $\pi(f) = \sum_{U \in \mathcal{U}} \psi(\sqrt{r_U^* f}) \psi(\sqrt{r_U^* f})^*$ . The converse of the first statement follows from Definition 2.10 and Remark 2.16. The second statement follows easily from the construction of  $\mathcal{G}$ .  $\square$

**4.1. The  $C^*$ -algebra generated by a Toeplitz representation.** In this subsection we provide some technical results which may prove useful in using our descriptions of the  $C^*$ -algebras associated to topological graphs. Proposition 4.12 aids in constructing representations; and Proposition 4.16 provides a well-behaved collection of spanning elements for the image of any Toeplitz representation of  $E$ , and also a formula for computing products of these spanning elements.

To construct Toeplitz representations of a topological graph, one needs to build linear maps  $\psi : C_c(E^1) \rightarrow B$  that are  $\|\cdot\|_{C_0(E^0)}$ -bounded. The following technical result reduces the task to defining  $\psi$  on functions that are dense in supremum norm on  $C_0(U)$  for a suitable family of open  $s$ -sections  $U$ .

**Proposition 4.12.** *Let  $E$  be a topological graph, let  $\mathcal{B}$  be an open base for the topology on  $E^1$  consisting of  $s$ -sections, and let  $\mathcal{F} \subset C_c(E^1)$  be a collection of nonnegative functions such that  $\text{osupp}(x)$  is an  $s$ -section for all  $x \in \mathcal{F}$ . Suppose that for each  $U \in \mathcal{B}$ ,*

$$(4.3) \quad \text{span}\{x \in \mathcal{F} : \text{osupp}(x) \subset U\} \text{ is } \|\cdot\|_\infty\text{-dense in } C_0(U).$$

*Then  $X_0 := \text{span } \mathcal{F}$  is dense in  $X(E)$ . Let  $B$  be a  $C^*$ -algebra. Suppose that  $\psi_0 : X_0 \rightarrow B$  is a linear map, that  $\pi : C_0(E^0) \rightarrow B$  is a homomorphism, and that the pointwise product  $xy$  satisfies*

$$(4.4) \quad \pi(\widehat{\sqrt{xy}}) = \psi_0(x)^* \psi_0(y) \quad \text{for all } x, y \in \mathcal{F}.$$

*Then  $\psi_0$  extends uniquely to a bounded linear map  $\psi$  on  $C_c(E^1)$ . Suppose that  $\mathcal{G} \subset C_0(E^0)$  generates  $C_0(E^0)$ . If the extension  $\psi$  satisfies*

$$(4.5) \quad \psi(f \cdot x) = \pi(f) \psi_0(x) \quad \text{for all } f \in \mathcal{G} \text{ and } x \in \mathcal{F},$$

*then  $(\psi, \pi)$  is a Toeplitz representation of  $E$ .*

**Proof.** Fix  $x \in C_c(E^1)$ . Let  $K = \text{supp}(x)$ . For each  $e \in K$ , there exists an open  $s$ -section  $N_e$  containing  $e$ , such that  $N_e \in \mathcal{B}$ . Since  $K$  is compact, there is a finite subset  $F \subset K$ , such that  $\{N_e : e \in F\}$  covers  $K$ . By Remark 2.9, there exists a finite collection of functions  $\{x_e : e \in F\} \subset C_c(E^1)$ , such that  $\text{osupp}(x_e) \subset N_e \cap K$  for all  $e \in F$ , and  $\sum_{e \in F} x_e = x$ . Fix  $e \in F$ . Since  $\text{osupp}(x_e) \subset N_e$ , there exists a sequence  $(x_{e,n}) \subset X_0 \cap C_0(N_e)$  converging to  $x_e$  in supremum norm. That  $(x_{e,n})$  and  $x_e$  vanish off the  $s$ -section  $N_e$  imply that  $\|x_{e,n} - x_e\|_{C_0(E^0)} = \sup_{e \in E^1} |x_{e,n} - x_e|$ . Hence  $\sum_{e \in F} x_{e,n} \rightarrow x$  in  $\|\cdot\|_{C_0(E^0)}$  norm. Therefore  $X_0$  is dense in  $X(E)$ .

Fix  $x, y \in \mathcal{F}$ . Since  $x, y$  are nonnegative,  $\widehat{\sqrt{xy}} = \langle x, y \rangle_{C_0(E^0)}$ . Hence (4.4) implies that  $\pi(\langle x, y \rangle_{C_0(E^0)}) = \psi_0(x)^* \psi_0(y)$ . Linearity of  $\psi_0$  and  $\pi$  gives  $\pi(\langle x, y \rangle_{C_0(E^0)}) = \psi_0(x)^* \psi_0(y)$ , for all  $x, y \in X_0$ . By Remark 3.4,  $\psi_0$  is bounded, so extends uniquely to  $\tilde{\psi} : X(E) \rightarrow B$ . Then  $\psi := \tilde{\psi}|_{C_c(E^1)}$  is the required map. Continuity and (4.5) show that  $(\psi, \pi)$  is a Toeplitz representation of  $E$ .  $\square$

**Remark 4.13.** To prove Proposition 4.12 we showed that Equation (4.3) implies that  $X_0$  is dense in  $X(E)$  under the  $\|\cdot\|_{C_0(E^0)}$  norm, and then deduced that  $(\psi, \pi)$  extends to a Toeplitz representation of  $E$ . So replacing Equation (4.3) with the hypothesis that  $X_0$  is dense in  $X(E)$  would yield a formally stronger result. However, Equation (4.3) is in many instances easier to check.

Our next proposition provides a description of the  $C^*$ -algebra generated by a Toeplitz representation of  $E$  in terms of a spanning family which captures many of the key properties of the usual spanning family in the Toeplitz algebra of a directed graph. We first need some notation and two technical lemmas.

Recall that  $E^n$  denotes the space

$$\{\mu = \mu_1 \dots \mu_n : \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\}$$

of paths of length  $n$  in  $E$ . We define  $r, s : E^n \rightarrow E^0$  by  $r(\mu) = r(\mu_1)$  and  $s(\mu) = s(\mu_n)$ , and we give  $E^n$  the relative topology inherited from the product space  $\prod_{i=1}^n E^1$ . For  $x \in C_c(E^n)$  and  $f \in C_0(E^0)$  we define  $f \cdot x, x \cdot f \in C_c(E^n)$  by  $(f \cdot x)(\mu) = f(r(\mu))x(\mu)$  and  $(x \cdot f)(\mu) = x(\mu)f(s(\mu))$ .

For  $x_1, \dots, x_n \in C_c(E^1)$ , we define  $x_1 \diamond \dots \diamond x_n \in C_c(E^n)$  by

$$(x_1 \diamond \dots \diamond x_n)(\mu) = \prod_{i=1}^n x_i(\mu_i) \quad \text{for } \mu = \mu_1 \dots \mu_n \in E^n.$$

We use the symbol  $\diamond$  for this operation to distinguish it from the pointwise product of elements of  $C_c(E^1)$  appearing in, for example, Equation (4.4).

The second assertion of the following technical result follows from the discussion preceding [9, Proposition 3.3] together with [5, Proposition 1.27] (see also [10, Proposition 9.7]). We include the result and a simple proof here for completeness.

Suppose that  $x, y \in C_c(E^n)$  are supported on  $s$ -sections. Then there is a unique  $H(x, y) \in C_c(E^0)$  that vanishes on  $E^0 \setminus \{s(\mu) : x(\mu)y(\mu) \neq 0\}$  and satisfies

$$H(x, y)(s(\mu)) = \overline{x(\mu)}y(\mu) \text{ whenever } x(\mu)y(\mu) \neq 0.$$

**Lemma 4.14.** *Let  $E$  be a topological graph. Suppose that  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  are supported on  $s$ -sections. Let  $(\psi, \pi)$  be a Toeplitz representation of  $E$ . Let  $x = x_1 \diamond \dots \diamond x_n$  and  $y = y_1 \diamond \dots \diamond y_n$ . Then*

$$\pi(H(x, y)) = \psi(x_n)^* \dots \psi(x_1)^* \psi(y_1) \dots \psi(y_n).$$

*If  $x = y$  then  $\prod_{i=1}^n \psi(x_i) = \prod_{i=1}^n \psi(y_i)$ .*

**Proof.** By Proposition 4.4,  $(\psi, \pi)$  extends to a Toeplitz representation of  $X(E)$ . We have  $H(x, y) = \langle x_n, H(x_1 \diamond \dots \diamond x_{n-1}, y_1 \diamond \dots \diamond y_{n-1}) \cdot y_n \rangle$ , and hence

$$\pi(H(x, y)) = \psi(x_n)^* \pi(H(x_1 \diamond \dots \diamond x_{n-1}, y_1 \diamond \dots \diamond y_{n-1})) \psi(y_n).$$



The first assertion now follows by induction. For the second assertion, we use the first to see that

$$\begin{aligned} \left( \prod_{i=1}^n \psi(x_i) - \prod_{i=1}^n \psi(y_i) \right)^* & \left( \prod_{i=1}^n \psi(x_i) - \prod_{i=1}^n \psi(y_i) \right) \\ & = \pi(H(x, x) - H(x, y) - H(y, x) + H(y, y)), \end{aligned}$$

which is equal to zero since  $x = y$ . □

**Lemma 4.15.** *Let  $E$  be a topological graph. Suppose that  $x_1, \dots, x_n \in C_c(E^1)$  are supported on  $s$ -sections, and fix  $f \in C_0(E^0)$ . Then there exists  $\tilde{f} \in C_c(E^0)$  such that*

$$(4.6) \quad \tilde{f}(s(\mu)) = f(r(\mu)) \quad \text{whenever } (x_1 \diamond \dots \diamond x_n)(\mu) \neq 0.$$

For any such  $\tilde{f}$ , we have  $f \cdot (x_1 \diamond \dots \diamond x_n) = (x_1 \diamond \dots \diamond x_n) \cdot \tilde{f}$ , and  $\pi(f) \prod_{i=1}^n \psi(x_i) = \prod_{i=1}^n \psi(x_i) \pi(\tilde{f})$  for any Toeplitz representation  $(\psi, \pi)$  of  $E$ .

**Proof.** The second assertion will follow from the first by definition of  $f \cdot (x_1 \diamond \dots \diamond x_n)$  and  $(x_1 \diamond \dots \diamond x_n) \cdot \tilde{f}$ . The final assertion will then follow from Lemma 4.14. So we just need to prove the first assertion. Let  $x := x_1 \diamond \dots \diamond x_n$ . Fix  $f \in C_0(E^0)$ . Since  $K := \text{supp}(x) \subseteq E^n$  is an  $s$ -section, there is a well-defined continuous function from  $s(K)$  to  $r(K)$  given by  $s(\mu) \mapsto r(\mu)$  for  $\mu \in K$ . So there is a continuous function  $f_0 \in C(s(K))$  given by  $f_0(s(\mu)) = f(r(\mu))$  for all  $\mu \in K$ . Since  $s(K)$  is compact, an application of the Tietze extension theorem shows that  $\tilde{f}$  has an extension  $\tilde{f} \in C_0(E^0)$ , which satisfies (4.6) by definition. □

**Proposition 4.16.** *Let  $E$  be a topological graph and let  $(\psi, \pi)$  be a Toeplitz representation of  $E$ . Let  $\mathcal{B}$  be a base for the topology on  $E^1$  consisting of open  $s$ -sections, and let  $\mathcal{F} \subseteq C_c(E^1)$  be a collection of nonnegative functions such that  $\text{supp}(f)$  is an  $s$ -section for each  $f \in \mathcal{F}$ . Then:*

- (1)  $C^*(\psi, \pi)$  is densely spanned by elements of the form

$$\psi(x_1) \cdots \psi(x_n) \pi(f) \psi(y_m)^* \cdots \psi(y_1)^*$$

where  $m, n \geq 0$ ,  $f \in C_c(E^0)$ , the  $x_i, y_j$  all belong to  $\mathcal{F}$ , each of the sets  $s(\text{osupp}(x_i)) \cap r(\text{osupp}(x_{i+1}))$  and  $s(\text{osupp}(y_i)) \cap r(\text{osupp}(y_{i+1}))$  is nonempty, and  $s(\text{osupp}(x_n)) \cap s(\text{osupp}(y_m)) \cap \text{osupp}(f) \neq \emptyset$ .

- (2) Consider a pair of spanning elements

$$\begin{aligned} & \psi(w_1) \cdots \psi(w_m) \pi(f) \psi(x_n)^* \cdots \psi(x_1)^*, \\ & \psi(y_1) \cdots \psi(y_p) \pi(g) \psi(z_q)^* \cdots \psi(z_1)^* \end{aligned}$$

as in (1) with  $p \geq n$ . Let  $x = x_1 \diamond \dots \diamond x_n$  and  $y = y_1 \diamond \dots \diamond y_n$ . Fix  $k \in C_0(E^0)$  such that  $(fH(x, y)) \cdot (y_{n+1} \diamond \dots \diamond y_p) = (y_{n+1} \diamond \dots \diamond y_p) \cdot k$

as in Lemma 4.15. Then

$$\begin{aligned} & (\psi(w_1) \dots \psi(w_m) \pi(f) \psi(x_n)^* \dots \psi(x_1)^*) \\ & \quad \cdot (\psi(y_1) \dots \psi(y_p) \pi(g) \psi(z_q)^* \dots \psi(z_1)^*) \\ & = \psi(w_1) \dots \psi(w_m) \psi(y_{n+1}) \dots \psi(y_p) \pi(kg) \psi(z_q)^* \dots \psi(z_1)^*. \end{aligned}$$

**Remark 4.17.** Consider the situation of Proposition 4.16(2) but with  $p < n$ . Let  $x = x_1 \diamond \dots \diamond x_p$  and  $y = y_1 \diamond \dots \diamond y_p$ , and fix  $k' \in C_0(E^0)$  such that  $(H(x, y)g) \cdot (x_{p+1} \diamond \dots \diamond x_n) = (x_{p+1} \diamond \dots \diamond x_n) \cdot k'$ . Taking adjoints in Proposition 4.16(2) gives

$$\begin{aligned} & (\psi(w_1) \dots \psi(w_m) \pi(f) \psi(x_n)^* \dots \psi(x_1)^*) \\ & \quad \cdot (\psi(y_1) \dots \psi(y_p) \pi(g) \psi(z_q)^* \dots \psi(z_1)^*) \\ & = \psi(w_1) \dots \psi(w_m) \pi(fk') \psi(x_n)^* \dots \psi(x_{p+1}) \psi(z_q)^* \dots \psi(z_1)^*. \end{aligned}$$

**Proof of Proposition 4.16.** (1) By Proposition 4.4,  $(\tilde{\psi}, \pi)$  is a Toeplitz representation of  $X(E)$ , where  $\tilde{\psi}$  is the unique extension of  $\psi$  to  $X(E)$ . The argument of [6, Proposition 2.7] shows that  $C^*(\psi, \pi)$  is densely spanned by elements of the form  $\psi(x_1) \dots \psi(x_n) \pi(f) \psi(y_m)^* \dots \psi(y_1)^*$  where each  $x_i, y_i \in C_c(E_1)$  and  $f \in C_c(E^0)$ . Fix  $x_1, x_2 \in C_c(E^1)$  with

$$s(\text{osupp}(x_1)) \cap r(\text{osupp}(x_2)) = \emptyset.$$

Then

$$\begin{aligned} \|\psi(x_1)\psi(x_2)\|^2 &= \|\psi(x_2)^*\psi(x_1)^*\psi(x_1)\psi(x_2)\| \\ &= \|\psi(x_2)^*\pi(\langle x_1, x_1 \rangle_{C_0(E^0)})\psi(x_2)\| \\ &= \|\psi(x_2)^*\psi(\langle x_1, x_1 \rangle_{C_0(E^0)} \cdot x_2)\| = 0. \end{aligned}$$

Similarly,  $\psi(x_1)\psi(x_2)^* = 0$  whenever  $s(\text{osupp}(x_1)) \cap s(\text{osupp}(x_2)) = \emptyset$ . So elements of the form

$$\psi(x_1) \dots \psi(x_m) \pi(f) \psi(y_m)^* \dots \psi(y_1)^*$$

where

- $f \in C_c(E^0)$ ,
- each  $s(\text{osupp}(x_i)) \cap r(\text{osupp}(x_{i+1})) \neq \emptyset$ ,
- each  $s(\text{osupp}(y_i)) \cap r(\text{osupp}(y_{i+1})) \neq \emptyset$ , and
- $s(\text{osupp}(x_n)) \cap s(\text{osupp}(y_m)) \cap \text{osupp}(f) \neq \emptyset$

span a dense subspace of  $C^*(\psi, \pi)$ . Since Proposition 4.12 implies that  $X_0 = \text{span } \mathcal{F}$  is dense in  $X(E)$  and hence in  $C_c(E^1)$ , the first assertion follows.

(2) Lemma 4.14 implies that

$$\pi(f)\psi(x_n)^* \dots \psi(x_1)^*\psi(y_1) \dots \psi(y_p) = \pi(fH(x, y))\psi(y_{n+1}) \dots \psi(y_p).$$

The result now follows from Lemma 4.15.  $\square$

**Example 4.18.** Let  $E$  be a directed graph regarded as a topological graph under the discrete topology. Let  $\mathcal{G} = \{\delta_v : v \in E^0\}$  and  $\mathcal{F} = \{\delta_e : e \in E^1\}$ . Then  $\mathcal{G}$  and  $\mathcal{F}$  satisfy the hypotheses of Proposition 4.12, so we recover as an immediate consequence the isomorphism of the Toeplitz algebra  $\mathcal{T}(E)$  of the graph bimodule (see [4] and [5]) with the Toeplitz algebra  $\mathcal{TC}^*(E)$  of the graph  $E$ . The usual spanning family and multiplication rule for  $\mathcal{TC}^*(E)$  follows from Proposition 4.16 applied to the same  $\mathcal{F}$ .

**Remark 4.19.** The multiplication formula of Proposition 4.16(2) has the drawback that the element  $k$  has no explicit formula in terms of the  $x_i$  the  $y_i$  and the function  $f$ ; it is obtained by an application of the Tietze extension theorem (see Lemma 4.15). However, in practise there will frequently be a natural choice for  $k$ . Suppose, for example, that  $E^1$  is totally disconnected. Then  $\mathcal{F}$  can be taken to consist of characteristic functions of compact open  $s$ -sections. We can then take  $k$  to be the function that is identically zero off  $s(\text{supp}(y_{n+1} \diamond \cdots \diamond y_p))$  and satisfies  $k(s(\mu)) = f(r(\mu))H(x, y)(r(\mu))$  whenever  $\mu \in \text{supp}(y_{n+1} \diamond \cdots \diamond y_p)$ ; this  $k$  is continuous because  $s(\text{supp}(y_{n+1} \diamond \cdots \diamond y_p))$  is clopen.

### 5. The topological graph arising from the shift map on the infinite-path space

In this section we discuss how our results apply to the topological graph  $\widehat{E}$  arising from the shift map on the infinite-path space  $E^\infty$  of a row-finite directed graph  $E$  with no sources. It is known that  $\mathcal{O}_{X(\widehat{E})}$  is isomorphic to  $C^*(E)$  (it could be recovered from [2, 3]) but existing proofs use the universal property of  $C^*(E)$  to induce a homomorphism from  $C^*(E)$  to  $\mathcal{O}_{X(\widehat{E})}$ , invoke the gauge-invariant uniqueness theorem for  $C^*(E)$  to establish injectivity, and then argue surjectivity by hand. It takes some work to show using the universal property of  $\mathcal{O}_{X(\widehat{E})}$  that there is a homomorphism going in the other way.

Let  $E$  be a row-finite directed graph with no sources. Let  $E^* = \bigcup_{n \geq 0} E^n$  and let  $E^\infty = \{z \in \prod_{i=1}^\infty E^1 : s(z_i) = r(z_{i+1}), \text{ for all } i = 1, 2, \dots\}$ . We view  $E^\infty$  as a topological space under the subspace topology coming from the ambient space  $\prod_{i=1}^\infty E^1$ . For any  $\mu \in E^* \setminus E^0$ , we define the cylinder set  $Z(\mu) = \{z \in E^\infty : z_1 = \mu_1, \dots, z_{|\mu|} = \mu_{|\mu|}\}$ . For  $v \in E^0$  we define  $Z(v) = \{z \in E^\infty : r(z_1) = v\}$ . Since  $E$  has no sources, each  $Z(\mu)$  is nonempty. The space  $E^\infty$  is a locally compact Hausdorff space with a base of compact open sets  $\{Z(\mu) : \mu \in E^*\}$  ([8, Corollary 2.2]).

We construct a topological graph  $\widehat{E} = (\widehat{E}^0, \widehat{E}^1, \widehat{r}, \widehat{s})$ . Let  $\widehat{E}^0 = \widehat{E}^1 = E^\infty$ . Define  $\widehat{r}$  to be the identity map, and define  $\widehat{s}(z) = (z_2, z_3, \dots)$  for all  $z \in \widehat{E}^1$ . Since  $Z(\mu)$  is a compact open  $\widehat{s}$ -section whenever  $\mu \notin E^0$ , the map  $\widehat{s}$  is a local homeomorphism and hence  $\widehat{E}$  is a topological graph.

For the following result recall that a Cuntz–Krieger  $E$ -family is a family  $\{p_v, s_e\}$  satisfying the relations described in the first paragraph of the introduction: that is, the numbered relations (1) and (2) and the additional relation  $p_v = \sum_{r(e)=v} s_e s_e^*$  whenever  $0 < |r^{-1}(v)| < \infty$ .

**Proposition 5.1.** *Let  $E$  be a row-finite directed graph with no sources, and let  $\widehat{E}$  be the topological graph described above.*

- (1) *Let  $(\psi, \pi)$  be a covariant Toeplitz representation of  $\widehat{E}$ . For  $v \in E^0$  define  $q_v := \pi(\chi_{Z(v)})$  and for  $e \in E^1$  define  $t_e := \psi(\chi_{Z(e)})$ . Then the  $q_v$  and the  $t_e$  form a Cuntz–Krieger  $E$ -family.*
- (2) *Let  $\{q_v : v \in E^0\}, \{t_e : e \in E^1\}$  be a Cuntz–Krieger  $E$ -family in a  $C^*$ -algebra  $B$ . Then there is a unique covariant Toeplitz representation  $(\psi, \pi)$  of  $\widehat{E}$  such that  $\psi(\chi_{Z(e\mu)}) = t_{e\mu} t_\mu^*$ , and  $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$  for all  $e \in E^1, \mu \in E^*$ .*
- (3) *Let  $(j_1, j_0)$  be the universal covariant Toeplitz representation of  $\widehat{E}$  in  $\mathcal{O}(\widehat{E})$ , and let  $\{p_v, s_e : v \in E^0, e \in E^1\}$  be the Cuntz–Krieger  $E$ -family generating  $C^*(E)$ . There is an isomorphism  $\mathcal{O}(\widehat{E}) \cong C^*(E)$  which carries each  $j_1(\chi_{Z(e\mu)})$  to  $s_{e\mu} s_\mu^*$ , and each  $j_0(\chi_{Z(\mu)})$  to  $s_\mu s_\mu^*$ .*

**Proof of Proposition 5.1(1).** The  $q_v$  are mutually orthogonal projections because the  $\chi_{Z(v)}$  are. For  $\mu \in E^* \setminus E^0$ , the set  $Z(\mu)$  is a compact open  $s$ -section. Thus for  $e \in E^1$ , relation (2) of Definition 2.2 implies that  $t_e^* t_e = \pi(\widehat{\chi_{Z(e)}}) = \pi(\chi_{Z(s(e))}) = q_{s(e)}$ . For  $\mu \in E^* \setminus E^0$ ,  $(\{Z(\mu)\}, Z(\mu))$  is an open local  $\widehat{r}$ -fibration. Since  $\text{supp}(\chi_{Z(\mu)}) = Z(\mu)$  and  $(\psi, \pi)$  is covariant, Corollary 2.15 implies that

$$(5.1) \quad \pi(\chi_{Z(\mu)}) = \psi\left(\sqrt{r_{Z(\mu)}^* \chi_{Z(\mu)}}\right) \psi\left(\sqrt{r_{Z(\mu)}^* \chi_{Z(\mu)}}\right)^* = \psi(\chi_{Z(\mu)}) \psi(\chi_{Z(\mu)})^*.$$

So for  $v \in E^0$ , we have

$$q_v = \pi(\chi_{Z(v)}) = \sum_{r(e)=v} \pi(\chi_{Z(e)}) = \sum_{r(e)=v} \psi(\chi_{Z(e)}) \psi(\chi_{Z(e)})^* = \sum_{r(e)=v} t_e t_e^*. \quad \square$$

**Proof of Proposition 5.1(2).** Let  $\mathcal{G} := \{\chi_{Z(\mu)} : \mu \in E^* \setminus E^0\} \subseteq C_0(\widehat{E}^0)$ . Then  $\text{span } \mathcal{G}$  is a dense  $*$ -subalgebra of  $C_0(\widehat{E}^0)$ . We aim to define a map  $\pi_0 : \text{span } \mathcal{G} \rightarrow B$  by  $\pi_0(\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)}) = \sum_{i=1}^n \alpha_i t_{\mu_i} t_{\mu_i}^*$ . We check that  $\pi_0$  is well-defined. It suffices to prove that  $\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)} = 0$  implies  $\sum_{i=1}^n \alpha_i t_{\mu_i} t_{\mu_i}^* = 0$ , where the  $\mu_i$  are distinct. Since  $E$  is row-finite and has no sources,

$$\begin{aligned} \pi_0\left(\sum_{e \in r^{-1}(s(\mu))} \chi_{Z(\mu e)}\right) &= \sum_{e \in r^{-1}(s(\mu))} t_{\mu e} t_{\mu e}^* \\ &= t_\mu \left(\sum_{e \in r^{-1}(s(\mu))} t_e t_e^*\right) t_\mu^* = t_\mu t_\mu^* = \pi_0(\chi_{Z(\mu)}), \end{aligned}$$

so we can assume that the  $\mu_i$  have the same length. It follows that the  $\chi_{Z(\mu_i)}$  are mutually orthogonal nonzero projections and hence each  $\alpha_i = 0$ . So  $\pi_0$  is well-defined. It is obvious that  $\pi_0$  is a linear map preserving the involution. [10, Corollary 1.15] implies that  $\pi_0$  is a homomorphism. Now we show that  $\pi_0$  is norm decreasing. Fix  $\mu_1, \dots, \mu_n$ . We can assume that the  $\mu_i$  are distinct and have the same length. Then

$$\begin{aligned} \left\| \pi_0 \left( \sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)} \right) \right\|^2 &= \left\| \sum_{i=1}^n |\alpha_i|^2 t_{\mu_i} t_{\mu_i}^* \right\|^2 \\ &\leq (\max_i |\alpha_i|^2) \left\| \sum_{i=1}^n t_{\mu_i} t_{\mu_i}^* \right\|^2 \\ &\leq \max_i |\alpha_i|^2 = \left\| \sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)} \right\|^2, \end{aligned}$$

so  $\pi_0$  is norm decreasing. Thus we obtain a unique homomorphism

$$\pi : C_0(\widehat{E}^0) \rightarrow C^*(E)$$

by  $\pi(\chi_{Z(\mu)}) = t_{\mu} t_{\mu}^*$  for all  $\mu \in E^*$ .

We next aim to define a linear map  $\psi : C_c(\widehat{E}^1) \rightarrow B$  by extension of the formula  $\psi(\chi_{Z(e\mu)}) = t_{e\mu} t_{\mu}^*$ , and to show that the pair  $(\psi, \pi)$  is a Toeplitz representation of  $\widehat{E}$ . To do so, we will apply Proposition 4.12, so we need to set up the rest of the elements of the statement. Let

$$\mathcal{B} := \{Z(\mu) : \mu \in E^* \setminus E^0\}$$

and let  $\mathcal{F} := \{\chi_{Z(e\mu)} : e \in E^1, \mu \in E^*\} \subseteq C_c(E^1)$ . Certainly  $\mathcal{F}$  and  $\mathcal{B}$  satisfy Equation (4.3). Similarly to the construction of  $\pi_0$ , there is a well-defined linear map  $\psi_0 : \text{span } \mathcal{F} \rightarrow B$  satisfying

$$\psi_0 \left( \sum_{i=1}^n \alpha_i \chi_{Z(e_i \mu_i)} \right) = \sum_{i=1}^n \alpha_i t_{e_i \mu_i} t_{\mu_i}^*.$$

Fix  $x = \chi_{Z(e\mu)}$  and  $y = \chi_{Z(f\nu)}$  in  $\mathcal{F}$ . We verify Equation (4.4). For this, observe that

$$\widehat{\sqrt{xy}} = \begin{cases} \chi_{Z(\mu)} & \text{if } e\mu = f\nu\mu' \\ \chi_{Z(\nu)} & \text{if } f\nu = e\mu\nu' \\ 0 & \text{otherwise.} \end{cases}$$

Then calculate:

$$(t_{e\mu} t_{\mu}^*)^* t_{f\nu} t_{\nu}^* = t_{\mu} t_{e\mu}^* t_{f\nu} t_{\nu}^* = \begin{cases} t_{\nu} t_{\nu}^* & \text{if } f\nu = e\mu\nu' \\ t_{\mu} t_{\mu}^* & \text{if } e\mu = f\nu\mu' \\ 0 & \text{otherwise.} \end{cases}$$

This establishes Equation (4.4). So Proposition 4.12 shows that  $\psi_0$  extends uniquely to a linear map  $\psi : C_c(\widehat{E}^1) \rightarrow B$ . A similar calculation establishes

Equation (4.5). Proposition 4.12 implies that  $(\psi, \pi)$  is a Toeplitz representation of  $E$ .

It remains to check covariance. Since the range map is a homeomorphism onto  $\widehat{E}^0$  we can apply Corollary 2.15 with  $\mathcal{G}$  as in the proof of part (2) and the local  $\widehat{r}$ -fibrations  $\{(\{Z(\mu)\}, Z(\mu)) : \mu \in E^* \setminus E^0\}$  to see that  $(\psi, \pi)$  is covariant.  $\square$

**Proof of Proposition 5.1(3).** We show  $C^*(E)$  has the universal property of  $\mathcal{O}(\widehat{E})$  and then invoke Theorem 2.13. Proposition 5.1(2) yields a covariant Toeplitz representation  $(\theta_1, \theta_0)$  of  $\widehat{E}$  in  $C^*(E)$  such that

$$\begin{aligned}\theta_1(\chi_{Z(e\mu)}) &= s_{e\mu}s_\mu^* && \text{for all } e \in E^1, \mu \in E^*, \\ \theta_0(\chi_{Z(\mu)}) &= s_\mu s_\mu^* && \text{for all } \mu \in E^*.\end{aligned}$$

Fix a covariant Toeplitz representation  $(\psi, \pi)$  of  $\widehat{E}$  in a  $C^*$ -algebra  $B$ . Then Proposition 5.1(1) gives a Cuntz–Krieger  $E$ -family

$$\{q_v := \pi(\chi_{Z(v)}), t_e := \psi(\chi_{Z(e)}) : v \in E^0, e \in E^1\}$$

in  $B$ . So [10, Proposition 1.21] gives a homomorphism  $\rho : C^*(E) \rightarrow B$  such that  $\rho(p_v) = q_v$ , and  $\rho(s_e) = t_e$ . An induction on the length of  $\mu$  using Equation (5.1) shows that  $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$ . For  $e \in E^1$ , and  $\mu \in E^*$ , we have  $\rho \circ \theta_1(\chi_{Z(e\mu)}) = t_e t_\mu t_\mu^* = \psi(\chi_{Z(e)})\pi(\chi_{Z(\mu)}) = \psi(\chi_{Z(e\mu)})$ . For  $\mu \in E^* \setminus E^0$ , we have  $\rho \circ \theta_0(\chi_{Z(\mu)}) = t_\mu t_\mu^* = \pi(\chi_{Z(\mu)})$ . Hence  $\rho \circ \theta_1 = \psi$ , and  $\rho \circ \theta_0 = \pi$ . Since the image of  $(\theta_1, \theta_0)$  generates  $C^*(E)$ , Theorem 2.13 implies that there is an isomorphism  $\mathcal{O}(E) \cong C^*(E)$  which carries each  $j_1(\chi_{Z(e\mu)})$  to  $s_{e\mu}s_\mu^*$ , and each  $j_0(\chi_{Z(\mu)})$  to  $s_\mu s_\mu^*$ .  $\square$

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This paper is available via <http://nyjm.albany.edu/j/2014/20-26.html>.