The equivalence relations of local homeomorphisms and Fell algebras

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ABSTRACT. We study the groupoid C^*-algebra associated to the equivalence relation induced by a quotient map on a locally compact Hausdorff space. This C^*-algebra is always a Fell algebra, and if the quotient space is Hausdorff, it is a continuous-trace algebra. We show that the C^*-algebra of a locally compact, Hausdorff and principal groupoid is a Fell algebra if and only if the groupoid is one of these relations, extending a theorem of Archbold and Somerset about étale groupoids. The C^*-algebras of these relations are, up to Morita equivalence, precisely the Fell algebras with trivial Dixmier–Douady invariant as recently defined by an Huef, Kumjian and Sims. We use twisted groupoid algebras to provide examples of Fell algebras with nontrivial Dixmier–Douady invariant.

CONTENTS

1. Introduction

Important classes of type I C^*-algebras include the continuous-trace algebras and the liminary or CCR algebras, and lately there has been renewed interest in a family of liminary algebras called Fell algebras [12, 5, 1, 13]. A C^*-algebra is a *Fell algebra* if for each $\pi \in \hat{A}$, there exists a positive element a such that $\rho(a)$ is a rank-one projection for all ρ in a neighbourhood of π in $\hat{A}$. Fell algebras were named by Archbold and Somerset [2] in respect of Fell's contributions [9], and had been previously studied by Pedersen as "algebras of type I_0" [25, §6]. Proposition 4.5.4 of [7] says that a C^*-algebra has continuous trace if and only if it is a Fell algebra and its spectrum is Hausdorff.

The Dixmier–Douady class $\delta_{\mathrm{DD}}(A)$ of a continuous-trace algebra A identifies A up to Morita equivalence, and $\delta_{\mathrm{DD}}(A) = 0$ if and only if A is Morita equivalent to a commutative C^*-algebra [29, Theorem 5.29]. An Huef, Kumjian and Sims have recently developed an analogue of the Dixmier–Douady classification for Fell algebras [13]. Their Dixmier–Douady invariant $\delta(A)$ vanishes if and only if A is Morita equivalent to the groupoid C^*-algebra $C^*(R(\psi))$ of the equivalence relation associated to a local homeomorphism ψ of a Hausdorff space onto the spectrum $\hat{A}$. (Since this theorem was not explicitly stated in [13], we prove it here as Theorem 6.1.)

So the theory in [13] identifies the C^*-algebras $C^*(R(\psi))$ as an interesting family of model algebras. Here we investigate the structure of the algebras $C^*(R(\psi))$ and their twisted analogues, and use them to provide examples of Fell algebras exhibiting certain kinds of behaviour. In particular, we will produce some concrete examples of Fell algebras with nonvanishing Dixmier–Douady invariant.

After some background in §2, we discuss in §3 the topological spaces that arise as the spectra of Fell algebras. In §4 we study the locally compact Hausdorff equivalence relation $R(\psi)$ associated to a surjection $\psi : Y \to X$ defined on a locally compact Hausdorff space Y. We show how extra properties of ψ influence the structure of the groupoid $R(\psi)$. We show in particular that if ψ is a surjective local homeomorphism of Y onto a topological space X, then $R(\psi)$ is étale, principal and Cartan, with orbit space naturally homeomorphic to X. We also show that every principal Cartan groupoid has the form $R(\psi)$ for some quotient map ψ.

In §5, we show that the twisted groupoid C^*-algebras of the groupoids $R(\psi)$ give many examples of Fell algebras. We also show, extending a result of Archbold and Somerset for étale groupoids [2], that the C^*-algebra $C^*(G)$ of a principal groupoid G is a Fell algebra if and only if G is topologically isomorphic to the relation $R(q)$ determined by the quotient map q of the unit space $G^{(0)}$ onto $G^{(0)}/G$. We then illustrate our results with a discussion of the path groupoids of directed graphs, and an example from [11] which fails to be Fell in a particularly delicate way.

In Theorem 6.1, we prove that the Dixmier–Douady class $\delta(A)$ of a Fell algebra A vanishes if and only if A is Morita equivalent to some $C^*(R(\psi))$. We then use this to partially resolve a problem left open in [13, Remark 7.10]: when A has continuous trace, how is $\delta(A)$ related to the usual Dixmier–Douady invariant $\delta_{\mathrm{DD}}(A)$ of [8, 7, 29]? In Corollary 6.3, we show that when A has continuous trace, $\delta(A) = 0$ if and only if $\delta_{\mathrm{DD}}(A) = 0$. We also show that if A is Fell and $\delta(A) = 0$, then every ideal I in A with continuous trace has $\delta_{\mathrm{DD}}(I) = 0$. Since δ_{DD} is computable, this allows us to recognise some Fell algebras whose invariant is nonzero.

In §7, we describe two examples of Fell algebras which we have found instructive. The first is a Fell algebra whose spectrum fails to be paracompact in any reasonable sense, even though the algebra is separable. The second set of examples are Fell algebras A with nonzero Dixmier–Douady invariant $\delta(A)$ and non-Hausdorff spectrum (so that they are not continuous-trace algebras). We close §7 with a brief epilogue on how we found these examples and what we have learned from them.

We finish with two short appendices. The first concerns the different twisted groupoid algebras appearing in this paper. We mainly use Renault's algebras associated to a 2-cocycle σ on G from [30], but the proof of Theorem 6.1 uses the twisted groupoid algebra associated to a twist Γ over G from [18], and the proof of Theorem 5.1 uses yet another version from [21]. In Appendix A, we show that when Γ is the twist associated to a continuous cocycle, the three reduced C^*-algebras are isomorphic. In the last appendix, we describe the continuous-trace ideal in an arbitrary C^*-algebra. In the end, we did not need this result, but we think it may be of some general interest: we found it curious that the ideas which work for transformation group algebras in [10, Corollary 18] and [15, Theorem 3.10] work equally well in arbitrary C^*-algebras.

2. Notation and background

A *groupoid* G is a small category in which every morphism is invertible. We write s and r for the domain and range maps in G. The set $G^{(0)}$ of objects in G is called the *unit space*, and we frequently identify a unit with the identity morphism at that unit. A groupoid is *principal* if there is at most one morphism between each pair of units.

A *topological groupoid* is a groupoid equipped with a topology on the set of morphisms such that the composition and inverse maps are continuous. A topological groupoid G is *étale* if the map r (equivalently, s) is a local homeomorphism. The unit space of an étale groupoid is open in G, and the sets $s^{-1}(u)$ and $r^{-1}(u)$ are discrete for every $u \in G^{(0)}$.

Suppose G is a topological groupoid. Then the *orbit* of $u \in G^{(0)}$ is $[u] := r(s^{-1}(u))$. For $u, v \in G^{(0)}$ we write $u \sim v$ if $[u] = [v]$, and then $\sim$ is an equivalence relation on $G^{(0)}$. We write $q : G^{(0)} \to G^{(0)}/G := G^{(0)}/\sim$ for

the quotient map onto the orbit space. If G is étale, then r is open, and then the quotient map is also open because $q^{-1}(q(U)) = r(s^{-1}(U))$ for $U \subset G^{(0)}$.

A topological groupoid G is *Cartan* if every unit $u \in G^{(0)}$ has a neighbourhood N in $G^{(0)}$ which is *wandering* in the sense that $s^{-1}(N) \cap r^{-1}(N)$ has compact closure.

Let G be a locally compact Hausdorff groupoid with left Haar system $\lambda = \{\lambda^u : u \in G^{(0)}\}$. We also need to use the corresponding right Haar system $\{\lambda_u\}$ defined by $\lambda_u(E) = \lambda^u(E^{-1})$. A 2-*cocycle* on G is a function $\sigma : G^{(2)} \to \mathbb{T}$ such that $\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$. As in [4], we assume that all our cocycles are continuous and normalised in the sense that $\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))$, and we write $Z^2(G, \mathbb{T})$ for the set of such cocycles. For such σ, there are both full and reduced twisted groupoid C^*-algebras. Here we work primarily with the reduced version, though in fact the full and reduced groupoid C^*-algebras coincide for the groupoids of interest to us (see Theorem 5.1). Let $C_c(G, \sigma)$ be $C_c(G)$ with involution and convolution given by $f^*(\alpha) = \overline{f(\alpha^{-1})}\sigma(\alpha, \alpha^{-1})$ and

$$(f * g)(\alpha) = \int_G f(\alpha\gamma)g(\gamma^{-1})\sigma(\alpha\gamma, \gamma^{-1}) \, d\lambda^{s(\alpha)}(\gamma);$$

it is shown in [30, Proposition II.1.1] that $C_c(G, \sigma)$ is a $*$-algebra. The invariance of the Haar system gives

$$(2.1) \qquad (f * g)(\alpha) = \int_G f(\beta)g(\beta^{-1}\alpha)\sigma(\beta, \beta^{-1}\alpha) \, d\lambda^{r(\alpha)}(\beta).$$

For $u \in G^{(0)}$, we write Ind_u^σ for the *induced representation* of $C_c(G, \sigma)$ on $L^2(s^{-1}(u), \lambda_u)$ given as follows: for $f \in C_c(G, \sigma)$ and $\xi \in L^2(s^{-1}(u), \lambda_u)$,

$$(\mathrm{Ind}_u^\sigma(f)\xi)(\alpha) = \int_G f(\beta)\xi(\beta^{-1}\alpha)\sigma(\beta, \beta^{-1}\alpha) \, d\lambda^{r(\alpha)}(\beta).$$

As in [30, §II.2], the *reduced twisted groupoid C^*-algebra* $C_\mathrm{r}^*(G, \sigma)$ is the completion of $C_c(G, \sigma)$ with respect to the reduced norm

$$\|f\|_\mathrm{r} = \sup_{u \in G^{(0)}} \|\mathrm{Ind}_u^\sigma(f)\|.$$

As usual, we write $C_\mathrm{r}^*(G)$ for $C_\mathrm{r}^*(G, 1)$ and Ind_u for Ind_u^1.

If G is étale then $r^{-1}(r(\beta))$ is discrete, and (2.1), for example, reduces to

$$f * g(\alpha) = \sum_{r(\alpha)=r(\beta)} f(\beta)g(\beta^{-1}\alpha)\sigma(\beta, \beta^{-1}\alpha) = \sum_{\alpha=\beta\gamma} f(\beta)g(\gamma)\sigma(\beta, \gamma).$$

3. Topological preliminaries

By the standard definition, a topological space X is *locally compact* if every point of X has a compact neighbourhood. When X is Hausdorff, this is equivalent to asking that every point has a neighbourhood base of compact sets [23, Theorem 29.2]. For general, possibly non-Hausdorff spaces, we say that X is *locally locally-compact* if every point of X has a neighbourhood

basis of compact sets (Lemma 3.1 below explains our choice of name). The spectrum of a C^*-algebra is always locally locally-compact (see Lemma 3.2), so this "neighbourhood basis" version of local compactness has attractions for operator algebraists. It also has the advantage, as Munkres points out in [23, page 185], that it is more consistent with other uses of the word "local" in topology. It has been adopted without comment as *the* definition of local compactness in [3, page 149, problem 29][1] and in [33, Definition 1.16]. However, many topology books, such as [16] and [23], and many real-analysis texts, such as [31] and [24], use the standard "every point has a compact neighbourhood" definition, and we will go along with them.

The proof of the following lemma is straightforward.

Lemma 3.1. *A topological space X is locally locally-compact if and only if every open subset of X is locally compact.*

A topological space X is *locally Hausdorff* if every point of X has a Hausdorff neighbourhood. It is straightforward to verify that a locally Hausdorff space is T_1. The following result explains our interest in locally locally-compact and locally Hausdorff spaces.

Lemma 3.2. *If A is a Fell algebra, then the spectrum of A is locally locally-compact and locally Hausdorff.*

Proof. Corollary 3.3.8 of [7] implies that $\hat{A}$ is locally locally-compact, and Corollary 3.4 of [2] that $\hat{A}$ is locally Hausdorff. $\qquad\Box$

Lemma 3.2 has a converse: every second-countable, locally locally-compact and locally Hausdorff space is the spectrum of some Fell algebra [13, Theorem 6.6(2)]. We will later give a shorter proof of this result (see Corollary 5.5).

We warn that a locally locally-compact and locally Hausdorff space may not be paracompact, that compact subsets may not be closed, and that the intersection of two compact sets may not be compact (for example, in the spectrum of the Fell C^*-algebra described in §7.1). So we have found our usual, Hausdorff-based intuition to be distressingly misleading, and we have tried to exercise extreme caution in matters topological.

Locally locally-compact and locally Hausdorff spaces have the following purely topological characterisation.

Proposition 3.3.   (a) *Let $\psi : Y \to X$ be a local homeomorphism of a locally compact Hausdorff space Y onto a topological space X. Then X is locally locally-compact and locally Hausdorff. If Y is second-countable, so is X.*

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[1]Modulo Bourbaki's use of the word "quasi-compact" to mean what we call compact. Dixmier follows Bourbaki, as one should be aware when reading [7].

(b) *Let X be a locally locally-compact and locally Hausdorff space. Then there are a locally compact Hausdorff space Y and a local homeomorphism ψ of Y onto X. If X is second-countable, then we can take Y to be second-countable.*

Proof. For (a) suppose that $\psi : Y \to X$ is a local homeomorphism. Fix $x \in X$ and an open neighbourhood W of x in X. Let $y \in \psi^{-1}(x)$. Since $\psi^{-1}(W)$ is an open neighbourhood of y and ψ is a local homeomorphism, there is a neighbourhood U of y contained in $\psi^{-1}(W)$ such that $\psi|_U$ is a homeomorphism. Since Y is locally compact and Hausdorff, it is locally locally-compact by [23, Theorem 29.2], and there is a compact neighbourhood K of y contained in U. Then $\psi(K)$ is compact and Hausdorff, and because ψ is open, it is a neighbourhood of x contained in W. This proves both that X is locally locally-compact and that X is locally Hausdorff.

Since ψ is continuous and open, the image of a basis for the topology on Y is a basis for the topology on X. Thus X is second-countable if Y is.

For (b), suppose that X is locally locally-compact and locally Hausdorff. Choose an open cover $\mathcal{U}$ of X by Hausdorff sets. Let $Y := \bigsqcup_{U \in \mathcal{U}} U$, and topologise Y by giving each U the subspace topology from X and making each U open and closed in Y. Then Lemma 3.1 implies that Y is locally compact and Hausdorff, and the inclusion maps $U \to X$ combine to give a surjective local homeomorphism $\psi : Y \to X$. If X is second-countable, then we can take the cover to be countable, and Y is also second-countable. $\square$

4. The groupoid associated to a local homeomorphism

Let ψ be a surjective map from a topological space Y to a set X, and take

$$R(\psi) = Y \times_\psi Y := \{(y, z) \in Y \times Y : \psi(y) = \psi(z)\}.$$

With the subspace topology and the operations $r(y, z) = y$, $s(y, z) = z$ and $(x, y)(y, z) = (x, z)$, $R(\psi)$ is a principal topological groupoid with unit space Y.

We want to examine the effect of properties of Y and ψ on the structure of $R(\psi)$. We begin by looking at the orbit space.

Lemma 4.1. *Let ψ be a surjective map from a topological space Y to a set X, and define $h : X \to Y/R(\psi)$ by $h(x) = \psi^{-1}(x)$.*

(a) *The function h is a bijection, and $h \circ \psi$ is the quotient map*

$$q : Y \to Y/R(\psi).$$

(b) *If X is a topological space and ψ is continuous, then h is open.*
(c) *Suppose that $\psi : Y \to X$ is a quotient map, in the sense that U is open in X if and only if $\psi^{-1}(U)$ is open. Then h is a homeomorphism of X onto $Y/R(\psi)$.*

Proof. (a) If $h(x) = h(x')$, then the surjectivity of ψ implies that there exists at least one $z \in \psi^{-1}(x) = \psi^{-1}(x')$, and then $x = \psi(z) = x'$. So h is one-to-one. Surjectivity is easy: every orbit $\psi^{-1}(\psi(y)) = h(\psi(y))$. The same formula $h(\psi(y)) = \psi^{-1}(\psi(y))$ shows that $h \circ \psi(y)$ is the orbit $q(y)$ of y.

For (b), take U open in X. Then $q^{-1}(h(U)) = \psi^{-1}(h^{-1}(h(U))) = \psi^{-1}(U)$ is open because ψ is continuous, and then $h(U)$ is open by definition of the quotient topology. For (c), take V open in $Y/R(\psi)$. Then $\psi^{-1}(h^{-1}(V)) = q^{-1}(V)$ is open in Y, and $h^{-1}(V)$ is open because ψ is a quotient map. $\square$

Lemma 4.2. *Suppose that* $\psi : Y \to X$ *is a quotient map. Then* $R(\psi)$ *is étale if and only if* ψ *is a local homeomorphism.*

Proof. Suppose that ψ is a local homeomorphism and $(y, z) \in R(\psi)$. There are open neighbourhoods U of y and V of z such that $\psi|_U$ and $\psi|_V$ are homeomorphisms onto open neighbourhoods of $\psi(y) = \psi(z)$. By shrinking if necessary, we may suppose $\psi(U) = \psi(V)$. Now $W := (U \times V) \cap R(\psi)$ is an open neighbourhood of (y, z), and the function $w \mapsto (w, (\psi|_V)^{-1} \circ \psi|_U(w))$ is a continuous inverse for $r|_W$. Thus r is a local homeomorphism, and $R(\psi)$ is étale.

Conversely, suppose that $R(\psi)$ is étale and $y \in Y$. Then $(y, y) \in R(\psi)$. Since r is a local homeomorphism, there is a neighbourhood W of (y, y) in $R(\psi)$ such that $r|_W$ is a homeomorphism. By shrinking, we can assume that $W = (U \times U) \cap R(\psi)$ for some open neighbourhood U of y in Y. We claim that ψ is one-to-one on U. Suppose $y_1, y_2 \in U$, and $\psi(y_1) = \psi(y_2)$. Then (y_1, y_1) and (y_1, y_2) are both in W. Now $r(y_1, y_1) = y_1 = r(y_1, y_2)$, the injectivity of $r|_W$ implies that $(y_1, y_1) = (y_1, y_2)$, and $y_1 = y_2$. Thus $\psi|_U$ is one-to-one, as claimed. Since the orbit map q in an étale groupoid is open, and h is a homeomorphism with $h \circ \psi = q$, it follows that $\psi(U) = h^{-1}(q(U))$ is open. Thus ψ is a local homeomorphism. $\square$

Proposition 4.3. *Suppose that* Y *and* X *are topological spaces with* Y *locally compact Hausdorff, and* $\psi : Y \to X$ *is a surjective local homeomorphism. Then* $R(\psi)$ *is locally compact, Hausdorff, principal, étale and Cartan.*

Proof. The groupoid $R(\psi)$ is principal because it is an equivalence relation, and is Hausdorff because Y is Hausdorff. Since ψ is a local homeomorphism, it is a quotient map, and hence Lemma 4.2 implies that $R(\psi)$ is étale. Let $(y, z) \in R(\psi)$. Then there is an open neighbourhood U of (y, z) such that $r|_U$ is a homeomorphism onto an open neighbourhood of y. Since locally compact Hausdorff spaces are locally locally-compact, $r(U)$ contains a compact neighbourhood K of y. Now $(r|_U)^{-1}(K)$ is a compact neighbourhood of (y, z) in $R(\psi)$, and we have shown that $R(\psi)$ is locally compact.

Since locally compact Hausdorff spaces are regular [24, 1.7.9], the following lemma tells us that $R(\psi)$ is Cartan, and hence completes the proof of

Proposition 4.3. The extra generality in the lemma will be useful in the proof of Proposition 4.5. $\qquad\square$

Lemma 4.4. *Suppose that Y is a regular topological space and $\psi : Y \to X$ is a surjection. If $R(\psi)$ is locally compact, then $R(\psi)$ is Cartan.*

Proof. Let $y \in Y$. We must find a wandering neighbourhood W of y in Y, that is, a neighbourhood W such that $(W \times W) \cap R(\psi)$ has compact closure in $R(\psi)$. Since $R(\psi)$ is locally compact, there exists a compact neighbourhood K of (y, y) in $R(\psi)$. Since K is a neighbourhood, the interior $\operatorname{int} K$ is an open set containing (y, y), and there exists an open set $O \subset Y \times Y$ such that $\operatorname{int} K = O \cap R(\psi)$.

Choose open neighbourhoods U_1, U_2 of y in Y such that $U_1 \times U_2 \subset O$. Since Y is regular, there are open neighbourhoods V_i of y such that $\overline{V}_i \subset U_i$ for $i = 1, 2$. Let $C := \overline{V}_1 \cap \overline{V}_2$. Then C is a closed neighbourhood of y in Y, and

$$(C \times C) \cap R(\psi) \subset (U_1 \times U_2) \cap R(\psi) \subset O \cap R(\psi) \subset K.$$

Since $(C \times C) \cap R(\psi)$ is closed in $R(\psi)$ and K is compact, $(C \times C) \cap R(\psi)$ is compact, and C is the required neighbourhood of y. $\qquad\square$

Proposition 4.3 has an intriguing converse. Suppose that G is a locally compact, Hausdorff and principal groupoid. We will see that if G is Cartan, then G has the form $R(q)$, where $q : G^{(0)} \to G^{(0)}/G$ is the quotient map. The key idea is that, because G is principal, the map $r \times s : \gamma \mapsto (r(\gamma), s(\gamma))$ is a groupoid isomorphism of G onto $R(q)$. The map $r \times s$ is also continuous for the product topology on $R(q)$, but it is not necessarily open, and hence is not necessarily an isomorphism of topological groupoids (see Example 5.8 below). But:

Proposition 4.5. *Suppose G is a locally compact, Hausdorff and principal groupoid which admits a Haar system. Then G is Cartan if and only if $r \times s$ is a topological isomorphism of G onto $R(q)$.*

For the proof we need a technical lemma.

Lemma 4.6. *Let G be a locally compact Hausdorff groupoid which admits a Haar system. If G is Cartan, then $r \times s$ is a closed map onto its image $r \times s(G)$.*

Proof. Let C be a closed subset of G and (u, v) be a limit point of $r \times s(C)$ in $r \times s(G)$. Then there exists $\gamma \in G$ such that $r \times s(\gamma) = (u, v)$ and a net $\{\gamma_i\}$ in C such that $r \times s(\gamma_i) \to (u, v)$. It suffices to show that $\{\gamma_i\}$ has a convergent subnet. Indeed, if $\gamma_{i_j} \to \gamma'$ then $\gamma' \in C$ because C is closed, $r \times s(\gamma') = (u, v)$ by continuity, and $(u, v) \in r \times s(C)$.

Since G is Cartan, u has a neighbourhood N in $G^{(0)}$ such that $U := (r \times s)^{-1}(N \times N)$ is relatively compact in G. Let V be a relatively compact neighbourhood of γ. We may assume by shrinking V that $V \subset r^{-1}(N)$, and

hence that $r(V) \subset N$. The continuity of multiplication implies that UV is relatively compact.

We claim that $\gamma_i \in UV$ eventually. To see this, we observe that the existence of the Haar system implies that s is open [32, Corollary, page 118], and hence $s(V)$ is a neighbourhood of $v = s(\gamma)$. Thus there exists i_0 such that $s(\gamma_i) \in s(V)$ and $r(\gamma_i) \in N$ for all $i \geq i_0$. For each $i \geq i_0$ there exists $\beta_i \in V$ such that $s(\gamma_i) = s(\beta_i)$. Now $s(\gamma_i \beta_i^{-1}) = r(\beta_i) \in r(V) \subset N$ and $r(\gamma_i \beta_i^{-1}) = r(\gamma_i) \in N$, so $\alpha_i := \gamma_i \beta_i^{-1}$ is in U. Thus $\gamma_i = \alpha_i \beta_i \in UV$ for $i \geq i_0$, as claimed. Now $\{\gamma_i : i \geq i_0\}$ is a net in a relatively compact set, and hence has a convergent subnet, as required. $\qquad\square$

Proof of Proposition 4.5. Suppose that G is Cartan. The map $r \times s$ is always a continuous surjection onto $R(q)$, and it is injective because G is principal. Since G is Cartan, Lemma 4.6 implies that $r \times s$ is closed as a map onto its image $R(q)$. A bijection is open if and only if it is closed, so $r \times s$ is open. Hence $r \times s$ is a homeomorphism onto $R(q)$.

Conversely, suppose that $r \times s$ is a homeomorphism onto $R(q)$. Then $r \times s$ is an isomorphism of topological groupoids, and since G is locally compact, so is $R(q)$. Thus $R(q)$ is Cartan by Lemma 4.4, and so is G. $\qquad\square$

Concluding discussion. Lemma 4.4 shows that, if the groupoid $R(\psi)$ associated to a quotient map is locally compact, then $R(\psi)$ is Cartan. On the other hand, Proposition 4.5 says that, if there is a topology on $R(\psi)$ which makes it into a locally compact Cartan groupoid, then that topology has to be the relative topology from the product space $Y \times Y$. So one is tempted to seek conditions on ψ which ensure that the subset $R(\psi) \subset Y \times Y$ is locally compact. By Proposition 4.3, it suffices for ψ to be a local homeomorphism. This is not a necessary condition: for example, if (Y, H) is a free Cartan transformation group with H nondiscrete, then the transformation groupoid $Y \times H$ is Cartan in our sense, but the quotient map $q : Y \to Y/H$ is not locally injective. (In [12] there is a specific example of a free Cartan transformation group which illustrates this.) However, the following example shows that something extra is needed.

Example 4.7. Take $Y = [0, 1]$ and $X = \{a, b\}$ with the topology $\{X, \{a\}, \emptyset\}$, and define $\psi : Y \to X$ by

$$\psi(y) = \begin{cases} a & \text{if } t > 0 \\ b & \text{if } t = 0. \end{cases}$$

Then ψ is an open quotient map, but $R(\psi) = \big((0, 1] \times (0, 1]\big) \cup \{(0, 0)\}$ is not a locally compact subset of $[0, 1] \times [0, 1]$ because $(0, 0)$ does not have a compact neighbourhood.

5. The groupoids whose C^*-algebras are Fell algebras

We begin this section by summarising results from [5, 6, 4, 22] about the twisted groupoid algebras of principal Cartan groupoids.

Theorem 5.1. *Suppose that G is a second-countable, locally compact, Hausdorff and principal groupoid which admits a Haar system. Then the following statements are equivalent:*

(a) *G is Cartan.*
(b) *$C^*(G)$ is a Fell algebra.*
(c) *For all $\sigma \in Z^2(G, \mathbb{T})$, $C^*(G, \sigma)$ is a Fell algebra.*

If items (a)–(c) *are satisfied, then $u \mapsto \operatorname{Ind}_u^\sigma$ induces a homeomorphism of $G^{(0)}/G$ onto $C^*(G, \sigma)^\wedge$, and $C^*(G, \sigma) = C_{\mathrm{r}}^*(G, \sigma)$.*

Proof. The implication (c) $\Longrightarrow$ (b) is trivial, the equivalence of (a) and (b) is Theorem 7.9 of [5], and the implication (b) $\Longrightarrow$ (c) is part (ii) of [4, Proposition 3.10 (a)]. So it remains to prove the assertions in the last sentence.

The orbits in a Cartan groupoid are closed [5, Lemma 7.4], so the orbit space $Y/R(\psi)$ is T_1, and it follows from [6, Proposition 3.2] that the map $y \mapsto [L^u]$ described there induces a homeomorphism of $G^{(0)}/G$ onto the spectrum of the twisted groupoid C^*-algebra $C^*(G^{\overline{\sigma}}; G)^{\mathfrak{MW}}$ of Muhly and Williams (to see why the complex conjugate appears see [22] or Appendix A). By [4, Lemma 3.1], $C^*(G^{\overline{\sigma}}; G)^{\mathfrak{MW}}$ is isomorphic to $C^*(G, \sigma)$, and by Lemma A.1, this isomorphism carries the equivalence class of L^u to the class of $\operatorname{Ind}_u^\sigma$. We deduce that $u \mapsto \operatorname{Ind}_u^\sigma$ induces a homeomorphism, as claimed. This implies in particular that all the irreducible representations of $C^*(G, \sigma)$ are induced, so for $f \in C_c(G, \sigma)$ we have

$$\|f\| = \sup\{\| \operatorname{Ind}_y(f)\| : y \in Y\} =: \|f\|_{\mathrm{r}},$$

and $C^*(G, \sigma) = C_{\mathrm{r}}^*(G, \sigma)$. $\qquad\square$

For our first application of Theorem 5.1, we observe that putting the equivalence of (a) and (b) together with Proposition 4.5 gives the following improvement of a result of Archbold and Somerset [2, Corollary 5.9]. (We discuss the precise connection with [2] in Remark 5.3.)

Corollary 5.2. *Suppose that G is a second-countable, locally compact, Hausdorff and principal groupoid which admits a Haar system λ, and $q : G^{(0)} \to G^{(0)}/G$ is the quotient map. Then $C^*(G, \lambda)$ is a Fell algebra if and only if $r \times s : G \to G^{(0)} \times G^{(0)}$ is a topological isomorphism of G onto $R(q)$.*

Remark 5.3. The "separated topological equivalence relations" R studied in [2, §5] are the second-countable, locally compact, Hausdorff and principal groupoids that are étale. When they say in [2, Corollary 5.9] that "the topologies τ_p and τ_0 coincide," they mean precisely that the map $r \times s$ is a homeomorphism for the original topology τ_0 on R and the product topology on $R^{(0)} \times R^{(0)}$. Theorem 5.1 implies that the full algebra above and the reduced algebra in [2] coincide. So Corollary 5.2 extends [2, Corollary 5.9] from principal ètale groupoids to principal groupoids which admit a Haar

system. This is a substantial generalisation since, for example, locally compact transformation groups always admit a Haar system [30, page 17] even though the associated transformation groupoids may not be étale. Our proof of Corollary 5.2 seems quite different from the representation-theoretic arguments used in [2].

Next we apply Theorem 5.1 to the groupoid associated to a local homeomorphism.

Corollary 5.4. *Let $\psi : Y \to X$ be a surjective local homeomorphism of a second-countable, locally compact and Hausdorff space Y onto a topological space X, and let $\sigma : R(\psi)^{(2)} \to \mathbb{T}$ be a continuous normalised 2-cocycle. Then $C^*(R(\psi), \sigma)$ is a Fell algebra with spectrum homeomorphic to X, and $C^*(R(\psi), \sigma) = C_{\mathrm{r}}^*(R(\psi), \sigma)$.*

Proof. Proposition 4.3 implies that $R(\psi)$ satisfies all the hypotheses of Theorem 5.1. Thus $C^*(R(\psi), \sigma) = C_{\mathrm{r}}^*(R(\psi), \sigma)$ is a Fell algebra, and $y \mapsto \mathrm{Ind}_y^\sigma$ induces a homeomorphism of $Y/R(\psi)$ onto $C^*(R(\psi), \sigma)^\wedge$. Since ψ is a quotient map, the map $x \mapsto \psi^{-1}(x)$ is a homeomorphism of X onto $Y/R(\psi)$ by Lemma 4.1(c). Given $x \in X$, choose $y_x \in \psi^{-1}(x)$; then $x \mapsto [\mathrm{Ind}_{y_x}^\sigma]$ is the required homeomorphism of X onto $C^*(R(\psi), \sigma)^\wedge$. $\square$

When X is Hausdorff, we know from [17] that $C^*(R(\psi))$ has continuous trace; twisted versions were used in [27] to provide examples of continuous-trace algebras with nonzero Dixmier–Douady class.

We now give our promised shorter proof of the converse of Lemma 3.2.

Corollary 5.5. *Let X be a second-countable, locally locally-compact and locally Hausdorff topological space. Then there is a separable Fell C^*-algebra with spectrum X.*

Proof. By Proposition 3.3, there are a second-countable, locally compact and Hausdorff space Y and a surjective local homeomorphism $\psi : Y \to X$. Consider the topological relation $R(\psi)$. Corollary 5.4 says that $C^*(R(\psi))$ is a separable Fell algebra with spectrum homeomorphic to X. $\square$

Next we consider a row-finite directed graph E with no sources, using the conventions of [26]; we also use the more recent convention that, for example,

$$vE^n w = \{\alpha \in E^n : r(\alpha) = v,\ s(\alpha) = w\}.$$

The infinite-path space E^∞ has a locally compact Hausdorff topology with basis the cylinder sets

$$Z(\alpha) = \{\alpha x :\ x \in E^\infty \text{ and } r(x) = s(\alpha)\}.$$

[20, Corollary 2.2]. The set

$$G_E = \{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \text{there is } n \text{ such that } x_i = y_{i+k} \text{ for } i \geq n\}$$

is a groupoid with unit space $G_E^{(0)} = E^\infty$, and this groupoid is locally compact, Hausdorff and étale in a topology which has a neighbourhood basis consisting of the sets

$$Z(\alpha, \beta) = \{(\alpha z, |\beta| - |\alpha|, \beta z) : z \in E^\infty, \ r(z) = s(\alpha)\}$$

parametrised by pairs of finite paths $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta)$ [20, Proposition 2.6]. We know from [11, Proposition 8.1] that G_E is principal if and only if E has no cycles (in which case we say E is *acyclic*).

We write $x \sim y$ to mean $(x, k, y) \in G_E$ for some k; this equivalence relation on E^∞ is called *tail equivalence with lag*.

Proposition 5.6. *Suppose that E is an acyclic row-finite directed graph with no sources. Then G_E is Cartan if and only if the quotient map $q : E^\infty \to E^\infty/\sim$ is a local homeomorphism.*

Proof. Suppose that G_E is Cartan. The quotient space $E^\infty/\sim$ is the orbit space of G_E. Thus Proposition 4.5 implies that $r \times s$ is an isomorphism of topological groupoids of G_E onto $R(q)$. Since G_E is étale, so is $R(q)$. Thus Lemma 4.2 implies that q is a local homeomorphism.

Conversely, suppose that q is a local homeomorphism. We claim that $r \times s : G_E \to R(q)$ is an isomorphism of topological groupoids. Then, since we know from Proposition 4.3 that $R(q)$ is Cartan, we can deduce that G_E is Cartan too.

To prove the claim, we observe first that $r \times s$ is an isomorphism of algebraic groupoids because G_E is principal, and is continuous because r and s are. So we take $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta)$, and prove that $r \times s(Z(\alpha, \beta))$ is open. A typical element of $r \times s(Z(\alpha, \beta))$ has the form $(\alpha z, \beta z)$ for some $z \in E^\infty$ with $r(z) = s(\alpha)$. Since q is a local homeomorphism there is an initial segment μ of z such that $q|_{Z(\mu)}$ is one-to-one. We will prove that $(Z(\alpha\mu) \times Z(\beta\mu)) \cap R(q)$ is contained in $r \times s(Z(\alpha, \beta))$.

Let $(x, y) \in (Z(\alpha\mu) \times Z(\beta\mu)) \cap R(q)$. Then there are $x', y' \in E^\infty$ such that $x = \alpha\mu x'$ and $y = \beta\mu y'$, and $q(x) = q(y)$ implies $q(\mu x') = q(\mu y')$. Both $\mu x'$ and $\mu y'$ are in $Z(\mu)$, so injectivity of $q|_{Z(\mu)}$ implies that $\mu x' = \mu y'$, and $x' = y'$. But now

$$(x, y) = (\alpha\mu x', \beta\mu x') = r \times s(\alpha\mu x', |\beta| - |\alpha|, \beta\mu x')$$

belongs to $r \times s(Z(\alpha, \beta))$. Thus $r \times s(Z(\alpha, \beta))$ contains a neighbourhood of $(\alpha z, \beta z)$, and $r \times s(Z(\alpha, \beta))$ is open. Now we have proved our claim, and the result follows.   □

The groupoid G_E was originally invented as a groupoid whose C^*-algebra is the universal algebra $C^*(E)$ generated by a Cuntz-Krieger E-family [20, Theorem 4.2]. In view of Corollary 5.2, Proposition 5.6 implies that $C^*(E) = C^*(G_E)$ is a Fell algebra if and only if $q : E^\infty \to E^\infty/\sim$ is a local homeomorphism. So it is natural to ask whether we can identify this property at the level of the graph. We can:
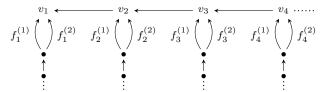
Proposition 5.7. *Suppose that E is an acyclic row-finite directed graph with no sources. Then the quotient map $q : E^\infty \to E^\infty/\sim$ is a local homeomorphism if and only if, for every $x \in E^\infty$, there exists n such that*

$$(5.1) \qquad s(x_n)E^*s(\mu) = \{\mu\} \text{ for every } \mu \in E^* \text{ with } r(\mu) = s(x_n).$$

Proof. Suppose first that q is a local homeomorphism, and $x \in E^\infty$. Then there is an initial segment $\mu = x_1 x_2 \cdots x_n$ of x such that $q|_{Z(\mu)}$ is injective. We claim that this n satisfies (5.1). Suppose not. Then there exist $\alpha, \beta \in s(x_n)E^*$ such that $\alpha \neq \beta$ and $s(\alpha) = s(\beta)$. Neither can be an initial segment of the other, since this would give a cycle at $s(\alpha)$. So there exists $i \leq \min(|\alpha|, |\beta|)$ such that $\alpha_i \neq \beta_i$. Then for any $y \in E^\infty$ with $r(y) = s(\alpha)$, $\mu\alpha y$ and $\mu\beta y$ are distinct paths in $Z(\mu)$ with $q(\mu\alpha y) = q(\mu\beta y)$, which contradicts the injectivity of $q|_{Z(\mu)}$.

Conversely, suppose that E has the property described, and let $x \in E^\infty$. Take n satisfying (5.1), and $\mu := x_1 x_2 \cdots x_n$. We claim that $q|_{Z(\mu)}$ is injective. Suppose $y = \mu y'$, $z = \mu z' \in Z(\mu)$ and $q(y) = q(z)$. Then $q(y') = q(z')$, and there exist paths γ, δ in $s(x_n)E^*$ such that $y' = \gamma y''$, $z' = \delta y''$, say. The existence of y'' forces $s(\gamma) = s(\delta)$, and (5.1) implies that $\gamma = \delta$, $y' = z'$ and $y = z$. Thus q is locally injective. Since G_E is étale, the quotient map q is open, and hence q is a local homeomorphism. $\qquad\square$

Example 5.8. Consider the following graph E from [11, Example 8.2]:

Let G_E be the path groupoid. Let z be the infinite path with range v_1 that passes through each v_n, and, for $n \geq 1$, let x_n be the infinite path with range v_1 that includes the edge $f_n^{(1)}$. It is shown in [11, Example 8.2] that the sequence $\{x_n\}$ "converges 2-times in E^∞/G_E to z", and it then follows from [6, Lemma 5.1] that G_E is not Cartan. Applying the criteria of Proposition 5.7 seems to give an easier proof of this: let z be as above. For each n, $s(z_n) = v_{n+1}$ and $s(z_n)E^*s(f_{n+1}^{(1)}) = \{f_{n+1}^{(1)}, f_{n+1}^{(2)}\}$. Thus $q : E^\infty \to E^\infty/\sim$ is not a local homeomorphism by Proposition 5.7 and hence G_E is not Cartan by Proposition 5.6. Proposition 4.5 implies that $r \times s$ is not open.

6. Fell algebras with trivial Dixmier–Douady invariant

If A is a Fell algebra, we write $\delta(A)$ for its Dixmier–Douady invariant, as defined in [13, Section 7]. If A is a continuous-trace algebra, then $\delta(A)$ makes sense, and A also has a Dixmier–Douady invariant $\delta_{\mathrm{DD}}(A)$ as in [29, §5.3], for example; as pointed out in [13, Remark 7.10], it is not clear whether these invariants are the same. Recall that the properties of being a Fell

algebra or having continuous trace are preserved under Morita equivalence by [14, Corollary 14] and [34, Corollary 3.5].

The following is implicitly assumed in [13].

Theorem 6.1. *Let A be a separable Fell algebra. Then the Dixmier–Douady invariant $\delta(A)$ of A is 0 if and only if there is is a local homeomorphism ψ of a second-countable, locally compact and Hausdorff space onto a topological space such that A is Morita equivalent to $C^*_r(R(\psi))$.*

Proof. We start by recalling how $\delta(A)$ is defined in [13]. By [13, Theorem 5.17], A is Morita equivalent to a C^*-algebra C which has a diagonal C^*-subalgebra D; let $h : \hat{C} \to \hat{A}$ be an associated Rieffel homeomorphism. By [18, Theorem 3.1], there is a twist $\Gamma \to R$ over an étale and principal groupoid R such that C is isomorphic to Kumjian's C^*-algebra $C^*(\Gamma; R)^{\mathrm{Kum}}$ of the twist. Since C is a Fell algebra we may by [13, Proposition 6.3] assume that $R = R(\psi)$, where $\psi : \hat{D} \to \hat{C}$ is the spectral map, which is a local homeomorphism by [13, Theorem 5.14].

By [19, Remark 2.9], there is an extension $\underline{\Gamma} \to R(\psi)$ where $\underline{\Gamma}$ is the groupoid consisting of germs of continuous local sections of the surjection $\Gamma \to R(\psi)$. Such extensions are called sheaf twists, and the group of their isomorphism classes is denoted by $T_{R(\psi)}(\mathcal{S})$, where $\mathcal{S}$ is a sheaf of germs of continuous functions. Let $H^2(R(\psi), \mathcal{S})$ be the second equivariant sheaf cohomology group. The long exact sequence of [19, Theorem 3.7] yields a boundary map ∂^1 from $T_{R(\psi)}(\mathcal{S})$ to $H^2(R(\psi), \mathcal{S})$. Finally, set

$$\delta(A) = (\pi^*_{h \circ \psi})^{-1}(\partial^1([\underline{\Gamma}])) \in H^2(\hat{A}, \mathcal{S})$$

where $\pi_{h \circ \psi} : R(h \circ \psi) \to \hat{A}$ is given by $(y, z) \mapsto h \circ \psi(y)$ [13, Definition 7.9]. Quite a bit of the work in [13, §7] is to show that $\delta(A)$ is well-defined.

Now suppose that $\delta(A) = 0$. Let Γ be a twist associated to A. Then $\partial^1([\underline{\Gamma}]) = 0$. Let $\Lambda := \mathbb{T} \times R(\psi)$ so that $\Lambda \to R(\psi)$ is the trivial twist. Then the associated sheaf twist $\underline{\Lambda}$ is also trivial, whence $\partial^1([\underline{\Lambda}]) = 0$. Now $0 = \partial^1([\underline{\Gamma}])$ is sent to $0 = \partial^1([\underline{\Lambda}])$ under a certain natural isomorphism (see [13, Corollary 7.6]), and [13, Lemma 7.12] implies that $\Gamma \to R(\psi)$ and $\Lambda \to R(\psi)$ are equivalent twists. Thus $C^*(\Gamma; R(\psi))^{\mathrm{Kum}}$ and $C^*(\Lambda; R(\psi))^{\mathrm{Kum}}$ are Morita equivalent by [13, Lemma 6.5]. But now A and $C^*(\Lambda; R(\psi))^{\mathrm{Kum}}$ are Morita equivalent. Since $\Lambda \to R(\psi)$ is trivial, $C^*(\Lambda; R(\psi))^{\mathrm{Kum}}$ is isomorphic to $C^*_r(R(\psi))$ by [13, Lemma A.1]. Thus A and $C^*_r(R(\psi))$ are Morita equivalent.

Conversely, suppose that A is Morita equivalent to $C^*_r(R(\psi))$ for some ψ. By Lemma A.1 of [13], $C^*_r(R(\psi))$ is isomorphic to the C^*-algebra of the trivial twist $\Lambda := \mathbb{T} \times R(\psi) \to R(\psi)$. The associated sheaf twist $\underline{\Lambda}$ is also trivial, so $\partial^1([\underline{\Lambda}]) = 0$, and $\delta(C^*_r(R(\psi))) = 0$. Since A and $C^*_r(R(\psi))$ are Morita equivalent, by Theorem 7.13 of [13] there is a homeomorphism $k : \hat{A} \to C^*_r(R(\psi))^\wedge$ such that the induced isomorphism k^* of $H^2(C^*_r(R(\psi)^\wedge, \mathcal{S})$ onto $H^2(\hat{A}, \mathcal{S})$ carries $0 = \delta(C^*_r(R(\psi)))$ to $\delta(A)$. Thus $\delta(A) = 0$. $\square$

Proposition 6.2. *Let A be a separable Fell algebra.*

(a) *If $\delta(A) = 0$ then $\delta_{\mathrm{DD}}(I) = 0$ for every ideal I of A with continuous trace.*

(b) *If A has continuous trace and $\delta_{\mathrm{DD}}(A) = 0$, then $\delta(A) = 0$.*

Proof. Let B_1 and B_2 be C^*-algebras with continuous trace and paracompact spectrum X. Propositions 5.32 and 5.33 of [29] together say that $\delta_{\mathrm{DD}}(B_1) = \delta_{\mathrm{DD}}(B_2)$ if and only if B_1 and B_2 are Morita equivalent. Below we consider an ideal I in a separable Fell algebra A such that I has continuous trace. Then $\hat{I}$ is second-countable, locally compact and Hausdorff, and hence is σ-compact. By [24, Proposition 1.7.11], for example, $\hat{I}$ is paracompact. Thus Propositions 5.32 and 5.33 of [29] apply to continuous-trace C^*-algebras with spectrum homeomorphic to $\hat{I}$.

(a) Suppose $\delta(A) = 0$, and let I be an ideal of A with continuous trace. By Theorem 6.1, A is Morita equivalent to $C_{\mathrm{r}}^*(R(\psi))$ for some local homeomorphism $\psi : Y \to X$. Then $C_{\mathrm{r}}^*(R(\psi))$ is also a Fell algebra, and I is Morita equivalent to an ideal J of $C_{\mathrm{r}}^*(R(\psi))$ with continuous trace. By Corollary 5.4, $C_{\mathrm{r}}^*(R(\psi)) = C^*(R(\psi))$. Since $R(\psi)$ is principal and $C^*(R(\psi))$ is liminary, by [5, Proposition 6.1] there exists an open invariant subset U of the unit space Y of $R(\psi)$ such that J is isomorphic to the C^*-algebra of $R(\psi)|_U := \{\gamma \in R(\psi) : r(\gamma), s(\gamma) \in U\}$. Since $R(\psi)|_U$ is principal and $C^*(R(\psi)|_U)$ has continuous trace, $R(\psi)|_U$ is a proper groupoid by [21, Theorem 2.3], and $\delta_{\mathrm{DD}}(C^*(R(\psi)|_U)) = 0$ by [21, Proposition 2.2]. Since $C^*(R(\psi)|_U)$ and I are Morita equivalent, $\delta_{\mathrm{DD}}(I) = 0$ by [29, Proposition 5.32].

(b) Suppose $\delta_{\mathrm{DD}}(A) = 0$. Then $\delta_{\mathrm{DD}}(A) = \delta_{DD}(C_0(\hat{A}))$, and A and $C_0(\hat{A})$ are Morita equivalent by [29, Proposition 5.33]. But $C_0(\hat{A})$ is isomorphic to $C^*(R(\psi)) = C_{\mathrm{r}}^*(R(\psi))$ where $\psi : \hat{A} \to \hat{A}$ is the identity. Thus $\delta(A) = 0$ by Theorem 6.1. $\qquad\square$

The following corollary is immediate from Proposition 6.2.

Corollary 6.3. *Suppose A is a separable C^*-algebra with continuous trace. Then $\delta(A) = 0$ if and only if $\delta_{\mathrm{DD}}(A) = 0$.*

7. Examples

A standard example of a Fell algebra which does not have continuous trace is the algebra

$$A_3 = \{f \in C([0,1], M_2(\mathbb{C})) : f(1) \text{ is diagonal}\}$$

discussed in [29, Example A.25]. Here we describe two variations on this construction. It seems clear to us that our constructions could be made much more general, for example by doubling up along topologically nontrivial subspaces rather than at a single point.

7.1. A Fell algebra with trivial Dixmier–Douady invariant. Write $\{\xi_i\}_{i \in \mathbb{N}}$ for the usual orthonormal basis in $\ell^2(\mathbb{N})$ and Θ_{ij} for the rank-one operator $\Theta_{ij} : h \mapsto (h \,|\, \xi_j)\xi_i$ on $\ell^2(\mathbb{N})$. We write $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$, and
(7.1)
$$A := \{a \in C([0,1], \mathcal{K}) : a(1) \text{ is diagonal, that is } (a(1)\xi_i \,|\, \xi_j) = 0 \text{ for } i \neq j\}.$$

We write e_{ij} for the constant function $t \mapsto \Theta_{ij}$ in $C([0,1], \mathcal{K})$.

We let Y be the disjoint union $\bigsqcup_{i \in \mathbb{N}}[0,1] = \bigcup_{i \in \mathbb{N}}[0,1] \times \{i\}$. Then Y is a locally compact Hausdorff space with the topology in which each $[0,1] \times \{i\}$ is open, closed and homeomorphic to $[0,1]$. There is an equivalence relation $\sim$ on Y such that $(s,i) \sim (s,j)$ for all $i, j \in \mathbb{N}$ and $s \in [0,1)$, and the $(1,i)$ are equivalent only to themselves.

We claim that the quotient map $\psi : Y \to X := Y/{\sim}$ is open. To see this, it suffices to take an open set $U = W \times \{i\}$ contained in one level $[0,1] \times \{i\}$, and see that $\psi(U)$ is open. By definition of the quotient topology, we have to show that $\psi^{-1}(\psi(U))$ is open. If $(1,i)$ is not in U, then

$$\psi^{-1}(\psi(U)) = \{(s,j) : (s,i) \in U, \ j \in \mathbb{N}\} = \bigcup_{j \in \mathbb{N}} W \times \{j\},$$

which is open. If $(1,i) \in U$, then

$$\psi^{-1}(\psi(U)) = (W \times \{i\}) \cup \left(\bigcup_{j \neq i}((W \setminus \{1\}) \times \{j\})\right),$$

which is open. Thus ψ is open, as claimed. Since $[0,1] \times \{i\}$ is open and $\psi|_{[0,1] \times \{i\}}$ is injective, ψ is a surjective local homeomorphism.

We now consider $R(\psi)$, and write

$$V_{ij} = \big(([0,1] \times \{i\}) \times ([0,1] \times \{j\})\big) \cap R(\psi).$$

Then for $i \neq j$, the map $\psi_{ij} : ((s,i),(s,j)) \mapsto s$ is a homeomorphism of V_{ij} onto $[0,1)$; for $i = j$, the similarly defined ψ_{ii} is a homeomorphism of V_{ii} onto $[0,1]$. Thus for each $f \in C_c(R(\psi))$, the compact set $\operatorname{supp} f$ meets only finitely many V_{ij}. Define $f_{ij}(s) = f((s,i),(s,j))$. Then $f_{ij} \in C_c([0,1))$ (for $i \neq j$) and $f_{ii} \in C([0,1])$, and f can be recovered as a finite sum $\sum_{\{(i,j):\operatorname{supp} f \cap V_{ij} \neq \emptyset\}}(f_{ij} \circ \psi_{ij})\chi_{V_{ij}}$. By viewing functions in $C_c([0,1))$ as functions on $[0,1]$ which vanish at 1, we can define $\rho : C_c(R(\psi)) \to A$ by

$$\rho(f) = \sum_{\{(i,j):\operatorname{supp} f \cap V_{ij} \neq \emptyset\}} f_{ij}e_{ij}.$$

Proposition 7.1. *The function $\rho : C_c(R(\psi)) \to A$ extends to an isomorphism of $C^*(R(\psi))$ onto A.*

First we check that ρ is a homomorphism on the convolution algebra $C_c(R(\psi))$. Let $f, g \in C_c(R(\psi))$, and observe that for each s,

$$(7.2) \qquad (f * g)((s,i),(s,j)) = \sum_k f((s,i),(s,k))g((s,k),(s,j))$$

$$= \sum_k f_{ik}(s)g_{kj}(s)$$

has only finitely many nonzero terms (and just one if $s = 1$ and $i = j$). Since the e_{ij} are matrix units in $C([0,1], \mathcal{K})$, and the operations in A are those of $C([0,1], \mathcal{K})$, Equation 7.2 implies that ρ is multiplicative. Thus ρ is a $*$-homomorphism.

To see that ρ extends to $C^*(R(\psi))$, we show that ρ is isometric for the reduced norm on $C_c(R(\psi))$, which by Corollary 5.4 is the same as the enveloping norm. The norm in $C([0,1], \mathcal{K})$ satisfies $\|a\| = \sup\{\|a(t)\| : t \in [0,1)\}$, and since $\bigcup_{i \in \mathbb{N}}[0,1) \times \{i\}$ is also dense in Y, the reduced norm on $C_c(R(\psi))$ satisfies

$$\|f\| = \sup\{\|\operatorname{Ind}_{(s,i)}(f)\| : s \in [0,1), i \in \mathbb{N}\}.$$

Thus the following lemma implies that ρ is isometric.

Lemma 7.2. For $s \in [0,1)$, we define $\epsilon_s : A \to B(\ell^2(\mathbb{N}))$ by $\epsilon_s(f) = f(s)$. Then the representation $\epsilon_s \circ \rho$ of $C_c(R(\psi))$ is unitarily equivalent to $\operatorname{Ind}_{(s,i)}$.

Proof. For each $i \in \mathbb{N}$ we have $s^{-1}((s,i)) = \{((s,j),(s,i)) : j \in \mathbb{N}\}$, so there is a unitary isomorphism U_s of $\ell^2(s^{-1}((s,i)))$ onto $\ell^2(\mathbb{N})$ which carries the point mass $\delta_{((s,j),(s,i))}$ to the basis vector ξ_j. We will prove that U_s intertwines $\epsilon_s \circ \rho$ and $\operatorname{Ind}_{(s,i)}$. Let $f \in C_c(R(\psi))$. On one hand, we have

$$\epsilon_s \circ \rho(f)(U_s \delta_{((s,j),(s,i))}) = \sum_{k,l} f_{kl}(s)\Theta_{kl}\xi_j = \sum_k f_{kj}(s)\xi_k.$$

On the other hand, the induced representation satisfies

$$\big(\operatorname{Ind}_{(s,i)}(f)\delta_{((s,j),(s,i))}\big)((s,k),(s,i))$$

$$= \sum_l f((s,k),(s,l))\delta_{((s,j),(s,i))}((s,l),(s,i))$$

$$= f((s,k),(s,j)) = f_{kj}(s),$$

and hence

$$U_s\big(\operatorname{Ind}_{(s,i)}(f)\delta_{((s,j),(s,i))}\big) = U_s\Big(\sum_k f_{kj}(s)\delta_{((s,k),(s,i))}\Big) = \sum_k f_{kj}(s)\xi_k. \qquad \square$$

It remains for us to see that ρ is surjective. Since $\rho(C^*(R(\psi)))$ is a C^*-algebra, it suffices to show that $\rho(C_c(R(\psi)))$ is dense in A. Indeed, because the $q_N = \sum_{i=1}^N e_{ii}$ form an approximate identity for A, it suffices to take $b \in q_N A q_N$ and show that we can approximate b by some $\rho(f)$. Fix $\epsilon > 0$. We can write $b = \sum_{i,j \le N} b_{ij}e_{ij}$ with $b_{ij} \in C([0,1])$ and $b_{ij}(1) = 0$ for $i \ne j$. Set $c := b - \sum_{i=0}^N b_{ii}e_{ii}$, and observe that $c(1) = 0$. Choose $\delta \in (0,1)$ such

that $\sup_{t \in [\delta, 1)} \|c(t)\| < \epsilon$, and choose a continuous function $h : [0, 1] \to [0, 1]$ such that $h = 1$ on $[0, \delta]$ and $h = 0$ near 1. Then $\|c - hc\|_\infty < \epsilon$. Set $d := \sum_{i=0}^{N} b_{ii} e_{ii} + hc$, and then d has the form $\sum_{i=0}^{N} d_{ij} e_{ij}$, where $d_{ij} \in C_c([0, 1))$ if $i \neq j$ and $d_{ii} \in C([0, 1])$. Set $f = \sum_{i=0}^{N} d_{ij} \chi_{V_{ij}}$. Then $f \in C_c(R(\psi))$, $\rho(f) = d$, and

$$\|b - \rho(f)\|_\infty = \left\| b - \sum_{i=1}^{n} b_{ii} e_{ii} - hc \right\|_\infty = \|c - hc\|_\infty < \epsilon.$$

This completes the proof of Proposition 7.1.

Corollary 7.3. *The C^*-algebra A in (7.1) is a Fell algebra which does not have continuous trace, and the Dixmier–Douady invariant of A is 0.*

Proof. By Proposition 7.1, A is isomorphic to $C^*(R(\psi))$ where ψ is a surjective local homeomorphism. Now Corollary 5.4 implies that A is a Fell algebra with spectrum X, and Theorem 6.1 implies that its Dixmier–Douady invariant vanishes. Because X is not Hausdorff, A does not have continuous trace. □

Remark 7.4. Paracompactness is usually defined only for Hausdorff spaces, and the example of this section confirms that things can go badly wrong for the sorts of non-Hausdorff spaces of interest to us. For the spectrum X of our algebra A, the sets $\psi([0, 1] \times \{i\})$ form an open cover of X, but every neighbourhood of the point $\psi(1, 1)$, for example, meets every neighbourhood of every other $\psi(1, i)$, so there cannot be a locally finite refinement.

7.2. A Fell algebra with nontrivial Dixmier–Douady invariant. We describe a Fell algebra A which does not have continuous trace, and has Dixmier–Douady class $\delta(A) \neq 0$. We do this by combining the construction of [27, §1] (see also [29, Example 5.23]) with that of the algebra A_3 at the start of §7, and then applying Proposition 6.2. We adopt the notation of [29, Chapter 5].

We start with a compact Hausdorff space S, a finite open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of S, and an alternating cocycle $\lambda_{ijk} : U_{ijk} \to \mathbb{T}$ whose class $[\lambda_{ijk}]$ in $H^2(S, \mathcal{S})$ is nonzero. By the argument of [28, Lemma 3.4], for example, we may multiply λ by a coboundary and assume that $\lambda_{ijk} \equiv 1$ whenever two of i, j, k coincide.

The algebra $A(\mathcal{U}, \lambda_{ijk})$ in [29, Example 5.23] has underlying vector space

$$A(\mathcal{U}, \lambda_{ijk}) = \{(f_{ij}) \in M_n(C(S)) : f_{ij} = 0 \text{ on } S \setminus U_{ij}\}.$$

The product in $A(\mathcal{U}, \lambda_{ijk})$ is defined by $(f_{ij})(g_{kl}) = (h_{il})$, where

$$(7.3) \qquad h_{il}(s) = \begin{cases} \sum_{\{j : s \in U_{ijl}\}} \overline{\lambda_{ijl}(s)} f_{ij}(s) g_{jl}(s) & \text{if } s \in U_{ijl} \text{ for some } j \\ 0 & \text{otherwise,} \end{cases}$$

and the involution is defined by $(f_{ij})^* = (\overline{f_{ji}})$. For $s \in S$, we take $I_s := \{i : s \in U_i\}$, and for $i \in I_s$ define $\pi_{i,s} : A(\mathcal{U}, \lambda_{ijk}) \to M_{I_s}$ by

$$(7.4) \qquad \pi_{i,s}\big((f_{jk})_{j,k\in I_s}\big) = \big(\overline{\lambda_{ijk}(s)}f_{jk}(s)\big).$$

It is shown in [29, Example 5.23] that $A(\mathcal{U}, \lambda_{ijk})$ is a C^*-algebra with

$$\big\|(f_{jk})\big\| = \sup_{s,i\in I_s} \big\|\pi_{i,s}\big((f_{jk})\big)\big\|.$$

In fact, and we shall need this later, $A(\mathcal{U}, \lambda_{ijk})$ is a continuous-trace algebra with spectrum S and Dixmier–Douady class $\delta_{\mathrm{DD}}(A(\mathcal{U}, \lambda_{ijk})) = [\lambda_{ijk}]$ (see [29, Proposition 5.40], which simplifies in our case because S is compact and the cover is finite).

For our new construction, we fix a point $*$ in U_1, and suppose that $*$ is not in any other U_i (we can ensure this is the case by replacing U_i with $U_i \setminus \{*\}$). We add a copy U_0 of U_1 to our cover, and set

$$Y := \bigsqcup_{i=0}^{n} U_i = \bigcup_{i=0}^{n} \big\{(s,i) : s \in U_i\big\}.$$

We define a relation $\sim$ on Y by $(s,i) \sim (s,j)$ if $s \neq *$, $(*,1) \sim (*,1)$, and $(*,0) \sim (*,0)$. This is an equivalence relation, and we define X to be the quotient space and $\psi : Y \to X$ to be the quotient map. Thus X consists of a copy of $S \setminus \{*\}$ with the subspace topology, and two closed points $\psi(*,0)$, $\psi(*,1)$ whose open neighbourhoods are the images under ψ of open sets $U \times \{0\}$ and $U \times \{1\}$, respectively.

Lemma 7.5. *The function $\psi : Y \to X$ is a surjective local homeomorphism.*

Proof. Quotient maps are always continuous and surjective, and ψ is injective on each $U_i \times \{i\}$. So it suffices to see that ψ is open, and for this, it suffices to see that for each open set W in U_i, $\psi(W \times \{i\})$ is open in X. By definition of the quotient topology, we need to show that $\psi^{-1}(\psi(W \times \{i\}))$ is open in Y. If $*$ is not in W, then

$$\psi^{-1}(\psi(W \times \{i\})) = \bigcup_{\{j:U_j\cap W\neq\emptyset\}} (W \cap U_j) \times \{j\}.$$

If $* \in W$ and $i = 0$, then

$$\psi^{-1}(\psi(W \times \{0\})) = \{W \times \{0\}\} \cup \left(\bigcup_{\{j:U_j\cap W\neq\emptyset\}} ((W \setminus \{*\}) \cap U_j) \times \{j\} \right)$$

is open, and similarly for $* \in W$ and $i = 1$. So $\psi^{-1}(\psi(W \times \{i\}))$ is always open, as required. $\qquad \square$

We extend λ to an alternating cocyle on the cover $\{U_0, U_1, \cdots, U_n\}$ by setting $\lambda_{0jk} = \lambda_{1jk}$. Then the formula

$$\sigma\big(((s,i)(s,j)), ((s,j)(s,k))\big) = \overline{\lambda_{ijk}(s)}$$

defines a continuous 2-cocycle σ on $R(\psi)$. Since ψ is a surjective local homeomorphism it follows from Proposition 4.3 that

$$R(\psi) := \{(y, z) \in Y \times Y : \psi(y) = \psi(z)\}$$

is a locally compact, Hausdorff, principal, étale and Cartan groupoid. By Corollary 5.4, $C^*(R(\psi), \sigma) = C_r^*(R(\psi), \sigma)$ is a Fell algebra with spectrum homeomorphic to X.

The $*$-algebra structure on $C_c(R(\psi), \sigma)$ is given by

$$(7.5) \qquad f^*((s, i), (s, j)) = \overline{f((s, j), (s, i))\lambda_{iji}(s)} = \overline{f((s, j), (s, i))}$$

$$(f * g)((s, i), (s, j)) = \sum_{\{k : s \in U_k\}} f((s, i), (s, k))g((s, k), (s, j))\overline{\lambda_{ikj}(s)},$$

and if (s, i) is a unit in $R(\psi)$ then the induced representation $\operatorname{Ind}_{(s,i)}^\sigma$ acts in $\ell^2(s^{-1}((s, i))) = \ell^2(\{((s, j), (s, i)) : s \in U_{ij}\})$ according to the formula

$$(7.6) \qquad (\operatorname{Ind}_{(s,i)}^\sigma(f)\xi)(s, j) = \sum_{\{k : s \in U_k\}} f((s, j), (s, k))\xi((s, k), (s, i))\overline{\lambda_{jki}(s)}.$$

Lemma 7.6. *There is a homomorphism* $\pi_0 : C^*(R(\psi), \sigma) \to \mathbb{C}$ *such that* $\pi_0(f) = f((*, 0), (*, 0))$ *for* $f \in C_c(R(\psi), \sigma)$.

Proof. The inverse image

$$s^{-1}((*, 0)) = \{((*, 0), (*, 0))\}$$

has just one point, so the Hilbert space $\ell^2(s^{-1}(*, 0))$ is one-dimensional, and $\operatorname{Ind}_{(*,0)}^\sigma(f)$ is multiplication by the complex number

$$f((*, 0), (*, 0))\overline{\lambda_{000}(1)} = f((*, 0), (*, 0)).$$

In other words, the representation $\operatorname{Ind}_{(*,0)}^\sigma$ of $C_c(R(\psi), \sigma)$ has the property we require of π_0. Since the reduced norm is $f \mapsto \sup_{y \in Y}\{\| \operatorname{Ind}_y^\sigma(f)\|\}$, the map $\operatorname{Ind}_{(*,0)}^\sigma$ is bounded for the reduced norm, and extends to a representation on $C_r^*(R(\psi), \sigma) = C^*(R(\psi), \sigma)$. $\qquad \square$

Lemma 7.7. *Let* $\mathcal{V} := \{V_0, V_1, \cdots, V_n\}$, *where* $V_0 := U_0 \setminus \{*\}$ *and* $V_i := U_i$ *for* $i \geq 1$. *Then the ideal* $\ker \pi_0$ *is isomorphic to the* C^*-*algebra* $A(\mathcal{V}, \lambda_{ijk})$ *of* [29, Example 5.23].

Proof. The maps defined by $\phi_{ij} : ((s, i), (s, j)) \mapsto s$ are homeomorphisms of $(V_i \times \{i\}) \times_\psi (V_j \times \{j\})$ onto V_{ij}. Thus for $f \in C_c(R(\psi))$ such that $\pi_0(f) = 0$, we can define $f_{ij} : V_{ij} \to \mathbb{C}$ by $f_{ij} = f \circ \phi_{ij}^{-1}$. For $\{i, j\} \neq \{0, 1\}$, the function f_{ij} has compact support, and extends uniquely to a continuous function f_{ij} on S with support in V_{ij}; because $f((*, 0), (*, 0)) = \pi_0(f) = 0$, the function f_{01} vanishes on the boundary of $V_{01} = V_0$, and extends to a continuous function f_{01} on S which vanishes off V_{01}.

We have constructed a map $\phi : f \mapsto (f_{ij})$ of $I_0 := C_c(R(\psi)) \cap \ker \pi_0$ into the underlying set of $A(\mathcal{V}, \lambda_{ijk})$. Since $f_{ij}(s) = f((s, i), (s, j))$, a comparison

of (7.5) with (7.3) shows that ϕ is a homomorphism. It is also $*$-preserving. If $s \in V_i$, then a comparison of (7.6) with (7.4), and an argument similar to the proof of Lemma 7.2, show that $\pi_{i,s} \circ \phi$ is unitarily equivalent to the representation $\mathrm{Ind}_{(s,i)}^{\sigma}$. Thus ϕ is isometric for the reduced norm on I_0, and since the range is dense in $A(\mathcal{V}, \lambda_{ijk})$, ϕ extends to an isomorphism of the closure $\ker \pi_0$ onto $A(\mathcal{V}, \lambda_{ijk})$. $\square$

Remark 7.8. If we delete the point $(*, 0)$ from X, we recover the original space S, the groupoid $R(\psi)$ is the one associated to the cover $\mathcal{V}$ of S in Remark 3 on page 399 of [27], and the isomorphism Φ of $C_{\mathrm{r}}^*(R(\psi), \sigma)$ with $A(\mathcal{V}, \lambda_{ijk})$ is discussed in that remark.

Theorem 1 of [27] (or Proposition 5.40 of [29]) implies that $A(\mathcal{V}, \lambda_{ijk})$ is a continuous-trace algebra with Dixmier–Douady class

$$\delta_{\mathrm{DD}}(A(\mathcal{V}, \lambda_{ijk})) = [\lambda_{ijk}] \neq 0.$$

This implies that the ideal $\ker \pi_0$ in $C^*(R(\psi), \sigma)$ has $\delta_{\mathrm{DD}}(\ker \pi_0) \neq 0$. Now Proposition 6.2 implies that $\delta(C^*(R(\psi), \sigma)) \neq 0$.

7.3. Epilogue. We started this project looking for a cocycle-based version of the Dixmier–Douady invariant of [13], which would enable us to resolve the issue about compatibility of $\delta(A)$ and $\delta_{\mathrm{DD}}(A)$ in [13, Remark 7.10], and to construct concrete families of Fell algebras as in [27]. Since the spectrum X of a Fell algebra is locally locally-compact and locally Hausdorff, it always has covers by open Hausdorff subsets such that the overlaps U_{ij} where cocycles live lie inside large Hausdorff subsets of X. So it seemed reasonable that cocycle-based arguments might work.

As we progressed, we realised how crucially the steps by which one refines covers, as in the proof of [29, Proposition 5.24], for example, depend on the existence of locally finite refinements. In the example of §7.1, this local finiteness fails spectacularly. So even though we know that that the algebra in §7.1 is a Fell algebra, and even though we know it must have vanishing Dixmier–Douady invariant, it is hard to see how a cocycle-based theory could accommodate it.

The second part of our project has worked to some extent, in that we can see how to build lots of Fell algebras from ordinary Čech cocycles. However, we can also see that the possibilities are almost limitless, and at this stage there seems little hope of finding a computable invariant.

Appendix A. Twisted groupoid C^*-algebras

There are several different ways of twisting the construction of a groupoid C^*-algebra. They include:

(a) Renault's $C^*(G, \sigma)$ associated to a 2-cocycle $\sigma : G^{(2)} \to \mathbb{T}$ on a groupoid G from [30, II.1] (which we discuss in §2 and use in §5 and §7);

  (b) Kumjian's $C^*(\Gamma; G)^{\text{Kum}}$ associated to a twist Γ over a principal, étale groupoid G in [18, §2] (which we use in §6);

  (c) Muhly and Williams' $C^*(E; G)^{\mathfrak{MW}}$ associated to an extension E of a groupoid G by $\mathbb{T}$ in [21, §2] (which we use in the proof of Theorem 5.1);

  (d) the reduced C^*-algebras $C_r^*(G, \sigma)$ and $C_r^*(E; G)^{\mathfrak{MW}}$ corresponding to (a) and (c), respectively.

Here we only consider second-countable, locally compact, Hausdorff and principal groupoids G with a left Haar system $\lambda = \{\lambda^u\}$. Let $\sigma : G^{(2)} \to \mathbb{T}$ be a continuous normalised 2-cocycle on G. Following [30, page 73] we denote by G^σ the associated extension of G by $\mathbb{T}$: thus G^σ is the groupoid $\mathbb{T} \times G$ with the product topology, with range and source maps $r(z, \alpha) = (1, r(\alpha))$ and $s(z, \alpha) = (1, s(\alpha))$, multiplication $(w, \alpha)(z, \beta) = (wz\sigma(\alpha, \beta), \alpha\beta)$ and inverse $(z, \alpha)^{-1} = (z^{-1}\sigma(\alpha, \alpha^{-1})^{-1}, \alpha^{-1})$. Then G^σ is a locally compact Hausdorff groupoid with left Haar system $\mu = \{\mu^u\}$, where μ^u is the product of the normalised Haar measure on $\mathbb{T}$ and λ^u.

    A twist Γ over a principal étale groupoid G has an underlying principal $\mathbb{T}$-bundle over G, and in [18, page 985] Kumjian observes that the twists whose underlying bundle is trivial are in one-to-one correspondence with continuous 2-cocycles σ.

    Set

$$C_c(G^\sigma; G) := \{f \in C_c(G^\sigma) : f(z \cdot \gamma) = zf(\gamma) \text{ for } z \in \mathbb{T}, \gamma \in G^\sigma\}.$$

It is easy to check that $C_c(G^\sigma; G)$ is a $*$-subalgebra of the usual convolution algebra $C_c(G^\sigma)$. The C^*-algebra $C^*(G^\sigma; G)^{\mathfrak{MW}}$ is by definition the completion of $C_c(G^\sigma; G)$ in the supremum norm $\|f\| = \sup\{\|L(f)\|\}$, where L ranges over a collection of appropriately continuous $*$-representations of $C_c(G^\sigma; G)$. By [4, Proposition 3.7], $C^*(G^\sigma; G)^{\mathfrak{MW}}$ is a direct summand of $C^*(G^\sigma)$.

    We know from [4, Lemma 3.1] that the map $\rho : C_c(G^\sigma; G) \to C_c(G, \overline{\sigma})$ defined by $\rho(f)(\alpha) = f(1, \alpha)$ is a $*$-isomorphism of $C_c(G^\sigma; G)$ onto Renault's twisted convolution algebra $C_c(G, \overline{\sigma})$ (the algebra $C_c(G^\sigma; G)$ is denoted $C_c(G^\sigma, -1)$ in [4]), and that ρ extends to an isomorphism

$$C^*(G^\sigma; G)^{\mathfrak{MW}} \cong C^*(G, \overline{\sigma}).$$

    Let $u \in G^{(0)}$. For the proof of Theorem 5.1 we need to know that the isomorphism ρ sends the class of certain irreducible representations L^u of $C^*(G^\sigma; G)^{\mathfrak{MW}}$ defined in [21, §3] to the class of the representation $\text{Ind}_u^{\overline{\sigma}}$ of $C^*(G, \overline{\sigma})$. The Hilbert space of L^u is the completion H_u of

$$H_u^0 := \{g \in C_c(G^\sigma; G) : \text{supp } g \subset \mathbb{T} \times s^{-1}(u)\}$$

with respect to the inner product $(f \mid g) = \int_G f(1, \alpha)\overline{g(1, \alpha)} \, d\lambda_u(\alpha)$. Let $f \in C_c(G^\sigma; G)$ and $g \in H_u^0$. Then $L^u(f)g = f * g$, where the convolution takes place in $C_c(G^\sigma; G)$. We compute using the formulas in [4, Remark 2.3]

and the cocycle identity for the triple $(\beta, \beta^{-1}, \alpha)$:

$$(A.1) \quad (L^u(f)g)(1, \alpha)$$

$$= \int_G f(1, \beta)g\big((1, \beta)^{-1}(1, \alpha)\big)\, d\lambda^{r(\alpha)}(\beta)$$

$$= \int_G f(1, \beta)g\big(\overline{\sigma(\beta, \beta^{-1})}\sigma(\beta^{-1}, \alpha), \beta^{-1}\alpha\big)\, d\lambda^{r(\alpha)}(\beta)$$

$$= \int_G f(1, \beta)g\big(\overline{\sigma(\beta, \beta^{-1}\alpha)}\sigma(\beta\beta^{-1}, \alpha), \beta^{-1}\alpha\big)\, d\lambda^{r(\alpha)}(\beta)$$

$$= \int_G f(1, \beta)g(1, \beta^{-1}\alpha)\overline{\sigma(\beta, \beta^{-1}\alpha)}\, d\lambda^{r(\alpha)}(\beta).$$

Define $U : C_c(G^\sigma; G) \to L^2(s^{-1}(u), \lambda_u)$ by $(Ug)(\alpha) = g(1, \alpha)$. Then U extends to a unitary U from the Hilbert space H_u of L^u onto the Hilbert space $L^2(s^{-1}(u), \lambda_u)$ of $\mathrm{Ind}_u^{\overline{\sigma}}$. We have

$$(\mathrm{Ind}_u^{\overline{\sigma}}(\rho(f))U(g))(\alpha) = \int_G \rho(f)(\beta)U(g)(\beta^{-1}\alpha)\overline{\sigma(\beta, \beta^{-1}\alpha)}\, d\lambda^{r(\alpha)}(\beta)$$

$$= \int_G f(1, \beta)g(1, \beta^{-1}\alpha)\overline{\sigma(\beta, \beta^{-1}\alpha)}\, d\lambda^{r(\alpha)}(\beta),$$

which is $(U(L^u(f)g))(\alpha)$ using (A.1). We have proved:

Lemma A.1. *Let G be a principal groupoid with a Haar system and $\sigma : G^{(2)} \to \mathbb{T}$ a continuous normalised 2-cocycle. Then for each unit u, $\mathrm{Ind}_u^{\overline{\sigma}} \circ \rho$ is unitarily equivalent to L^u.*

Now let G be a principal and étale groupoid. As shown in [18, pages 977–8], there is a positive-definite $C_0(G^{(0)})$-valued inner product on $C_c(G^\sigma; G)$, and the action of $C_c(G^\sigma; G)$ by left multiplication on itself extends to an action by adjointable operators on the Hilbert-module completion $H(G^\sigma)$. The C^*-algebra $C^*(G^\sigma; G)^{\mathrm{Kum}}$ is by definition the completion of $C_c(G^\sigma; G)$ in $\mathcal{L}(H(G^\sigma))$.

Proposition A.2. *Suppose that G is an étale and principal groupoid, and $\sigma : G^{(2)} \to \mathbb{T}$ is a continuous normalised 2-cocycle. The homomorphism ρ of $C_c(G^\sigma; G)$ onto $C_c(G, \overline{\sigma})$ extends to an isomorphism of $C^*(G^\sigma; G)^{\mathrm{Kum}}$ onto the reduced crossed product $C_r^*(G, \overline{\sigma})$.*

In view of what we already know about ρ from [4], it suffices to check that ρ is isometric for the given norm on $C_c(G^\sigma) \subset \mathcal{L}(H(G^\sigma))$ and the reduced norm on $C_c(G, \overline{\sigma})$. The general theory of Hilbert bimodules says that, if π is a faithful representation of $C_0(G^{(0)})$, then the induced representation $H(G^\sigma)$-Ind π is faithful on $\mathcal{L}(H(G^\sigma))$. We can in particular take π to be the atomic representation $\bigoplus_{u \in G^{(0)}} \epsilon_u$, and then $H(G^\sigma)$-Ind $\pi = \bigoplus_{u \in G^{(0)}} H(G^\sigma)$-Ind ϵ_u. So Proposition A.2 follows from the following lemma.

Lemma A.3. *Let G be an étale and principal groupoid, and $\sigma : G^{(2)} \to \mathbb{T}$ a continuous normalised 2-cocycle. For each $u \in G^{(0)}$, the representation $H(G^\sigma)$-Ind ϵ_u is unitarily equivalent to the representation $(\mathrm{Ind}_u^{\overline{\sigma}}) \circ \rho$ of $C_c(G^\sigma; G)$.*

Proof. As G is étale, the representation $\mathrm{Ind}_u^{\overline{\sigma}}$ of $C^*(G, \overline{\sigma})$ acts on $\ell^2(s^{-1}(u))$ by the formula

$$(A.2) \qquad (\mathrm{Ind}_u^{\overline{\sigma}}(h)\xi)(\alpha) = \sum_{r(\beta)=r(\alpha)} h(\beta)\xi(\beta^{-1}\alpha)\overline{\sigma(\beta, \beta^{-1}\alpha)}$$

for $h \in C_c(G)$, $\xi \in \ell^2(s^{-1}(u))$, $\alpha \in G$. The representation $H(G^\sigma)$-Ind ϵ_u acts on (the completion of) $C_c(G^\sigma; G) \otimes_{C_0(G^{(0)})} \mathbb{C}$, which is $C_c(G^\sigma; G)$ with the inner product $(f \mid g) = (g^* * f)(u)$. The Haar system on G^σ is the product of the normalised Haar measure on $\mathbb{T}$ and the counting measure on $s^{-1}(u)$. Let $f, g \in C_c(G^\sigma; G)$. We have

$$(f \mid g) = (g^* * f)(u)$$

$$= \int_\mathbb{T} \left(\sum_{s(\alpha)=u} g^*(\overline{z}, \alpha^{-1})f(z, \alpha) \right) dz = \int_\mathbb{T} \left(\sum_{s(\alpha)=u} \overline{g(z, \alpha)}f(z, \alpha) \right) dz$$

$$= \int_\mathbb{T} \left(\sum_{s(\alpha)=u} \overline{zg(1, \alpha)}zf(1, \alpha) \right) dz = \sum_{s(\alpha)=u} f(1, \alpha)\overline{g(1, \alpha)},$$

and it follows that the Hilbert space of $H(G^\sigma)$-Ind ϵ_u is the space H_u. The action of $H(G^\sigma)$-Ind $\epsilon_u(f)$ on g is by multiplication, so $H(G^\sigma)$-Ind $\epsilon_u(f) = L^u(f)$. Now the result follows from Lemma A.1. $\qquad \square$

Corollary A.4. *Suppose that G is a second-countable, locally compact, Hausdorff, étale, principal and Cartan groupoid, and that σ is a continuous normalised 2-cocycle on G. Then*

$$(A.3) \qquad C^*(G^\sigma; G)^{\mathfrak{MW}} \cong C^*(G, \overline{\sigma}) = C_r^*(G, \overline{\sigma}) \cong C^*(G^\sigma; G)^{\mathrm{Kum}}.$$

Proof. The first isomorphism is from [4, Lemma 3.1], the equality in the middle is from Theorem 5.1, and the second isomorphism is from Proposition A.2. $\qquad \square$

Remark A.5. We can relax the hypothesis "G is Cartan" in Corollary A.4. That hypothesis was used in the proof of Theorem 5.1 to see that $u \mapsto [L^u]$ is a homeomorphism of $G^{(0)}/G$ onto $(C^*(G^\sigma; G)^{\mathfrak{MW}})^\wedge$; then Lemma A.1 implies that every irreducible representation of $C^*(G, \sigma)$ is induced, and we have equality in the middle of (A.3). However, if we merely know that "$G^{(0)}/G$ is T_0", then we can use [6, Theorem 3.4] in place of [6, Proposition 3.2] in the proof of Theorem 5.1, still deduce that $u \mapsto [L^u]$ is a homeomorphism, and follow the rest of the argument to get $C^*(G, \overline{\sigma}) = C_r^*(G, \overline{\sigma})$.

Appendix B. The ideal of continuous-trace elements

If A is a C^*-algebra, we write $\mathfrak{m}(A)$ for the ideal spanned by the positive elements a such that $\pi \mapsto \mathrm{tr}(\pi(a))$ is continuous on $\hat{A}$, as in [7, §4.5.2]. The closure of $\mathfrak{m}(A)$ is an ideal in A, which we call the *ideal of continuous-trace elements*.

Proposition B.1. *Suppose that A is a C^*-algebra. Then*

$$U := \{\pi \in \hat{A} : \pi \text{ has a closed Hausdorff neighbourhood}\}$$

is an open Hausdorff subset of $\hat{A}$.

Proof. Let $x \in U$, and choose a closed Hausdorff neighbourhood N of x. Then for each point y in the interior $\mathrm{int}\, N$, N is a closed Hausdorff neighbourhood of y, and hence $y \in U$. Thus U is open.

To see that U is Hausdorff, let $x, y \in U$, and choose closed Hausdorff neighbourhoods N_x of x and N_y of y. If $x \in \mathrm{int}\, N_y$, then since N_y is Hausdorff we can choose open sets V_x and V_y in N_y such that $x \in V_x$, $y \in V_y$ and $V_x \cap V_y = \emptyset$; then $W_x := V_x \cap (\mathrm{int}\, N_y)$ and $W_y = V_y \cap (\mathrm{int}\, N_y)$ are open subsets of $\mathrm{int}\, N_y$ with the same property, and they are open in $\hat{A}$. A similar argument works if $y \in N_x$. It remains to deal with the case where $x \notin \mathrm{int}\, N_y$ and $y \notin \mathrm{int}\, N_x$. Since $\hat{A}$ is locally locally-compact, $\mathrm{int}\, N_y$ contains a compact neighbourhood K of y. Since N_y is Hausdorff, K is closed in N_y, and since N_y is closed in $\hat{A}$, K is closed in $\hat{A}$. Now $(\mathrm{int}\, N_x) \setminus K$ is open in $\hat{A}$, and contains x because $x \notin \mathrm{int}\, N_y$. Thus $(\mathrm{int}\, N_x) \setminus K$ and $\mathrm{int}\, K$ are disjoint open neighbourhoods of x and y. $\qquad\square$

Corollary B.2. *Suppose that A is a C^*-algebra. Then*

$$V := \{\pi \in \hat{A} : \pi \text{ is a Fell point and has a closed Hausdorff neighbourhood}\}$$

is an open Hausdorff subset of $\hat{A}$.

Proof. The set of Fell points is an open subset of $\hat{A}$, so V is the intersection of two open sets. It is Hausdorff because it is a subset of the Hausdorff set U of Proposition B.1. $\qquad\square$

Corollary B.3. *Suppose that A is a C^*-algebra. Then the set V of Corollary B.2 is the spectrum of the ideal of continuous-trace elements of A.*

Proof. We show that $\pi \in V$ if and only if π satisfies conditions (i) and (ii) of [15, Lemma 2.2], and then our corollary follows from that lemma.

First suppose that $\pi \in V$. Then since V is open it is the spectrum of an ideal J of A. Since π is a Fell point of A it is also a Fell point of J. To see this, let $a \in A^+$ and W an open neighbourhood of π in $\hat{A}$ such that $\rho(a)$ is a rank-one projection for all $\rho \in W$. Let $f \in C_c(\hat{J})^+$ such that f is identically one on a neighbourhood W' of π in $\hat{J}$. Using the Dauns–Hofmann theorem, $f \cdot a$ is in J^+, and $\rho(f \cdot a) = f(\rho)\rho(a) = \rho(a)$ is a rank-one projection for

$\rho \in W \cap W'$. Thus J is a continuous-trace algebra by [7, Proposition 4.5.4], and π satisfies (i).

To verify (ii), note that π has a closed Hausdorff neighbourhood N, and a neighbourhood base of compact sets [7, Corollary 3.3.8]. Since $\pi \in \operatorname{int} N$, the compact neighbourhoods K with $K \subset \operatorname{int} N$ also form a neighbourhood base. Since N is Hausdorff and closed, such K are also closed in $\hat{A}$.

Next suppose that π satisfies (i) and (ii). Then π belongs to the spectrum of a continuous-trace ideal J, and is a Fell point in $\hat{J}$; since $\hat{J}$ is open in $\hat{A}$, it trivially follows that π is a Fell point in $\hat{A}$ and that $\hat{J}$ is an open Hausdorff neighbourhood of π. Now (ii) implies that $\hat{J}$ contains a closed neighbourhood, and $\pi \in V$. □

References

[1] ARCHBOLD, ROBERT; AN HUEF, ASTRID. Strength of convergence and multiplicities in the spectrum of a C^*-dynamical system. *Proc. London Math. Soc.* (3) **96** (2008), no. 3, 545–581. MR2407813 (2009c:46092), Zbl 1155.46037, arXiv:math/0612296, doi: 10.1112/plms/pdm038.

[2] ARCHBOLD, R. J.; SOMERSET, D. W. B. Transition probabilities and trace functions for C^*-algebras. *Math. Scand.* **73** (1993), no. 1, 81–111. MR1251700 (95f:46095), Zbl 0792.46043.

[3] BOURBAKI, NICOLAS. General topology. *Hermann, Paris*, 1966.

[4] BROWN, JONATHAN H.; AN HUEF, ASTRID. Decomposing the C^*-algebras of groupoid extensions. To appear in *Proc. Amer. Math. Soc.* arXiv:1105.0718.

[5] CLARK, LISA ORLOFF. Classifying the types of principal groupoid C^*-algebras. *J. Operator Theory* **57** (2007), no. 2, 251–266. MR2328998 (2008i:46050), Zbl 1120.46036, arXiv:math/0511450.

[6] CLARK, LISA ORLOFF; AN HUEF, ASTRID. The representation theory of C^*-algebras associated to groupoids. *Math. Proc. Cambridge Philos. Soc.* **153** (2012), no. 1, 167–191. MR3043775, Zbl 06054438, arXiv:1007.2331, doi: 10.1017/S0305004112000047.

[7] DIXMIER, JACQUES. C^*-algebras. Translated from the French by Francis Jellett. North-Holland Mathematical Library, 15. *North-Holland Publishing Co., Amsterdam-New York-Oxford*, 1977. xiii+492 pp. ISBN: 0-7204-0762-1. MR0458185 (56 #16388), Zbl 0657.46040.

[8] DIXMIER, JACQUES; DOUADY, ADRIEN. Champs continus d'espaces hilbertiens et de C^*-algèbres. *Bull. Soc. Math. France* **91** (1963), 227–284. MR0163182 (29 #485), Zbl 0127.33102.

[9] FELL, J. M. G. The structure of algebras of operator fields. *Acta Math.* **106** (1961), 233–280. MR0164248 (29 #1547), Zbl 0101.09301, doi: 10.1007/BF02545788.

[10] GREEN, PHILIP. C^*-algebras of transformation groups with smooth orbit space. *Pacific J. Math.* **72** (1977), no. 1, 71–97. MR0453917 (56 #12170), Zbl 0374.46047.

[11] HAZLEWOOD, ROBERT; AN HUEF, ASTRID. Strength of convergence in the orbit space of a groupoid. *J. Math. Anal. Appl.* **383** (2011), no. 1, 1–24. MR2812714 (2012f:22004), Zbl 1232.46053, arXiv:1006.3115.

[12] AN HUEF, ASTRID. The transformation groups whose C^*-algebras are Fell algebras. *Bull. London Math. Soc.* **33** (2001), no. 1, 73–76. MR1798578 (2001k:46087), Zbl 1034.46060, arXiv:math/9907153, `http://journals.cambridge.org/article_S0024609300007554`.

[13] AN HUEF, ASTRID; KUMJIAN, ALEX; SIMS, AIDAN. A Dixmier–Douady theorem for Fell algebras. *J. Funct. Anal.* **260** (2011), no. 5, 1543–1581. MR2749438 (2012i:46066), Zbl 1251.46028, doi: 10.1016/j.jfa.2010.11.011.

[14] AN HUEF, ASTRID; RAEBURN, IAIN; WILLIAMS, DANA P. Properties preserved under Morita equivalence of C^*-algebras. *Proc. Amer. Math. Soc.* **135** (2007), no. 5, 1495–1503. MR2276659 (2008d:46071), Zbl 1118.46050, arXiv:math/0512159, doi: 10.1090/S0002-9939-06-08625-4.

[15] AN HUEF, ASTRID; WILLIAMS, DANA P. Ideals in transformation-group C^*-algebras. *J. Operator Theory* **48** (2002), no. 3, 535–548. MR1962470 (2004a:46065), Zbl 1029.46081, arXiv:math/0008013.

[16] KELLEY, JOHN L. General topology. *D. Van Nostrand Company, Inc. Toronto-New York-London*, 1955. xiv+298 pp. MR0070144 (16,1136c), Zbl 0066.16604.

[17] KUMJIAN, ALEXANDER. Preliminary algebras arising from local homeomorphisms. *Math. Scand.* **52** (1983), no. 2, 269–278. MR0702958 (85b:46078), Zbl 0541.46047.

[18] KUMJIAN, ALEXANDER. On C^*-diagonals. *Canad. J. Math.* **38** (1986), no. 4, 969–1008. MR0854149 (88a:46060), Zbl 0627.46071, doi: 10.4153/CJM-1986-048-0.

[19] KUMJIAN, ALEX. On equivariant sheaf cohomology and elementary C^*-bundles. *J. Operator Theory* **20** (1988), no. 2, 207–240. MR1004121 (90h:46102), Zbl 0692.46066.

[20] KUMJIAN, ALEX; PASK, DAVID; RAEBURN, IAIN; RENAULT, JEAN. Graphs, groupoids and Cuntz–Krieger algebras. *J. Funct. Anal.* **144** (1997), no. 2, 505–541. MR1432596 (98g:46083), Zbl 0929.46055, doi: 10.1006/jfan.1996.3001.

[21] MUHLY, PAUL S.; WILLIAMS, DANA P. Continuous trace groupoid C^*-algebras. *Math. Scand.* **66** (1990), no. 2, 231–241. MR1075140 (91j:46081), Zbl 0782.46048.

[22] MUHLY, PAUL S.; WILLIAMS, DANA P. Continuous trace groupoid C^*-algebras. II. *Math. Scand.* **70** (1992), no. 1, 127–145. MR1174207 (93i:46117), Zbl 0783.46028.

[23] MUNKRES, JAMES R. Topology. 2nd edition. *Prentice Hall, Upper Saddle River, NJ*, 2000. Zbl 0951.54001.

[24] PEDERSEN, GERT K. Analysis now. Graduate Texts in Mathematics, 118. *Springer-Verlag, New York*, 1989. xiv+277 pp. ISBN: 0-387-96788-5. MR0971256 (90f:46001), Zbl 0668.46002.

[25] PEDERSEN, GERT K. C^*-algebras and their automorphism groups. London Mathematical Society Monographs, 14. *Academic Press, Inc. London-New York*, 1979. ix+416 pp. ISBN: 0-12-549450-5. MR0548006 (81e:46037), Zbl 0416.46043.

[26] RAEBURN, IAIN. Graph algebras. CBMS Regional Conference Series in Mathematics, 103. *American Mathematical Society, Providence, RI*, 2005. vi+113 pp. ISBN: 0-8218-3660-9. MR2135030 (2005k:46141), Zbl 1079.46002.

[27] RAEBURN, IAIN; TAYLOR, JOSEPH L. Continuous trace C^*-algebras with given Dixmier–Douady class. *J. Austral. Math. Soc (Ser. A)* **38** (1985), no. 3, 394–407. MR0779202 (86g:46086), Zbl 0627.46068.

[28] RAEBURN, IAIN; WILLIAMS, DANA P. Topological invariants associated with the spectrum of crossed product C^*-algebras. *J. Funct. Anal.* **116** (1993), no. 2, 245–276. MR1239072 (95b:46096), Zbl 0813.46061, doi: 10.1006/jfan.1993.1113.

[29] RAEBURN, IAIN; WILLIAMS, DANA P. Morita equivalence and continuous-trace C^*-algebras. Mathematical Surveys and Monographs, 60. *American Mathematical Society, Providence, RI*, 1998. xiv+327 pp. ISBN: 0-8218-0860-5. MR1634408 (2000c:46108), Zbl 0922.46050.

[30] RENAULT, JEAN. A groupoid approach to C^*-algebras. Lecture Notes in Mathematics, 793. *Springer, Berlin*, 1980. ii+160 pp. ISBN: 3-540-09977-8. MR0584266 (82h:46075), Zbl 0433.46049.

[31] RUDIN, WALTER. Real and complex analysis. Second edition. *McGraw-Hill Book Co., Boston*, 1991.

[32] SEDA, ANTHONY KAREL. On the continuity of Haar measure on topological groupoids. *Proc. Amer. Math. Soc.* **96** (1986), no. 1, 115–120. MR0813822 (87m:46146), Zbl 0593.28015, doi: 10.1090/S0002-9939-1986-0813822-2.

[33] WILLIAMS, DANA P. Crossed products of C^*-algebras. Mathematical Surveys and Monographs, 134. *American Mathematical Society, Providence, RI*, 2007. xvi+528 pp. ISBN: 978-0-8218-4242-3; 0-8218-4242-0. MR2288954 (2007m:46003), Zbl 1119.46002.

[34] ZETTL, HEINRICH H. Ideals in Hilbert modules and invariants under strong Morita equivalence of C^*-algebras. *Arch. Math. (Basel)* **39** (1982), no. 1, 69–77. MR0674535 (84i:46060), Zbl 0498.46034, doi: 10.1007/BF01899246.

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