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Semigroups in which all strongly summable ultrafilters are sparse

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Abstract. We show that if $(S,+)$ is a commutative semigroup which can be embedded in the circle group $\mathbb{T}$, in particular if $S=(\mathbb{N},+)$, then all nonprincipal, strongly summable ultrafilters on S are sparse and can be written as sums in βS only trivially. We develop a simple condition on a strongly summable ultrafilter which guarantees that it is sparse and show that this holds for many ultrafilters on semigroups which are embeddable in the direct sum of countably many copies of $\mathbb{T}$.

Contents

1. Introduction

In 1972 the first author of this paper became aware of a question of Fred Galvin. This question was whether there exists an ultrafilter p on the set $\mathbb{N}$ of positive integers such that, for any $A\in p$, $\{x\in\mathbb{N}:-x+A\in p\}\in p$. Galvin called such an ultrafilter *almost translation invariant*. (One can appreciate the terminology if one views an ultrafilter as a $\{0,1\}$-valued measure on $\mathcal{P}(\mathbb{N})$.) Galvin wanted to know because he knew that the existence of such an ultrafilter trivially implied the validity of what was then called the *Graham–Rothschild conjecture*. That is, if $r\in\mathbb{N}$ and $\mathbb{N}=\bigcup_{i=1}^{r}A_i$, then there exist $i\in\{1,2,\ldots,r\}$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty})\subseteq A_i$, where $FS(\langle x_n\rangle_{n=1}^{\infty})=\{\sum_{t\in F}x_t : F\in\mathcal{P}_f(\mathbb{N})\}$ and given a set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X. In [5] that author showed that, assuming

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the continuum hypothesis, the Graham–Rothschild conjecture implies the existence of almost translation invariant ultrafilters. With the subsequent proof [6] of the Graham–Rothschild conjecture, almost translation invariant ultrafilters became a figment of the continuum hypothesis.

Of course Galvin wanted to know whether almost translation invariant ultrafilters really existed. He subsequently ran into Steven Glazer who knew of a result of Robert Ellis [3, Lemma 1] that a compact right topological semigroup has an idempotent and who also knew that, given a discrete semigroup $(S, +)$, there is a natural extension of the operation on S to its Stone-Čech compactification βS making βS a compact right topological semigroup, where βS is taken to be the set of ultrafilters on S. Further, an idempotent $p \in \beta\mathbb{N}$ is precisely an almost translation invariant ultrafilter. We need to know very little about the operation on βS in this paper, and then only in the proof of Theorem 4.8. That is the fact that, given $p, q \in \beta S$ and $A \subseteq S$, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$. For more background information, the interested reader is referred to [7] for more information than she could possibly want to know.

Consequently, there seemed to no longer be anything of interest in [5]. However, in 1985 Eric van Douwen pointed out in conversation that the almost translation invariant ultrafilters produced there had a stronger property. That is, they had a basis consisting of sets of the form $FS(\langle x_n \rangle_{n=1}^{\infty})$.

Definition 1.1. Let $(S, +)$ be a commutative semigroup and let p be an ultrafilter on S. Then p is *strongly summable* if and only if for every $A \in p$ there exists $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$ and $FS(\langle x_n \rangle_{n=1}^{\infty}) \in p$.

Naturally, van Douwen wanted to know whether the existence of strongly summable ultrafilters on $\mathbb{N}$ could be established in ZFC. In fact, their existence implies the existence of P-points in $\beta\mathbb{N}$ [2, Theorem 3], and so cannot be established in ZFC. An important tool in the verification of this fact was the notion of *union ultrafilter* introduced by Blass in [1].

Given a sequence $\langle F_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$, we let

$$FU(\langle F_n \rangle_{n=1}^{\infty}) = \left\{ \bigcup_{t \in H} F_t : H \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

Definition 1.2. A *union ultrafilter* is an ultrafilter $\mathcal{U}$ on $\mathcal{P}_f(\mathbb{N})$ with the property that for each $\mathcal{A} \in \mathcal{U}$, there is a sequence $\langle F_n \rangle_{n=1}^{\infty}$ of pairwise disjoint members of $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle F_n \rangle_{n=1}^{\infty}) \subseteq \mathcal{A}$ and $FU(\langle F_n \rangle_{n=1}^{\infty}) \in \mathcal{U}$.

It was shown in [1] that Martin's Axiom implies the existence of union ultrafilters and in [2] that the existence of union ultrafilters is equivalent to the existence of strongly summable ultrafilters on $\mathbb{N}$.

An important fact about strongly summable ultrafilters on $\mathbb{N}$ is that, in one sense at least, they are badly named. That is, [7, Theorem 12.42], if p is a strongly summable ultrafilter and $q, r \in \mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ are such that

$q + r = r + q = p$, then in fact $q, r \in \mathbb{Z} + p$. So the largest subgroup of $\beta\mathbb{N}$ with p as identity is just a copy of $\mathbb{Z}$. In fact, some strongly summable ultrafilters are even harder to write as sums.

Definition 1.3. Let $(S, +)$ be a countable commutative semigroup and let p be an ultrafilter on S. Then p is a *sparse strongly summable ultrafilter* if and only if for every $A \in p$, there exist a sequence $\langle x_n \rangle_{n=1}^{\infty}$ and a subsequence $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$, $FS(\langle y_n \rangle_{n=1}^{\infty}) \in p$, and $\{x_n : n \in \mathbb{N}\} \setminus \{y_n : n \in \mathbb{N}\}$ is infinite.

We view the circle group $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$ and represent its elements by points in $(-\frac{1}{2}, \frac{1}{2}]$. (So when we say $t \in \mathbb{T}$ we really mean that $t + \mathbb{Z} \in \mathbb{T}$.) By S_d we mean the set S with the discrete topology.

Theorem 1.4. *Let S be a countable subsemigroup of $\mathbb{T}$, let G be the subgroup of $\mathbb{T}$ generated by S, and let p be a sparse strongly summable ultrafilter on S. If $q, r \in \beta G_d$ and $q + r = p$, then $q, r \in G + p$.*

Proof. It is routine to show that p is a sparse strongly summable ultrafilter on G. (To be precise, the ultrafilter on G generated by p is a sparse strongly summable ultrafilter on G.) So [8, Theorem 4.5] applies. □

In [9] Peter Krautzberger established that all strongly summable ultrafilters on $\mathbb{N}$ have a property that he called *special*. And in a personal communication he easily showed that any special strongly summable ultrafilter on $\mathbb{N}$ is sparse. (And, less easily, the two notions are equivalent for strongly summable ultrafilters on $\mathbb{N}$.) Had he been aware of [8, Theorem 4.5] he could have included the following theorem.

Theorem 1.5 (Krautzberger)**.** *Let p be a strongly summable ultrafilter on $\mathbb{N}$ and let $q, r \in \beta\mathbb{Z}$ such that $q + r = p$. Then $q, r \in \mathbb{Z} + q$.*

Proof. Since all strongly summable ultrafilters on $\mathbb{N}$ are sparse, Theorem 1.4 applies. □

In Section 2 of this paper we will show that there does not exist a union ultrafilter on $\mathcal{P}_f(\mathbb{N})$ such that for all $\mathcal{A} \in \mathcal{U}$, $\mathbb{N} \setminus \bigcup \mathcal{A}$ is finite. In fact this is a special case of [9, Theorem 4]. However, we believe our proof is much simpler than the proof of that theorem.

In Section 3 we will show that any strongly summable ultrafilter on a commutative semigroup $(S, +)$ which satisfies a strong uniqueness of finite sums condition is sparse.

In Section 4 we will show that if S is a countable subsemigroup of $\mathbb{T}$ and p is a strongly summable ultrafilter on S, then p is sparse. Consequently, by Theorem 1.4 any strongly summable ultrafilter on S can only be written as a sum trivially. In this section we will also show that if S is a countable subsemigroup of $\bigoplus_{n=1}^{\infty} \mathbb{T}$, p is a strongly summable ultrafilter on S, and the set of points whose first nonzero coordinate is $\frac{1}{2}$ is not a member of p, then p is sparse.

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2. Nonexistence of certain union ultrafilters

In [9, Theorem 4], Krautzberger showed that if $\mathcal{U}$ is a union ultrafilter and L is an infinite subset of $\mathbb{N}$, then there is a member $\mathcal{A}$ of $\mathcal{U}$ such that $L \setminus \bigcup \mathcal{A}$ is infinite. We establish in this section the (at least superficially) weaker assertion that for any union ultrafilter $\mathcal{U}$, there is a member $\mathcal{A}$ of $\mathcal{U}$ such that $\mathbb{N} \setminus \bigcup \mathcal{A}$ is infinite. We do this because we believe our proof is significantly simpler than the proof of [9, Theorem 4].

Definition 2.1. Let $X \in \mathcal{P}_f(\mathbb{N})$.

(a) A block of X is a maximal interval contained in X.
(b) $\Pi(X) = \{I : I \text{ is a block of } X\}$.
(c) $\varphi(X) = |\Pi(X)|$.
(d) $P : \mathcal{P}_f(\mathbb{N}) \to \{0,1\}$ is defined by $P(X) \equiv \varphi(X) \pmod 2$.
(e) If $Y \in \mathcal{P}_f(\mathbb{N})$ and $X \subseteq Y$, then $L(X,Y) = \{x \in X : x - 1 \in \mathbb{N} \setminus Y\}$ and $R(X,Y) = \{x \in X : x + 1 \in \mathbb{N} \setminus Y\}$.

It is trivial that there is no pairwise disjoint sequence $\langle F_n \rangle_{n=1}^{\infty}$ with P constantly equal to 1 on $FU(\langle F_n \rangle_{n=1}^{\infty})$.

Lemma 2.2. *Let $A, B \in \mathcal{P}_f(\mathbb{N})$ with $A \cap B = \emptyset$. Then*

$$\begin{aligned}\varphi(A \cup B) \equiv \varphi(A) + \varphi(B) &+ |\{x \in A : x - 1 \in B\}| \\ &+ |\{x \in A : x + 1 \in B\}| \pmod 2.\end{aligned}$$

Proof. Given $I \in \Pi(A \cup B)$, let

$$\tau(I) = |\{J : J \in \Pi(A) \text{ and } J \subseteq I\}| + |\{J : J \in \Pi(B) \text{ and } J \subseteq I\}|.$$

Suppose we have a counterexample and choose one with

$$k = \max\{\tau(I) : I \in \Pi(A \cup B)\}$$

as small as possible and with $|\{I \in \Pi(A \cup B) : \tau(I) = k\}|$ as small as possible.

Assume first that $k = 1$. Then $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ and

$$|\{x \in A : x - 1 \in B\}| = |\{x \in A : x + 1 \in B\}| = 0.$$

Thus $k > 1$. Pick $L \in \Pi(A \cup B)$ such that $\tau(L) = k$ and let $y = \max L$.

Case 1. $y \in A$. Pick $J \in \Pi(A)$ such that $y \in J$ and let $A' = A \setminus J$. Then, since $J \neq L$, it follows that $L \setminus J \neq \emptyset$ and either

$$\max\{\tau(I) : I \in \Pi(A' \cup B)\} < k \quad \text{or}$$
$$|\{I \in \Pi(A' \cup B) : \tau(I) = k\}| < |\{I \in \Pi(A \cup B) : \tau(I) = k\}|.$$

Therefore the minimality hypothesis implies that

$$\begin{aligned}\varphi(A\cup B) &= \varphi(A'\cup B)\\ &\equiv \varphi(A')+\varphi(B)+|\{x\in A' : x-1\in B\}|\\ &\quad + |\{x\in A' : x+1\in B\}| \pmod 2\\ &= \varphi(A)-1+\varphi(B)+|\{x\in A : x-1\in B\}|-1\\ &\quad + |\{x\in A : x+1\in B\}|.\end{aligned}$$

The second case is symmetric to the first. □

We omit the routine proof of the following lemma.

Lemma 2.3. *Let $\langle F_k\rangle_{k=1}^n$ be a disjoint sequence in $\mathcal{P}_f(\mathbb{N})$. Then*

$$L(F_1,F_1) = L(F_1,\textstyle\bigcup_{k=1}^n F_k)\cup\bigcup_{k=2}^n\{x\in F_1 : x-1\in F_k\},$$
$$R(F_1,F_1) = R(F_1,\textstyle\bigcup_{k=1}^n F_k)\cup\bigcup_{k=2}^n\{x\in F_1 : x+1\in F_k\}.$$

Lemma 2.4. *Let $G,H,X\in\mathcal{P}_f(\mathbb{N})$ and assume that $1\in G\subseteq H$, $X=\{1,2,\ldots,\max G+1\}\setminus H$, and, if $X\neq\emptyset$, then $\{1,2,\ldots,\max X+1\}\setminus X\subseteq G$. Then:*

(1) $L(G,H)=\{x\in\mathbb{N} : \big(\exists I\in\Pi(X)\big)(x-1=\max I)\}$.
(2) $R(G,H)=\{x\in\mathbb{N} : \big(\exists I\in\Pi(X)\big)(x+1=\min I)\}$.
(3) $|L(G,H)|=|R(G,H)|=\varphi(X)$.

Proof. This is routine. The fact that $|R(G,H)|=\varphi(X)$ uses the fact that $1\notin X$. □

Theorem 2.5. *There does not exist a pairwise disjoint sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\mathbb{N}\setminus\bigcup_{n=1}^\infty F_n$ is finite and P is constantly equal to 0 on $FU(\langle F_n\rangle_{n=1}^\infty)$.*

Proof. Suppose we have such a sequence and order it so that for all n, $\min F_n<\min F_{n+1}$. We first note that we can assume that $1\in F_1$. To see this, let $a=\min F_1$, assume that $a>1$, and pick $I\in\Pi(F_1)$ such that $a\in I$. Let $F_1'=F_1\cup\{1,2,\ldots,a-1\}$ and for each $n>1$, let $F_n'=F_n$. Then $I\cup\{1,2,\ldots,a-1\}\in\Pi(F_1')$ and for each $H\in\mathcal{P}_f(\mathbb{N})$, $\varphi(\bigcup_{n\in H}F_n')=\varphi(\bigcup_{n\in H}F_n)$.

Let $X=\mathbb{N}\setminus\bigcup_{n=1}^\infty F_n$. We claim that if $X\neq\emptyset$, then we can assume that $\{1,2,\ldots,\max X+1\}\setminus X\subseteq F_1$. To see this, let $m=\max X$, let $F_1'=\bigcup_{k=1}^{m+1}F_k$ and for $n>1$, let $F_n'=F_{m+n}$. Then for each $H\in\mathcal{P}_f(\mathbb{N})$,

$$\varphi(\textstyle\bigcup_{n\in H}F_n')=\varphi(\bigcup_{n\in K}F_n),$$

where $K=\{1,2,\ldots,m+1\}\cup\big\{m+n : n\in H\setminus\{1\}\big\}$ if $1\in H$, while $K=\{m+n : n\in H\}$ if $1\notin H$.

Now $|L(F_1,F_1)|=\varphi(F_1)-1$ is odd and $|R(F_1,F_1)|=\varphi(F_1)$ is even. Let $r=\max F_1$ and let $H=\bigcup_{k=1}^{r+1}F_k$. We have that $1\in F_1\subseteq H$, $\{1,2,\ldots,\max X+1\}\setminus X\subseteq F_1$, and

$$X=\{1,2,\ldots,\max F_1+1\}\setminus H$$

so by Lemma 2.4, $|L(F_1,H)| = |R(F_1,H)| = \varphi(X)$. Also by Lemma 2.3,

$$L(F_1,F_1) = L(F_1,H) \cup \bigcup_{k=2}^{r+1}\{x \in F_1 : x-1 \in F_k\},$$
$$R(F_1,F_1) = R(F_1,H) \cup \bigcup_{k=2}^{r+1}\{x \in F_1 : x+1 \in F_k\}.$$

Therefore

$$\begin{aligned}&|L(F_1,F_1)| + |R(F_1,F_1)|\\ &= 2\varphi(X) + \textstyle\sum_{k=2}^{r+1}(|\{x \in F_1 : x-1 \in F_k\}| + |\{x \in F_1 : x+1 \in F_k\}|)\\ &\equiv \textstyle\sum_{k=2}^{r+1}(|\{x \in F_1 : x-1 \in F_k\}| + |\{x \in F_1 : x+1 \in F_k\}|) \pmod 2\end{aligned}$$

so that $\sum_{k=2}^{r+1}(|\{x \in F_1 : x-1 \in F_k\}| + |\{x \in F_1 : x+1 \in F_k\}|)$ is odd so we may pick $k \in \{2,3,\ldots,r+1\}$ such that $|\{x \in F_1 : x-1 \in F_k\}| + |\{x \in F_1 : x+1 \in F_k\}|$ is odd.

By Lemma 2.2

$$\begin{aligned}\varphi(F_1 \cup F_k) \equiv{}& \varphi(F_1) + \varphi(F_k) + \\ & |\{x \in F_1 : x-1 \in F_k\}| + |\{x \in F_1 : x+1 \in F_k\}| \pmod 2.\end{aligned}$$

But this is a contradiction, because $\varphi(F_1 \cup F_k)$, $\varphi(F_1)$, and $\varphi(F_k)$ are all even. □

Theorem 2.6. *Let $\mathcal{U}$ be a union ultrafilter. Then there exists $\mathcal{A} \in \mathcal{U}$ such that $\mathbb{N} \setminus \bigcup \mathcal{A}$ is infinite.*

Proof. Suppose that $\mathcal{U}$ is a union ultrafilter such that for every $\mathcal{A} \in \mathcal{U}$ one has $\mathbb{N} \setminus \bigcup \mathcal{A}$ is finite. Pick $i \in \{0,1\}$ such that

$$\mathcal{B} = \{F \in \mathcal{P}_f(\mathbb{N}) : P(F) = i\} \in \mathcal{U}.$$

Pick a disjoint sequence $\langle F_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle F_n \rangle_{n=1}^\infty) \in \mathcal{U}$ and $FU(\langle F_n \rangle_{n=1}^\infty) \subseteq \mathcal{B}$. One cannot have $i = 1$ since we may choose a subsequence $\langle G_n \rangle_{n=1}^\infty$ with $\max G_n < \min G_{n+1}$ for each n. But since $FU(\langle F_n \rangle_{n=1}^\infty) \in \mathcal{U}$, we have $\mathbb{N} \setminus \bigcup_{n=1}^\infty F_n$ is finite, so by Theorem 2.5, we cannot have $i = 0$ either. □

3. Strong uniqueness of finite sums

We show in this section that any strongly summable ultrafilter which is generated by sums of sequences satisfying a simple condition is sparse.

Definition 3.1. A sequence $\langle x_n \rangle_{n=1}^\infty$ in a commutative semigroup $(S,+)$ satisfies *strong uniqueness of finite sums* if and only if for all $F, H \in \mathcal{P}_f(S)$:

(1) If $\sum_{t\in F} x_t = \sum_{t\in H} x_t$, then $F = H$.
(2) If $\sum_{t\in F} x_t + \sum_{t\in H} x_t \in FS(\langle x_n \rangle_{n=1}^\infty)$, then $F \cap H = \emptyset$.

It is a consequence of [7, Lemmas 12.20 and 12.34] that if p is a strongly summable ultrafilter on $\mathbb{N}$, then p has a filter base consisting of sets of the form $FS(\langle x_n \rangle_{n=1}^\infty)$ where $\langle x_n \rangle_{n=1}^\infty$ satisfies strong uniqueness of finite sums.

Theorem 3.2. *Let $(S,+)$ be a commutative semigroup and let p be an ultrafilter on S. If for every $A \in p$ there is a sequence $\langle x_n\rangle_{n=1}^\infty$ in S satisfying strong uniqueness of finite sums such that $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A$ and $FS(\langle x_n\rangle_{n=1}^\infty) \in p$, then p is a sparse strongly summable ultrafilter.*

Proof. Trivially p is strongly summable. Suppose that p is not sparse. Pick $A \in p$ such that for every sequence $\langle x_n\rangle_{n=1}^\infty$ in S and every subsequence $\langle y_n\rangle_{n=1}^\infty$ of $\langle x_n\rangle_{n=1}^\infty$, if $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A$ and $FS(\langle y_n\rangle_{n=1}^\infty) \in p$, then

$$\{x_n : n \in \mathbb{N}\} \setminus \{y_n : n \in \mathbb{N}\}$$

is finite. Pick a sequence $\langle x_n\rangle_{n=1}^\infty$ satisfying strong uniqueness of finite sums such that $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A$ and $FS(\langle x_n\rangle_{n=1}^\infty) \in p$. Given $B \in p$, let $\mathcal{F}(B) = \{F \in \mathcal{P}_f(\mathbb{N}) : \sum_{t\in F} x_t \in B\}$ and let $\mathcal{U} = \{\mathcal{F}(B) : B \in p\}$. It is routine to establish that $\mathcal{U}$ is an ultrafilter on $\mathcal{P}_f(\mathbb{N})$.

We claim that $\mathcal{U}$ is a union ultrafilter. To see this, let $B \in p$. We need to show that there is a pairwise disjoint sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle F_n\rangle_{n=1}^\infty) \subseteq \mathcal{F}(B)$ and $FU(\langle F_n\rangle_{n=1}^\infty) \in \mathcal{U}$.

Now $B \cap FS(\langle x_n\rangle_{n=1}^\infty) \in p$ so pick a sequence $\langle y_n\rangle_{n=1}^\infty$ in S such that $FS(\langle y_n\rangle_{n=1}^\infty) \in p$ and $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq B \cap FS(\langle x_n\rangle_{n=1}^\infty)$. For each $n \in \mathbb{N}$, pick $F_n \in \mathcal{P}_f(\mathbb{N})$ such that $y_n = \sum_{t\in F_n} x_t$. Since $\langle x_n\rangle_{n=1}^\infty$ satisfies strong uniqueness of finite sums, we have that $F_n \cap F_m = \emptyset$ when $n \neq m$.

We now claim that $FU(\langle F_n\rangle_{n=1}^\infty) = \mathcal{F}\big(FS(\langle y_n\rangle_{n=1}^\infty)\big)$ so that

$$FU(\langle F_n\rangle_{n=1}^\infty) \subseteq \mathcal{F}(B) \text{ and } FU(\langle F_n\rangle_{n=1}^\infty) \in \mathcal{U}$$

as required. To see this, first let $H \in \mathcal{P}_f(\mathbb{N})$ and let $K = \bigcup_{n\in H} F_n$. Then

$$\begin{aligned}\textstyle\sum_{t\in K} x_t &= \textstyle\sum_{n\in H}\sum_{t\in F_n} x_t \\ &= \textstyle\sum_{n\in H} y_n \in FS(\langle y_n\rangle_{n=1}^\infty)\end{aligned}$$

so $K \in \mathcal{F}\big(FS(\langle y_n\rangle_{n=1}^\infty)\big)$. For the other inclusion, let $K \in \mathcal{F}\big(FS(\langle y_n\rangle_{n=1}^\infty)\big)$ and pick $H \in \mathcal{P}_f(\mathbb{N})$ such that $\sum_{t\in K} x_t = \sum_{n\in H} y_n$. Let $L = \bigcup_{n\in H} F_n$. Then $\sum_{t\in K} x_t = \sum_{n\in H}\sum_{t\in F_n} x_t = \sum_{t\in L} x_t$. By the uniqueness of finite sums $K = L$.

Since $\mathcal{U}$ is a union ultrafilter, by Theorem 2.6 we may pick $\mathcal{A} \in \mathcal{U}$ such that $\mathbb{N} \setminus \bigcup\mathcal{A}$ is infinite. Pick $B \in p$ such that $\mathcal{A} = \mathcal{F}(B)$. As before pick a sequence $\langle y_n\rangle_{n=1}^\infty$ in S such that

$$FS(\langle y_n\rangle_{n=1}^\infty) \in p \text{ and } FS(\langle y_n\rangle_{n=1}^\infty) \subseteq B \cap FS(\langle x_n\rangle_{n=1}^\infty).$$

Also as before, pick $F_n \in \mathcal{P}_f(\mathbb{N})$ such that $y_n = \sum_{t\in F_n} x_t$. Let $L = \bigcup_{n=1}^\infty F_n$. Then $\langle x_t\rangle_{t\in L}$ is a subsequence of $\langle x_n\rangle_{n=1}^\infty$ and $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq FS(\langle x_t\rangle_{t\in L})$ so $FS(\langle x_t\rangle_{t\in L}) \in p$. But then $\mathbb{N}\setminus L$ is finite and $L = \bigcup FU(\langle F_n\rangle_{n=1}^\infty) \subseteq \bigcup\mathcal{A}$, so $\mathbb{N} \setminus \bigcup\mathcal{A} \subseteq \mathbb{N} \setminus L$, a contradiction. □

4. Semigroups embedded in the direct sum of circle groups

We begin this section by showing that countable semigroups that are embeddable in the circle group do not have any nonsparse strongly summable ultrafilters. As a consequence, any strongly summable ultrafilter on such a semigroup can be written as a sum only trivially.

Recall that we view the circle group as $\mathbb{R}/\mathbb{Z}$ and let $t \in (-\frac{1}{2}, \frac{1}{2}]$ represent the coset $t + \mathbb{Z}$.

Lemma 4.1. *Let S be a subsemigroup of $\mathbb{T}$ and let $\widetilde{\iota} : \beta S_d \to \mathbb{T}$ be the continuous extension of the inclusion map. If p is an ultrafilter on S with the property that each member of p contains $FS(\langle x_n \rangle_{n=1}^{\infty})$ for some sequence $\langle x_n \rangle_{n=1}^{\infty}$, in particular if p is strongly summable, then $\widetilde{\iota}(p) = 0$.*

Proof. Suppose that $\widetilde{\iota}(p) = x \neq 0$. Then (addition in $\mathbb{T}$) we have $x + x \neq x$ so pick a neighborhood U of x such that $(U + U) \cap U = \emptyset$. Pick $A \in p$ such that $\widetilde{\iota}[\,\overline{A}\,] \subseteq U$. Pick a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. Then $x_1 + x_2 \in (U + U) \cap U$, a contradiction. □

In view of Lemma 4.1, the following theorem follows from Theorem 4.5. However, its proof is simpler.

Theorem 4.2. *Let S be a countable subsemigroup of $\mathbb{T}$ and let p be a nonprincipal, strongly summable ultrafilter on S. Then p has a basis of sets of the form $FS(\langle x_n \rangle_{n=1}^{\infty})$ for a sequence which satisfies strong uniqueness of finite sums. Consequently, p is sparse.*

Proof. In view of Theorem 3.2, it suffices to show that p has a basis of sets of the form $FS(\langle x_n \rangle_{n=1}^{\infty})$ for a sequence which satisfies strong uniqueness of finite sums. Let $\widetilde{\iota} : \beta S_d \to \mathbb{T}$ be the continuous extension of the inclusion map. By Lemma 4.1 we know $\widetilde{\iota}(p) = 0$ so there is some $A \in p$ such that $\widetilde{\iota}[\,\overline{A}\,] \subseteq (-\frac{1}{4}, \frac{1}{4})$. Since p is nonprincipal, $\{0\} \notin p$ so either

$$\{x \in S : x \in (-\tfrac{1}{4}, 0)\} \in p \quad \text{or} \quad \{x \in S : x \in (0, \tfrac{1}{4})\} \in p.$$

Essentially without loss of generality we assume that $\{x \in S : x \in (0, \frac{1}{4})\} \in p$. For $j \in \{0, 1, 2\}$, let $X_j = \bigcup_{m=0}^{\infty} \big[\frac{1}{2^{3m+j+3}}, \frac{1}{2^{3m+j+2}}\big)$. Then $(0, \frac{1}{4}) = X_0 \cup X_1 \cup X_2$ so pick $j \in \{0, 1, 2\}$ such that $X_j \in p$.

Let $B \in p$. We need to show that there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \in p$, $FS(\langle x_n \rangle_{n=1}^{\infty} \subseteq B$, and $\langle x_n \rangle_{n=1}^{\infty}$ satisfies strong uniqueness of finite sums. Since p is strongly summable, pick a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \in p$ and $FS(\langle x_n \rangle_{n=1}^{\infty} \subseteq X_j \cap B$. Note that if $\frac{1}{2^{3m+j+3}} \leq a \leq b < \frac{1}{2^{3m+j+2}}$, then $\frac{1}{2^{3m+j+2}} \leq a + b < \frac{1}{2^{3m+j+1}}$ and so $a + b \notin X_j$. Thus there is at most one x_n in each interval of the form $\big[\frac{1}{2^{3m+j+3}}, \frac{1}{2^{3m+j+2}}\big)$. We may thus assume that the sequence $\langle x_n \rangle_{n=1}^{\infty}$ is strictly decreasing and thus for each $n \in \mathbb{N}$, $4x_{n+1} < x_n$. As a consequence we have for each $n \in \mathbb{N}$ that $x_n > 3\sum_{t=n+1}^{\infty} x_t$. From this we conclude easily that if $F, H \in \mathcal{P}_f(\mathbb{N})$, for each $n \in F$, $a_n \in \{1, 2\}$, for each $n \in H$, $b_n \in \{1, 2\}$, and $\sum_{n \in F} a_n x_n = \sum_{n \in H} b_n x_n$, then $F = H$ and for each $n \in F$, $a_n = b_n$. We then have directly that $\langle x_n \rangle_{n=1}^{\infty}$ satisfies the first requirement of Definition 3.1. To verify the second requirement, let

$F, H, K \in \mathcal{P}_f(\mathbb{N})$ and assume that $\sum_{t\in F} x_t + \sum_{t\in H} x_t = \sum_{t\in K} x_t$. Then $\sum_{t\in F\triangle H} x_t + \sum_{t\in F\cap H} 2x_t = \sum_{t\in K} x_t$, so $F \cap H = \emptyset$ as required. □

If S is any discrete commutative semigroup contained in a discrete group G, then $\beta S \subseteq \beta G$. By [7, Theorem 4.23] if p is any idempotent in βS and $x \in G$, then $(x + p) + (-x + p) = p$.

Corollary 4.3. *Let S be a countable subsemigroup of $\mathbb{T}$, let G be the group generated by S, and let p be a nonprincipal, strongly summable ultrafilter on S. If $x, y \in \beta S_d \setminus S$ and $x + y = p$, then $x, y \in G + p$.*

Proof. By Theorem 4.2 p is sparse. But then it is routine to verify that p is also sparse when viewed as an ultrafilter on G. (Precisely,

$$\{A \subseteq G : A \cap S \in p\}$$

is a sparse ultrafilter on G.) So [8, Theorem 4.5] applies. □

We will use the following simple lemma. Note however, that it is possible that $\widetilde{h}(p) = 0$ even when p is nonprincipal.

Lemma 4.4. *Let S and T be discrete semigroups, let $h : S \to T$ be a homomorphism, and let $\widetilde{h} : \beta S \to \beta T$ be its continuous extension. If p is a strongly summable ultrafilter on S, then $\widetilde{h}(p)$ is a strongly summable ultrafilter on T.*

Proof. Let $A \in \widetilde{h}(p)$. Then $\pi^{-1}[A] \in p$ so pick a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \in p$ and $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq h^{-1}[A]$. Since h is a homomorphism, we have $h[FS(\langle x_n\rangle_{n=1}^{\infty})] = FS(\langle h(x_n)\rangle_{n=1}^{\infty})$ so $FS(\langle h(x_n)\rangle_{n=1}^{\infty}) \in \widetilde{h}(p)$ and $FS(\langle h(x_n)\rangle_{n=1}^{\infty}) \subseteq A$. □

The proof of the following theorem is adapted from the proof of [8, Lemma 3.3]. Given $x \in (\bigoplus_{n=1}^{\infty} \mathbb{T}) \setminus \{0\}$ we let $\mathrm{supp}(x) = \{i \in \mathbb{N} : \pi_i(x) \neq 0\}$ and $\min(x) = \min \mathrm{supp}(x)$.

Theorem 4.5. *Let S be a countable subsemigroup of $\bigoplus_{n=1}^{\infty} \mathbb{T}$ and let p be a nonprincipal, strongly summable ultrafilter on S. If*

$$\{x \in S : \pi_{\min(x)}(x) \neq \tfrac{1}{2}\} \in p,$$

then there exists $X \in p$ such that any sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq X$ has the property that if $F, H \in \mathcal{P}_f(\mathbb{N})$, $a_n \in \{1, 2\}$ for each $n \in F$, $b_n \in \{1, 2\}$ for each $n \in H$, and $\sum_{n\in F} a_n x_n = \sum_{n\in H} b_n x_n$, then $F = H$ and $a_n = b_n$ for each $n \in F$. In particular each such $\langle x_n\rangle_{n=1}^{\infty}$ satisfies strong uniqueness of finite sums. Consequently, p has a basis of sets of the form $FS(\langle x_n\rangle_{n=1}^{\infty})$ for which $\langle x_n\rangle_{n=1}^{\infty}$ satisfies strong uniqueness of finite sums and p is sparse.

Proof. Again by virtue of Theorem 3.2, to see that p is sparse, it suffices to show that p has a basis of sets of the form $FS(\langle x_n\rangle_{n=1}^{\infty})$ for a sequence which satisfies strong uniqueness of finite sums. And trivially any sequence

$\langle x_n\rangle_{n=1}^\infty$ satisfies strong uniqueness of finite sums if it has the property that whenever $F,H \in \mathcal{P}_f(\mathbb{N})$, $a_n \in \{1,2\}$ for each $n \in F$, $b_n \in \{1,2\}$ for each $n \in H$, and $\sum_{n\in F} a_n x_n = \sum_{n\in H} b_n x_n$, one has $F = H$ and $a_n = b_n$ for each $n \in F$. So it suffices to produce a set $X \in p$ as in the statement of the theorem. Essentially without loss of generality, we may assume that $\{x \in S : 0 < \pi_{\min(x)}(x) < \frac{1}{2}\} \in p$.

For $j \in \{0,1,2\}$, let $X_j = \bigcup_{m=0}^\infty \left[\frac{1}{2^{3m+j+2}}, \frac{1}{2^{3m+j+1}}\right)$ and pick $j \in \{0,1,2\}$ such that $X = \{x \in S : \pi_{\min(x)}(x) \in X_j\} \in p$. Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in S such that $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq X$. For each $i \in \mathbb{N}$, let $M_i = \{n \in \mathbb{N} : \min(x_n) = i\}$.

We first note that if $n,t \in M_i$ and $n \neq t$, then $\min(x_n + x_t) = i$ because $0 < \pi_i(x_n) < \frac{1}{2}$ and $0 < \pi_i(x_t) < \frac{1}{2}$ so $\pi_i(x_n + x_t) \neq 0$. Assume that $\pi_i(x_n) \leq \pi_i(x_t)$. We cannot have some m such that $\frac{1}{2^{3m+j+2}} \leq \pi_i(x_n) \leq \pi_i(x_t) \leq \frac{1}{2^{3m+j+1}}$ and thus $4\pi_i(x_n) \leq \pi_i(x_t)$. Consequently, if $F \in \mathcal{P}_f(M_i)$, then $\min(\sum_{n\in F} x_n) = i$.

We now claim that if $y \in S$ and $y = \sum_{n\in F} a_n x_n$ where for each $n \in F$, $a_n \in \{1,2\}$, then for each $n \in F$, $\min(x_n) \geq \min(y)$. Suppose instead that $i = \min\{\min(x_n) : n \in F\} < \min(y)$. Let $H = \{n \in F \cap M_i : a_n = 2\}$. Since for $n \in F \setminus M_i$, $\min(x_n) > i$ we have

$$\begin{aligned} 0 &= \pi_i(y) \\ &= \textstyle\sum_{n\in F\cap M_i} a_n \pi_i(x_n) \\ &= \textstyle\sum_{n\in F\cap M_i} \pi_i(x_n) + \sum_{n\in H} \pi_i(x_n) \\ &= \pi_i(\textstyle\sum_{n\in F\cap M_i} x_n) + \pi_i(\sum_{n\in H} x_n). \end{aligned}$$

But this is impossible since both $\pi_i(\sum_{n\in F\cap M_i} x_n)$ and $\pi_i(\sum_{n\in H} x_n)$ are in $(0, \frac{1}{2})$.

To complete the proof, suppose that $\langle x_n\rangle_{n=1}^\infty$ does not have the property in the statement of the theorem and pick a counterexample with $|F|$ as small as possible. Let $y = \sum_{n\in F} a_n x_n = \sum_{n\in H} b_n x_n$ and let $i = \min(y)$. Then

$$\pi_i(y) = \textstyle\sum_{n\in F} a_n \pi_i(x_n) = \sum_{n\in F\cap M_i} a_n \pi_i(x_n)$$

since $\pi_i(x_n) = 0$ for $i \in F \setminus M_i$. Also $\pi_i(y) = \sum_{n\in H\cap M_i} b_n \pi_i(x_n)$. Essentially as in the proof of Theorem 4.2 we conclude that $F \cap M_i = H \cap M_i$ and $a_n = b_n$ for $n \in F \cap M_i$. And, of course, $F \cap M_i \neq \emptyset$ since $\pi_i(y) \neq 0$. If $F' = F \setminus M_i$ and $H' = H \setminus M_i$, we have that $\sum_{n\in F'} a_n x_n = \sum_{n\in H'} b_n x_n$. Since $|F'| < |F|$ we have that $F' = H'$ and $a_n = b_n$ for all $n \in F'$. □

Corollary 4.6. *Let $(S,+)$ be a countable, commutative, and cancellative semigroup, and let q be a nonprincipal, strongly summable ultrafilter on S. If q is not sparse, then $\{x \in S : (\exists n \in \mathbb{N})(o(x) = 2^n)\} \in q$.*

Proof. We may assume that there is a sequence of groups $\langle G_n\rangle_{n=1}^\infty$ such that the group $S - S \subseteq T = \bigoplus_{n=1}^\infty G_n$ where each G_n is a subgroup of $\mathbb{T}$ which is either isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}[p^\infty]$ for some prime p. (See for example [10, Assertions 4.1.5 and 4.1.6].) Let $M = \{n \in \mathbb{N} : G_n \not\approx \mathbb{Z}[2^\infty]\}$ and let $T_0 = \bigoplus_{n \in M} G_n$. (If $M = \emptyset$ then all elements of S are of order 2^n for some n.) Let $\pi : T \to T_0$ be the natural surjection and let $\widetilde{\pi} : \beta T_d \to \beta(T_0)_d$ be its continuous extension.

Let $q_0 = \widetilde{\pi}(q)$. By Lemma 4.4. q_0 is strongly summable. No coordinate of any member of T_0 is $\frac{1}{2}$ so by Theorem 4.5, if q_0 is nonprincipal, then it is sparse. We shall show that this leads to a contradiction, and consequently q_0 must be principal, and thus $q_0 = 0$. So suppose that q_0 is sparse and let $X \in q_0$ be as guaranteed by Theorem 4.5. We claim that q is sparse. To see this let $A \in q$. Then $\pi[A] \in q_0$. Pick a sequence $\langle x_n \rangle_{n=1}^\infty$ such that $FS(\langle x_n \rangle_{n=1}^\infty) \in q$ and $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A \cap \pi^{-1}[X]$. Then $FS(\langle \pi(x_n) \rangle_{n=1}^\infty) \subseteq X$, and $FS(\langle \pi(x_n) \rangle_{n=1}^\infty) \in q_0$. Pick a sequence $\langle a_n \rangle_{n=1}^\infty$ and a subsequence $\langle b_n \rangle_{n=1}^\infty$ of $\langle a_n \rangle_{n=1}^\infty$ such that $FS(\langle a_n \rangle_{n=1}^\infty) \subseteq FS(\langle \pi(x_n) \rangle_{n=1}^\infty)$, $FS(\langle b_n \rangle_{n=1}^\infty) \in q_0$, and $\{a_n : n \in \mathbb{N}\} \setminus \{b_n : n \in \mathbb{N}\}$ is infinite. For each $n \in \mathbb{N}$ pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $a_n = \sum_{t \in H_n} \pi(x_t)$. Since $\langle \pi(x_n) \rangle_{n=1}^\infty$ satisfies strong uniqueness of finite sums, $H_n \cap H_k = \emptyset$ when $n \neq k$. In particular $\langle a_n \rangle_{n=1}^\infty$ is injective. Thus if $b_n = a_{k(n)}$ for $n \in \mathbb{N}$, we have $\mathbb{N} \setminus \{k(n) : n \in \mathbb{N}\}$ is infinite. For each $n \in \mathbb{N}$, let $c_n = \sum_{t \in H_n} x_t$ and let $d_n = c_{k(n)}$. Since $\pi(c_n) = a_n$ we have that $\langle c_n \rangle_{n=1}^\infty$ is injective and therefore $\{c_n : n \in \mathbb{N}\} \setminus \{d_n : n \in \mathbb{N}\}$ is infinite. Now $FS(\langle c_n \rangle_{n=1}^\infty) \subseteq FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A$. Also $\pi[FS(\langle d_n \rangle_{n=1}^\infty)] = FS(\langle b_n \rangle_{n=1}^\infty) \in q_0$ so $FS(\langle d_n \rangle_{n=1}^\infty) \in q$. thus q is sparse as claimed.

Thus $q_0 = 0$ so $\pi^{-1}[\{0\}] \in q$. And $\pi^{-1}[\{0\}] \cap S \in q$ and

$$\pi^{-1}[\{0\}] \cap S \subseteq \{0\} \cup \{x \in S : (\exists n \in \mathbb{N})(o(x) = 2^n)\}$$

and so $\{x \in S : (\exists n \in \mathbb{N})(o(x) = 2^n)\} \in q$. □

Corollary 4.7. *Let G be a countable abelian group with only finitely many elements of order 2 and let p be a nonprincipal strongly summable ultrafilter on G. Then p is sparse.*

Proof. Suppose p is not sparse. Let H be the subgroup of G consisting of all elements whose order is a power of 2. By Corollary 4.6, $H \in p$. Pick $k \in \mathbb{N}$ such that if $o(x) = 2$, $\min(x) \leq k$. Note that if $x \in H \setminus \{0\}$, then $\min(x) \leq k$. Indeed for such x, $\{x^n : n \in \mathbb{N}\}$ is a subgroup of H which has a member y of order 2. Then $k \geq \min(y) \geq \min(x)$. Thus we may pick $i \in \{1, 2, \ldots, k\}$ such that $\{x \in H : \min(x) = i\} \in p$. Since there do not exist x_1 and x_2 such that $\{x_1, x_2, x_1 + x_2\} \subseteq \{x \in H : \min(x) = i \text{ and } x_{\min(x)} = \frac{1}{2}\}$, we must have that $\{x \in H : x_{\min(x)} \neq \frac{1}{2}\} \in p$. Therefore, by Theorem 4.5, p is sparse. □

Theorem 4.8. *Let $(S, +)$ be a commutative semigroup embedded in a group G, let p be a nonprincipal strongly summable ultrafilter on S, and assume that there is a sequence $\langle x_n \rangle_{n=1}^\infty$ such that $FS(\langle x_n \rangle_{n=1}^\infty) \in p$ and whenever $F, H \in \mathcal{P}_f(\mathbb{N})$, $a_n \in \{1, 2\}$ for each $n \in F$, $b_n \in \{1, 2\}$ for each $n \in H$, and*

$\sum_{n\in F} a_n x_n = \sum_{n\in H} b_n x_n$, one has $F = H$ and $a_n = b_n$ for each $n \in F$. If $q, r \in \beta G_d$ and $q + r = p$, then $q, r \in G + p$.

Proof. Pick a sequence $\langle x_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty) \in p$ and whenever $F, H \in \mathcal{P}_f(\mathbb{N})$, $a_n \in \{1, 2\}$ for each $n \in F$, $b_n \in \{1, 2\}$ for each $n \in H$, and $\sum_{n\in F} a_n x_n = \sum_{n\in H} b_n x_n$, one has $F = H$ and $a_n = b_n$ for each $n \in F$. Let $X = FS(\langle x_n\rangle_{n=1}^\infty)$. Notice that the sequence $\langle 3^n\rangle_{n=1}^\infty$ shares the above property with $\langle x_n\rangle_{n=1}^\infty$. That is, whenever $F, H \in \mathcal{P}_f(\mathbb{N})$, $a_n \in \{1, 2\}$ for each $n \in F$, $b_n \in \{1, 2\}$ for each $n \in H$, and $\sum_{n\in F} a_n 3^n = \sum_{n\in H} b_n 3^n$, one has $F = H$ and $a_n = b_n$ for each $n \in F$. Consequently, the following statements hold for any $F, H, L, K \in \mathcal{P}_f(\mathbb{N})$.

(1) $\sum_{n\in F} x_n + \sum_{n\in H} x_n = \sum_{n\in K} x_n + \sum_{n\in L} x_n$ if and only if $\sum_{n\in F} 3^n + \sum_{n\in H} 3^n = \sum_{n\in K} 3^n + \sum_{n\in L} 3^n$.
(2) $\sum_{n\in F} x_n - \sum_{n\in H} x_n = \sum_{n\in K} x_n - \sum_{n\in L} x_n$ if and only if $\sum_{n\in F} 3^n - \sum_{n\in H} 3^n = \sum_{n\in K} 3^n - \sum_{n\in L} 3^n$.
(3) $\sum_{n\in F} x_n - \sum_{n\in H} x_n + \sum_{n\in K} x_n = \sum_{n\in L} x_n$ if and only if $\sum_{n\in F} 3^n - \sum_{n\in H} 3^n + \sum_{n\in K} 3^n = \sum_{n\in L} 3^n$.

Indeed, both statements in (1) hold if and only if $F \triangle H = K \triangle L$ and $F \cap H = K \cap L$; both statements in (2) hold if and only if $F \triangle L = H \triangle K$ and $F \cap L = H \cap K$; and both statements in (3) hold if and only if $F \triangle K = H \triangle L$ and $F \cap K = H \cap L$.

By virtue of statement (2) we can define $\phi : X - X \to \mathbb{Z}$ by

$$\phi(\textstyle\sum_{n\in F} x_n - \sum_{n\in H} x_n) = \sum_{n\in F} 3^n - \sum_{n\in H} 3^n.$$

Note that $\phi[X] \subseteq \mathbb{N}$. For $y \in G \setminus (X - X)$, define $\phi(y) = 0$ (or any other value). Let $\widetilde{\phi} : \beta G_d \to \beta\mathbb{Z}$ be the continuous extension of ϕ.

We claim that $\widetilde{\phi}(p)$ is strongly summable on $\mathbb{N}$. To see this, let $A \in \widetilde{\phi}(p)$. Then $X \cap \phi^{-1}[A] \in p$ so pick a sequence $\langle y_n\rangle_{n=1}^\infty$ in S such that $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq X \cap \phi^{-1}[A]$ and $FS(\langle y_n\rangle_{n=1}^\infty) \in p$. For each $n \in \mathbb{N}$, pick $F_n \in \mathcal{P}_f(\mathbb{N})$ such that $y_n = \sum_{t\in F_n} x_t$. Given $n \neq k$ in $\mathbb{N}$, we have $y_n + y_k \in FS(\langle x_n\rangle_{n=1}^\infty)$ so $\sum_{t\in F_n} x_t + \sum_{t\in F_k} x_t = \sum_{t\in L} x_t$ for some $L \in \mathcal{P}_f(\mathbb{N})$, so $F_n \cap F_k = \emptyset$. Therefore we have that $\phi[FS(\langle y_n\rangle_{n=1}^\infty)] = FS(\langle \phi(y_n)\rangle_{n=1}^\infty)$ so $FS(\langle \phi(y_n)\rangle_{n=1}^\infty) \in p$ and $FS(\langle \phi(y_n)\rangle_{n=1}^\infty) \subseteq A$.

Assume now that we have $q, r \in \beta G_d$ such that $q+r = p$. We need to show that $q, r \in G + p$. We claim that we can make the additional assumption that $X \in r$. To see this, since $X \in p$, $\{b \in G : -b + X \in r\} \in q$ so pick some $b \in G$ such that $-b + X \in r$. If $r' = b + r$ and $q' = -b + q$, then $q' + r' = p$ and $X \in r'$. If we show that $q', r' \in G + p$ it follows immediately that $q, r \in G + p$. So we assume that $X \in r$.

Next we show that:

($*$) If $u, v, w \in \beta G_d$, $u + v = w$, and $v, w \in \overline{X}$, then $X - X \in u$.
($\dagger$) If $u, v \in \beta G_d$, $u + v = p$, $X - X \in u$, and $X \in v$, then

$$\widetilde{\phi}(u) + \widetilde{\phi}(v) = \widetilde{\phi}(p).$$

To verify $(*)$, let $B = \{a \in G : -a + X \in v\}$. Since $X \in w$, we have that $B \in u$. We claim that $B \subseteq X - X$. So let $a \in B$. Then $(-a + X) \cap X \in v$ so $(-a + X) \cap X \neq \emptyset$ and thus $a \in X - X$.

To verify $(\dagger)$, let $A \in \widetilde{\phi}(p)$. We shall show that $A \in \widetilde{\phi}(u) + \widetilde{\phi}(v)$. Let $B = X \cap \phi^{-1}[A]$. Then $B \in p = u + v$ so $\{s \in G : -s + B \in v\} \in u$. Let $C = \{s \in X - X : -s + B \in v\}$. Since $X - X \in u$, $C \in u$ and so $\phi[C] \in \widetilde{\phi}(u)$. We claim that $\phi[C] \subseteq \{k \in \mathbb{Z} : -k + \phi[B] \in \widetilde{\phi}(v)\}$, so that $\phi[B] \in \widetilde{\phi}(u) + \widetilde{\phi}(v)$ and therefore $A \in \widetilde{\phi}(u) + \widetilde{\phi}(v)$ as required. Let $s \in C$. We need to show that $-\phi(s) + \phi[B] \in \widetilde{\phi}(v)$. Now $X \cap (-s + B) \in v$ so it suffices to show that $\phi[X \cap (-s + B)] \subseteq -\phi(s) + \phi[B]$. So let $w \in X \cap (-s + B)$. Then $s + w \in B \subseteq X$ so by statement (3), $\phi(s) + \phi(w) = \phi(s + w)$ and thus $\phi(w) \in -\phi(s) + \phi[B]$.

By $(*)$, $X - X \in q$ and by $(\dagger)$, $\widetilde{\phi}(q) + \widetilde{\phi}(r) = \widetilde{\phi}(p)$ so by Theorem 1.5 pick $m \in \mathbb{Z}$ such that $\widetilde{\phi}(r) = m + \widetilde{\phi}(p)$. Now $\phi[X] \in \widetilde{\phi}(r)$ and $\phi[X] \in \widetilde{\phi}(p)$ so $m \in \phi[X] - \phi[X] = \phi[X - X]$. Pick $c \in X - X$ such that $m = \phi(c)$. We claim that $r = c + p$. To show this we let $A \in r$ and show that $-c + A \in p$, for which it suffices that $-c + (A \cap X) \in p$. Now $\phi[A \cap X] \in \widetilde{\phi}(r) = \phi(c) + \widetilde{\phi}(p)$ so $-\phi(c) + \phi[A \cap X] \in \widetilde{\phi}(p)$. Pick $B \in p$ with $B \subseteq X$ such that $\phi[B] \subseteq -\phi(c) + \phi[A \cap X]$. We claim that $B \subseteq -c + A$ so let $d \in B$. Then $\phi(c) + \phi(d) \in \phi[A \cap X]$ so pick $h \in A \cap X$ such that $\phi(c) + \phi(d) = \phi(h)$. Then by statement (2), $c + d = h$ so $d \in -c + A$ as required. Thus $r = c + p \in G + p$.

Finally, let $q' = c + q$. By [7, Theorem 6.54], the center of βG_d is G so $q' + p = c + q + (-c) + r = q + r = p$. Thus by $(*)$, $X - X \in q'$ and so by $(\dagger)$, $\widetilde{\phi}(q') + \widetilde{\phi}(p) = \widetilde{\phi}(p)$. So $\widetilde{\phi}(q') = \widetilde{\phi}(p)$. (By Theorem 1.5 $\widetilde{\phi}(q') = n + \widetilde{\phi}(p)$ for some $n \in \mathbb{Z}$. but then $\widetilde{\phi}(p) = \widetilde{\phi}(q') + \widetilde{\phi}(p) = n + \widetilde{\phi}(p) + \widetilde{\phi}(p) = n + \widetilde{\phi}(p)$, and thus $n = 0$.) By statement (2), ϕ is injective on $X - X$ so $\widetilde{\phi}$ is injective on $\overline{X - X}$ and thus $q' = p$. That is, $q = -c + p$. □

Corollary 4.9. *Let S be a countable subsemigroup of $\bigoplus_{n=1}^{\infty} \mathbb{T}$, let p be a nonprincipal, strongly summable ultrafilter on S, and let $G = S - S$. If $\{x \in S : \pi_{\min(x)}(x) \neq \frac{1}{2}\} \in p$, $q, r \in \beta G_d$, and $q + r = p$, then $q, r \in G + p$.*

Proof. Theorems 4.5 and 4.8. □

Question 4.10. *Can it be shown in ZFC that there is an infinite group G and an idempotent in G^* which can only be expressed trivially as a product in G^*?*

Question 4.11. *Does every strongly summable ultrafilter on a countable abelian group G have the property that it can only be expressed trivially as a product in G^*?*

Let $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. Then any strongly summable ultrafilter on G fails to satisfy the hypotheses of Theorem 4.5. And, in fact, there does not exist a sequence in G satisfying strong uniqueness of finite sums. On the other hand by [8, Theorem 2.8] Martin's Axiom does imply the existence of sparse strongly summable ultrafilters on G. (And [8, Theorem 3.6] establishes that one cannot prove in ZFC the existence of strongly summable ultrafilters on G.) In the originally submitted version of this paper we asked whether it is consistent with ZFC that there is a nonsparse strongly summable ultrafilter on G? This question has now been answered in the negative by David FernándezBretón [4] who showed that every strongly summable ultrafilter on G is sparse.

Question 4.12. *Is every strongly summable ultrafilter on a countable abelian group sparse?*

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