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# Sasaki–Einstein 5-manifolds associated to toric 3-Sasaki manifolds

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Abstract. We give a correspondence between toric 3-Sasaki 7-manifolds $\mathcal{S}$ and certain toric Sasaki–Einstein 5-manifolds $M$. These 5-manifolds are all diffeomorphic to $\#k(S^2 \times S^3)$, where $k = 2b_2(\mathcal{S}) + 1$, and are given by a pencil of Sasaki embeddings, where $M \subset \mathcal{S}$ is given concretely by the zero set of a component of the 3-Sasaki moment map. It follows that there are infinitely many examples of these toric Sasaki–Einstein manifolds $M$ for each odd $b_2(M) > 1$. This is proved by determining the invariant divisors of the twistor space $\mathcal{Z}$ of $\mathcal{S}$, and showing that the irreducible such divisors admit orbifold Kähler–Einstein metrics.

As an application of the proof we determine the local space of anti-self-dual structures on a toric anti-self-dual Einstein orbifold.

Contents



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## Introduction

Recall that a 3-Sasaki manifold $\mathcal{S}$ is a Seifert $S^1$-fibration over its twistor space $\mathcal{Z}$, which is complex contact Kähler–Einstein space, and $\mathcal{Z}$ is also the, usually singular, twistor space of an quaternion-Kähler orbifold $\mathcal{M}$. In this article we show that when $\mathcal{S}$ is 7-dimensional and toric, i.e., has a two-torus $T^2$ preserving the three Sasakian structures, there is a toric Sasaki–Einstein 5-manifold $M$ naturally associated to $\mathcal{S}$. There is a pencil of embeddings $M \subset \mathcal{S}$, which are equivariant with respect to the $T^3$ action on $M$ and respect the respective Sasaki structures. This is proved by determining the $T^2_{\mathbb{C}}$-invariant divisors of $\mathcal{Z}$. This gives a pencil with finitely many reducible elements. Away from the reducible elements we get a toric surface $X \subset \mathcal{Z}$ whose orbifold singularities are inherited from those of $\mathcal{Z}$. The general picture is given in (1), where the horizontal maps are embeddings and vertical maps are orbifold fibrations.

There is an elementary construction of infinitely many toric 3-Sasaki manifolds $\mathcal{S}$ for $b_2(\mathcal{S})$ any positive integer due to C. P. Boyer, K. Galicki, B. Mann, and E. G. Rees [14]. This is done by taking a 3-Sasaki version of a Hamiltonian reduction of $S^{4m+3}$ by a torus $T^{m-1}$. In the case of 7-dimensional quotients simple numerical criterion on the weight matrix $\Omega$ of the torus ensure that

$$\mathcal{S}_\Omega = S^{4m+3} /\!\!/ T^{m-1}$$

is smooth. It follows that the embedded $M \subset \mathcal{S}_\Omega$ is also smooth. Thus we get infinitely many smooth examples as in (1). Since $M$ is toric it is known from the classification of 5-manifolds that it is diffeomorphic to $k\#(S^2 \times S^3)$ where in this case $k = 2b_2(\mathcal{S}) + 1$.

$$\begin{array}{ccc} M & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{Z} \\ & & \downarrow \\ & & \mathcal{M}. \end{array} \tag{1}$$

The results of this article are not only intimately related to the examples of [14], but they also provide examples of Einstein manifolds of positive scalar curvature exhibiting similar non-finiteness properties in dimension 5 rather than 7. In the above article it was shown that there are compact Einstein 7-manifolds of positive scalar curvature with arbitrarily large total

Betti number. Also it was shown that there are infinitely many compact 7-manifolds which admit an Einstein metric of positive scalar curvature but do not admit a metric with nonnegative sectional curvature. The Sasaki–Einstein manifolds constructed here provide examples of both phenomena in dimension 5. In particular, we prove the following.

**Theorem 1.** *For each odd* $k \geq 3$ *there is a countably infinite number of toric Sasaki–Einstein structures on* $\#k(S^2 \times S^3)$.

The next result shows that the moduli space of Einstein metrics on $\#k(S^2 \times S^3)$ has infinitely many path components.

**Proposition 2.** *For* $M = \#k(S^2 \times S^3)$ *with* $k > 1$ *odd, let* $g_i$ *be the sequence of Einstein metrics in the theorem normalized so that* $\mathrm{Vol}_{g_i}(M) = 1$. *Then we have* $\mathrm{Ric}_{g_i} = \lambda_i g_i$ *with the Einstein constants* $\lambda_i \to 0$ *as* $i \to \infty$.

The result of M. Gromov [27] that a manifold which admits a metric of nonnegative sectional curvature satisfies a bound on the total Betti number depending only on the dimension implies the following.

**Theorem 3.** *There are infinitely many compact* 5*-dimensional Einstein manifolds of positive scalar curvature which do not admit metrics on nonnegative sectional curvature.*

The diagram (1) gives a correspondence in the sense that from one of the given spaces the remaining four are uniquely determined. Furthermore, $M$ is smooth precisely when $\mathcal{S}$ is. This is used in proving the above theorems, as numerical criteria is for constructing a smooth 3-Sasaki space $\mathcal{S}$ is known from [14].

In terms of toric geometry, the relation between $X$ and $M$ on the one hand and the righthand side of (1) on the other is elementary. The ASD Einstein space $\mathcal{M}$ is a simply connected 4-orbifold with a $T^2$ action and is thus characterized by the stabilizer groups along an exceptional set of 2-spheres. And $P = \mathcal{M}/T^2$ is a polygon with edges which can be labeled with $v_1, v_2, \ldots, v_\ell \in \mathbb{Z}^2$ which characterize the stabilizers. Note that they are not assumed to be primitive, as the metric may have a cone angle along the corresponding $S^2$. It follows from the existence of the positive scalar curvature ASD Einstein metric [18] that the vectors $v_1, v_2, \ldots, v_\ell, -v_1, -v_2, \ldots, -v_\ell \in \mathbb{Z}^2$ are vertices of a convex polytope in $\mathbb{R}^2$. Thus they define an augmented fan $\Delta^*$ which characterizes the toric Fano orbifold surface $X_{\Delta^*}$ obtained above.

In Section 1 we provide some necessary background on Sasaki and 3-Sasaki manifolds and related geometries. In Section 2 we prove the existence of an orbifold Kähler–Einstein metric on the divisor $X$. From this we get the Sasaki–Einstein structure on $M$. More generally a proof is given that any symmetric toric Fano orbifold admits a Kähler–Einstein metric. Here *symmetric* means that the normalizer $\mathcal{N}(T_{\mathbb{C}}) \subset \mathrm{Aut}(X)$ of the torus $T_{\mathbb{C}}$ acts on the characters of $T_{\mathbb{C}}$ fixing only the trivial character. It is a result of V. Batyrev and E. Selivanova [4] that a symmetric toric Fano manifold admits a Kähler–Einstein metric. It was then proved by X. Wang and X. Zhu [49] that every toric Fano manifold with vanishing Futaki invariant has a Kähler–Einstein metric. It was then shown by A. Futaki, H. Ono and G. Wang [25] that every toric Sasaki manifold with $a\omega^T \in c_1(\mathscr{F}_\xi),\ a > 0$, where $c_1(\mathscr{F}_\xi)$ is the first Chern class of the transversely holomorphic foliation, admits a transversal Kähler–Einstein metric. This latter result includes the orbifolds considered here. But the proof included here gives a lower bound on the Tian invariant, $\alpha_G(X) \geq 1$, where $G \subset \mathcal{N}(T_{\mathbb{C}})$ is a maximal compact group. For toric manifolds it is known that $\alpha_G(X) = 1$ if and only if $X$ is symmetric [4, 45].

In Section 3 we construct the correspondence and embeddings in (1). In particular, in Section 3.2 the existence of the pencil of embeddings $X \subset \mathcal{Z}$ is proved. All of the $T^2_{\mathbb{C}}$-invariant divisors of $\mathcal{Z}$ are determined, and in effect the entire orbit structure of $\mathcal{Z}$ is determined. The reducible $T^2_{\mathbb{C}}$-invariant divisors $X$ represent the complex contact bundle $\mathbf{L} = \mathbf{K}_{\mathcal{Z}}^{-\frac{1}{2}}$.

In Section 4.1 we prove the Sasaki manifold $M$ admits a Sasaki–Einstein metric, and prove the above theorems.

In Section 4.2 as an application of the results on the twistor space $\mathcal{Z}$ we prove that

$$\dim_{\mathbb{C}} H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}) = \dim_{\mathbb{C}} H^1(\mathcal{Z}, \Theta_{\mathcal{Z}})^{T^2} = b_2(\mathcal{Z}) - 2 = b_2(\mathcal{M}) - 1,$$

which gives the dimension of the local deformation space of ASD conformal structures on $\mathcal{M}$. This dimension $b_2(\mathcal{M}) - 1 = \ell - 3$, where $\ell$ is number of edges of the polygon $P = \mathcal{M}/T^2$ labeled by the 1-dimensional stabilizers in $T^2$. This is the same as the dimension of the space of deformations of $(\mathcal{M}, g)$ preserving the toric structure given by the Joyce ansatz [33]. In other words, locally every ASD deformation of the comformal metric $[g]$ is a Joyce metric. This is in contrast to the, in many respects similar, case of toric ASD structures on $\#m\overline{\mathbb{CP}}^2$ as there are many examples of deformation preserving only an $S^1 \subset T^2$ [36]. It is known that the virtual dimension of the moduli space of ASD conformal structures on $\#m\overline{\mathbb{CP}}^2$ is $7m-15$ plus the dimension of the conformal group. Thus in general the expected dimension of the deformation space will be much greater than the $m - 1$ dimensional space of Joyce metrics. The deformations of $\mathcal{Z}$ are also of interest for other work of the author. It is a consequence of results in [47] that the existence of the Kähler–Einstein metric is open under deformations of $\mathcal{Z}$.

## 1. Sasaki manifolds

We review the basics of Sasaki and 3-Sasaki manifolds in this section. See the monograph [11] for more details. The survey article [9] is a good introduction to 3-Sasakian geometry. These references are a good source of background on orbifolds and orbifold bundles which will be used in this article. In a few places we will make use of orbifold invariants $\pi_1^{\mathrm{orb}}(X)$, $H^*_{\mathrm{orb}}(X)$, etc., which make use of local classifying spaces $B(X)$ for orbifolds. An introduction to these topics can be found in the above references.

### 1.1. Sasaki structures.

**Definition 1.1.** A Riemannian manifold $(M, g)$ is a *Sasaki manifold*, or has a compatible Sasaki structure, if the metric cone

$$(C(M), \bar{g}) = (\mathbb{R}_{>0} \times M, dr^2 + r^2 g)$$

is Kähler with respect to some complex structure $I$, where $r$ is the usual coordinate on $\mathbb{R}_{>0}$.

Thus $M$ is odd and denoted $n = 2m + 1$, while $C(M)$ is a complex manifold with $\dim_{\mathbb{C}} C(M) = m + 1$.

Although, this is the simplest definition, Sasaki manifolds were originally defined as a special type of metric contact structure. We will identify $M$ with the $\{1\} \times M \subset C(M)$. Let $r\partial_r$ be the Euler vector field on $C(M)$, then it is easy to see that $\xi = Ir\partial_r$ is tangent to $M$. Using the warped product formulae for the cone metric $\bar{g}$ [41] it is easy check that $r\partial_r$ is real holomorphic, $\xi$ is Killing with respect to both $g$ and $\bar{g}$, and furthermore the orbits of $\xi$ are geodesics on $(M, g)$. Define $\eta = \frac{1}{r^2} \xi \lrcorner \bar{g}$, then we have

$$\eta = -\frac{I^* dr}{r} = d^c \log r, \tag{2}$$

where $d^c = \sqrt{-1}(\bar{\partial} - \partial)$. If $\omega$ is the Kähler form of $\bar{g}$, i.e., $\omega(X, Y) = \bar{g}(IX, Y)$, then $\mathcal{L}_{r\partial_r} \omega = 2\omega$ which implies that

$$\omega = \frac{1}{2} d(r\partial_r \lrcorner \omega) = \frac{1}{2} d(r^2 \eta) = \frac{1}{4} dd^c(r^2). \tag{3}$$

From (3) we have

$$\omega = r dr \wedge \eta + \frac{1}{2} r^2 d\eta. \tag{4}$$

We will use the same notation to denote $\eta$ and $\xi$ restricted to $M$. Then (4) implies that $\eta$ is a contact form with Reeb vector field $\xi$, since $\eta(\xi) = 1$ and $\mathcal{L}_\xi \eta = 0$. Let $D \subset TM$ be the contact distribution which is defined by

$$D_x = \ker \eta_x \tag{5}$$

for $x \in M$. Furthermore, if we restrict the almost complex structure to $D$, $J := I|_D$, then $(D, J)$ is a strictly pseudoconvex CR structure on $M$. We have a splitting of the tangent bundle $TM$

$$TM = D \oplus L_\xi, \tag{6}$$

where $L_\xi$ is the trivial subbundle generated by $\xi$. It will be convenient to define a tensor $\Phi \in \mathrm{End}(TM)$ by $\Phi|_D = J$ and $\Phi(\xi) = 0$. Then

$$\Phi^2 = -\mathbf{1} + \eta \otimes \xi. \tag{7}$$

Since $\xi$ is Killing, we have

$$d\eta(X,Y) = 2g(\Phi(X),Y), \quad \text{where } X,Y \in \Gamma(TM), \tag{8}$$

and $\Phi(X) = \nabla_X \xi$, where $\nabla$ is the Levi-Civita connection of $g$. Making use of (7) we see that

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

and one can express the metric by

$$g(X,Y) = \frac{1}{2}(d\eta)(X,\Phi Y) + \eta(X)\eta(Y). \tag{9}$$

We will denote a Sasaki structure on $M$ by $(g,\eta,\xi,\Phi)$. Although, the reader can check that merely specifying $(g,\xi)$, $(g,\eta)$, or $(\eta,\Phi)$ is enough to determine the Sasaki structure, it will be convenient to denote the remaining structure.

The action of $\xi$ generates a foliation $\mathscr{F}_\xi$ on $M$ called the *Reeb* foliation. Note that it has geodesic leaves and is a Riemannian foliation, that is has a $\xi$-invariant Riemannian metric on the normal bundle $\nu(\mathscr{F}_\xi)$. But in general the leaves are not compact. If the leaves are compact, or equivalently $\xi$ generates an $S^1$-action, then $(g,\eta,\xi,\Phi)$ is said to be a *quasi-regular* Sasaki structure, otherwise it is *irregular*. If this $S^1$ action is free, then $(g,\eta,\xi,\Phi)$ is said to be *regular*. In this last case $M$ is an $S^1$-bundle over a manifold $Z$, which we will see below is Kähler. If the structure if merely quasi-regular, then the leaf space has the structure of a Kähler orbifold $Z$.

The vector field $\xi - \sqrt{-1}I\xi = \xi + \sqrt{-1}r\partial_r$ is holomorphic on $C(M)$. If we denote by $\tilde{\mathbb{C}}^*$ the universal cover of $\mathbb{C}^*$, then $\xi + \sqrt{-1}r\partial_r$ induces a holomorphic action of $\tilde{\mathbb{C}}^*$ on $C(M)$. The orbits of $\tilde{\mathbb{C}}^*$ intersect $M \subset C(M)$ in the orbits of the Reeb foliation generated by $\xi$. We denote the Reeb foliation by $\mathscr{F}_\xi$. This gives $\mathscr{F}_\xi$ a transversely holomorphic structure.

We define a *transversely Kähler* structure on $\mathscr{F}_\xi$ with Kähler form and metric

$$\omega^T = \frac{1}{2}d\eta \tag{10}$$

$$g^T = \frac{1}{2}d\eta(\cdot, \Phi\cdot). \tag{11}$$

Though in general it is not the case, the examples in this article will be quasi-regular. Therefore, the transversely Kähler leaf space of $\mathscr{F}_\xi$ will be a Kähler orbifold $Z$. Up to a homothetic transformation all such examples are as in the following example.

**Example 1.2.** Let $\mathbf{F}$ be a negative holomorphic orbifold line bundle on a complex orbifold $Z$ and $h$ an Hermitian connection with negative curvature. Define $r^2 = h(w,w)$ where $w$ is the fiber coordinate. Then $\omega = \frac{1}{4}dd^c r^2$ is the Kähler form of a cone metric on the total space minus the zero section

$\mathbf{F}^\times$. Then $\eta = \frac{1}{2}d^c \log r^2$ is a contact form, and since $\mathbf{F}$ is negative

$$\omega^T = \frac{1}{2}d\eta = \frac{1}{4}dd^c \log r^2 = -\frac{1}{2}\Theta_{\mathbf{F}} > 0$$

gives the transversal Kähler metric. In this case $\omega^T$ is an orbifold Kähler metric on $Z$.

The following follows from O'Neill tensor computations for a Riemannian submersion. See [40] and [5, Ch. 9].

**Proposition 1.3.** *Let* $(M, g, \eta, \xi, \Phi)$ *be a Sasaki manifold of dimension* $n = 2m + 1$, *then:*

(i) $\mathrm{Ric}_g(X, \xi) = 2m\eta(X)$, *for* $X \in \Gamma(TM)$.
(ii) $\mathrm{Ric}^T(X, Y) = \mathrm{Ric}_g(X, Y) + 2g^T(X, Y)$, *for* $X, Y \in \Gamma(D)$.
(iii) $s^T = s_g + 2m$.

**Definition 1.4.** A *Sasaki–Einstein* manifold $(M, g, \eta, \xi, \Phi)$ is a Sasaki manifold with

$$\mathrm{Ric}_g = 2m\, g.$$

Note that by (i) the Einstein constant must be $2m$, and $g$ is Einstein precisely when the cone $(C(M), \bar{g})$ is Ricci-flat. Furthermore, the transverse Kähler metric is also Einstein

$$\mathrm{Ric}^T = (2m + 2)\, g^T. \tag{12}$$

Conversely, if one has a Sasaki structure $(g, \eta, \xi, \Phi)$ with $\mathrm{Ric}^T = \tau\, g^T$ with $\tau > 0$, then after a $D$-homothetic transformation one has a Sasaki–Einstein structure $(g', \eta', \xi', \Phi)$, where $\eta' = a\eta$, $\xi' = a^{-1}\xi$, and $g' = ag + a(a-1)\eta\otimes\eta$, with $a = \frac{\tau}{2m+2}$.

**1.2. 3-Sasaki and related structures.** Recall that a hyper-Kähler structure on a $4m$-dimensional manifold consists of a metric $g$ which is Kähler with respect to three complex structures $J_1, J_2, J_3$ satisfying the quaternionic relations

$$J_1^2 = J_2^2 = J_3^2 = -\mathbf{1}, \quad J_1 J_2 = -J_2 J_1 = J_3.$$

**Definition 1.5.** A Riemannian manifold $(\mathcal{S}, g)$ is 3-Sasaki if the metric cone $(C(\mathcal{S}), \bar{g})$ is hyper-Kähler. That is, $\bar{g}$ admits compatible almost complex structures $J_i,\ i = 1, 2, 3$ such that $(\bar{g}, J_1, J_2, J_3)$ is a hyper-Kähler structure on $C(\mathcal{S})$. Equivalently, $\mathrm{Hol}(C(\mathcal{S})) \subseteq \mathrm{Sp}(m)$.

A consequence of the definition is that $(\mathcal{S}, g)$ is equipped with three Sasaki structures $(g, \eta_i, \xi_i, \phi_i),\ i = 1, 2, 3$. The Reeb vector fields $\xi_i = J_i(r\partial_r),\ i = 1, 2, 3$ are orthogonal and satisfy $[\xi_i, \xi_j] = -2\varepsilon^{ijk}\xi_k$, where $\varepsilon^{ijk}$ is anti-symmetric in the indicies $i, j, k \in \{1, 2, 3\}$ and $\epsilon^{123} = 1$. The tensors $\phi_i,\ i = 1, 2, 3$ satisfy the identities

$$\phi_i(\xi_j) = \varepsilon^{ijk}\xi_k, \tag{13}$$

$$\phi_i \circ \phi_j = -\delta_{ij}\mathbf{1} + \epsilon^{ijk}\phi_k + \eta_j \otimes \xi_i. \tag{14}$$

It is easy to see that there is an $S^2$ of Sasaki structures with Reeb vector field $\xi_\tau = \tau_1\xi_1 + \tau_2\xi_2 + \tau_3\xi_3$ with $\tau \in S^2$.

The Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ generate a Lie algebra $\mathfrak{sp}(1)$, so there is an effective isometric action of either SO(3) or Sp(1) on $(\mathcal{S}, g)$. Both cases occur in the examples in this article. This action generates a foliation $\mathscr{F}_{\xi_1,\xi_2,\xi_3}$ with generic leaves either SO(3) or Sp(1).

If we set $D_i = \ker \eta_i \subset T\mathcal{S},\ i = 1, 2, 3$ to be the contact subbundles, then the complex structures $J_i,\ i = 1, 2, 3$ are recovered by

$$J_i(r\partial_r) = \xi_i, \quad J_i|_{D_i} = \phi_i. \tag{15}$$

Because a hyper-Käher manifold is always Ricci-flat we have the following.

**Proposition 1.6.** *A 3-Sasaki manifold $(\mathcal{S}, g)$ of dimension $4m + 3$ is Einstein with Einstein constant $\lambda = 4m + 2$.*

We choose a Reeb vector field $\xi_1$ fixing a Sasaki structure, then the leaf space $\mathscr{F}_{\xi_1}$ is a Kähler orbifold $\mathcal{Z}$ with respect to the transversal complex structure $J = \Phi_1$. But it has addition has a complex contact structure and a fibering by rational curves which we now describe. The 1-form $\eta^c = \eta_2 - \sqrt{-1}\eta_3$ is a $(1, 0)$-form with respect to $J$. But it is not invariant under the U(1) group generated by $\exp(t\xi_1)$. We have $\exp(t\xi_1)^*\eta^c = e^{2\sqrt{-1}t}\eta^c$. Let $\mathbf{L} = \mathcal{S} \times_{\mathrm{U}(1)} \mathbb{C}$, with U(1) action on $\mathbb{C}$ by be $e^{2\sqrt{-1}t}$. This is a holomorphic orbifold line bundle; in fact $C(\mathcal{S})$ is either $\mathbf{L}^{-1}$ or $\mathbf{L}^{\frac{1}{2}}$ minus the zero section. It is easy to see that each of these cases occur precisely where the Reeb vector fields generate an effective action of SO(3) and Sp(1) respectively. Then $\eta^c$ descends to an $\mathbf{L}$ valued holomorphic 1-form $\theta \in \Gamma\big(\Omega^{1,0}(\mathbf{L})\big)$. It follows easily from identities (14) that $d\eta^c$ restricted to $D_1 \cap \ker \eta^c$ is a nondegenerate type $(2, 0)$ form. Thus $\theta$ is a complex contact form on $\mathcal{Z}$, and $\theta \wedge (d\theta)^m \in \Gamma\big(\mathbf{K}_{\mathcal{Z}} \otimes \mathbf{L}^{m+1}\big)$ is a nonvanishing section. Thus $\mathbf{L} \cong \mathbf{K}_{\mathcal{Z}}^{-\frac{1}{m+1}}$.

Each leaf of $\mathscr{F}_{\xi_1,\xi_2,\xi_3}$ descends to a rational curve in $\mathcal{Z}$. Each curve is a $\mathbb{CP}^1$ but may have orbifold singularities for nongeneric leaves. We see that restricted to a leaf $\mathbf{L}|_{\mathbb{CP}^1} = \mathcal{O}(2)$.

The element $\exp(\frac{\pi}{2}\xi_2)$ acts on $\mathcal{S}$ taking $\xi_1$ to $-\xi$, thus it descends to an anti-holomorphic involution $\varsigma : \mathcal{Z} \to \mathcal{Z}$. This *real structure* is crucial to the twistor approach. Note that $\varsigma^*\theta = \bar{\theta}$.

This all depends on the choice $\xi_1 \in S^2$ of the Reeb vector field. But taking a different Reeb vector field gives an isomorphic twistor space under the transitive action of Sp(1).

Taking the quotient of $\mathcal{S}$ by Sp(1) gives the leaf space of $\mathscr{F}_{\xi_1,\xi_2,\xi_3}$ an orbifold $\mathcal{M}$. We now consider the orbifold $\mathcal{M}$ more closely. Let $(\mathcal{M}, g)$ be any $4m$ dimensional Riemannian orbifold. An *almost quaternionic* structure on $\mathcal{M}$ is a rank 3 V-subbundle $\mathcal{Q} \subset \mathrm{End}(T\mathcal{M})$ which is locally spanned by almost complex structures $\{J_i\}_{i=1,2,3}$ satisfying the quaternionic identities

$J_i^2 = -\mathbf{1}$ and $J_1J_2 = -J_2J_1 = J_3$. We say that $\mathcal{Q}$ is compatible with $g$ if $J_i^*g = g$ for $i = 1, 2, 3$. Equivalently, each $J_i, i = 1, 2, 3$ is skew symmetric.

**Definition 1.7.** A Riemannian orbifold $(\mathcal{M}, g)$ of dimension $4m$, $m > 1$ is *quaternion-Kähler* if there is an almost quaternionic structure $\mathcal{Q}$ compatible with $g$ which is preserved by the Levi-Civita connection.

This definition is equivalent to the holonomy of $(\mathcal{M}, g)$ being contained in $\mathrm{Sp}(1)\,\mathrm{Sp}(m)$. For orbifolds this is the holonomy on $\mathcal{M} \setminus S_{\mathcal{M}}$ where $S_{\mathcal{M}}$ is the singular locus of $\mathcal{M}$. Notice that this definition always holds on an oriented Riemannian 4-manifold ($m = 1$). This case requires a different definition. Consider the *curvature operator*

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

of an oriented Riemannian 4-manifold. With respect to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, we have

$$\mathcal{R} = \begin{pmatrix} W_g^+ + \frac{s_g}{12} & \overset{\circ}{\mathrm{Ric}}_g \\ \overset{\circ}{\mathrm{Ric}}_g & W_g^- + \frac{s_g}{12} \end{pmatrix}, \tag{16}$$

where $W_g^+$ and $W_g^-$ are the self-dual and anti-self-dual pieces of the Weyl curvature and $\overset{\circ}{\mathrm{Ric}}_g = \mathrm{Ric}_g - \frac{s_g}{4} g$ is the trace-free Ricci curvature. An oriented 4 dimensional Riemannian orbifold $(\mathcal{M}, g)$ is quaternion-Kähler if it is Einstein and anti-self-dual, meaning that $\overset{\circ}{\mathrm{Ric}}_g = 0$ and $W_g^+ = 0$.

One can prove that $\{\Phi_i\}_{i=1,2,3}$, restricted to $D_1 \cap D_2 \cap D_3$, the horizontal space to $\mathscr{F}_{\xi_1,\xi_2,\xi_3}$, defines a quaternion-Kähler structure on the leaf space of $\mathscr{F}_{\xi_1,\xi_2,\xi_3}$.

**Theorem 1.8** ([9]). *Let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold of dimension $n = 4m + 3$. Then there is a natural quaternion-Kähler structure on the leaf space of $\mathcal{F}_{\xi_1,\xi_2,\xi_3}$, $(\mathcal{M}, \breve{g})$, such that the orbifold map $\varpi : \mathcal{S} \to \mathcal{M}$ is a Riemannian submersion. Furthermore, $(\mathcal{M}, \breve{g})$ is Einstein with scalar curvature $s_{\breve{g}} = 16m(m + 2)$.*

The geometries associated to a 3-Sasaki manifold can be seen in Figure 1. Up to a finite cover, from each space in Figure 1 the other three spaces can be recovered. Unlike $\mathcal{S}$ the spaces $\mathcal{Z}$ and $\mathcal{M}$ are smooth in no more than finitely many cases for each $n \geq 1$. Furthermore, it is known that the only smooth $\mathcal{M}^{4n}$ for $n = 1, 2$ are symmetric spaces. That this is true for all $n$ is the famous LeBrun–Salamon conjecture. See [37].

We will need to distinguish when the fibering $\mathcal{S} \to \mathcal{M}$ has generic fiber $\mathrm{Sp}(1)$. The obstruction to this is the Marchiafava–Romani class. An almost quaternionic structure $\mathcal{Q}$ is a reduction of the frame bundle to an $\mathrm{Sp}(1)\,\mathrm{Sp}(m)$ bundle. Let $\mathcal{G}$ be the sheaf of germs of smooth maps to $\mathrm{Sp}(1)\,\mathrm{Sp}(m)$. An almost quaternionic structure is an element $s \in H^1_{\mathrm{orb}}(\mathcal{M}, \mathcal{G})$. Consider the

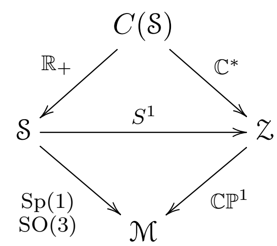


FIGURE 1. Related geometries

exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Sp}(1) \times \mathrm{Sp}(m) \to \mathrm{Sp}(1)\,\mathrm{Sp}(m) \to 1. \tag{17}$$

**Definition 1.9.** The *Marchiafava–Romani class* is $\varepsilon = \delta(s)$, where

$$\delta : H^1_{\mathrm{orb}}(\mathcal{M}, \mathcal{G}) \to H^2_{\mathrm{orb}}(\mathcal{M}, \mathbb{Z}_2)$$

is the connecting homomorphism.

One has that $\varepsilon$ is the orbifold Stiefel–Whitney class $w_2(\mathcal{Q})$. Also, $\varepsilon$ is the obstruction to the existence of a square root $\mathbf{L}^{\frac{1}{2}}$ of $\mathbf{L}$. In the four-dimensional case $n = 1$, $\varepsilon = w_2(\Lambda^2_+) = w_2(T\mathcal{M})$. When $\varepsilon = 0$ for the 3-Sasakian space $\mathcal{S}$ associated to $(\mathcal{M}, \check{g})$ we will always mean the one with Sp(1) generic fibres.

**1.3. Toric 3-Sasaki manifolds.** A 3-Sasaki manifold $\mathcal{S}$ with $\dim \mathcal{S} = 4m+3$ is *toric* if it admits an effective action of $T^{m+1} \subset \mathrm{Aut}(\mathcal{S}, g)$, where $\mathrm{Aut}(\mathcal{S}, g)$ is the group of 3-Sasaki automorphisms: isometries preserving $(g, \eta_i, \xi_i, \phi_i)$, $i = 1, 2, 3$. Equivalently, $C(\mathcal{S})$ is a toric hyper-Kähler manifold [7]. We will consider toric 3-Sasaki 7-manifolds which were constructed by a 3-Sasaki reduction procedure in [14]. This constructs infinitely many smooth 3-Sasaki 7-manifolds for each $b_2 \geq 1$. Subsequently it was proved by R. Bielawski [6] that up to a finite cover all toric examples are obtained this way.

Let $\mathrm{Aut}(\mathcal{S}, g)$ be the group 3-Sasaki automorphisms, that is isometries preserving $(g, \eta_i, \xi_i, \phi_i)$, $i = 1, 2, 3$. Given a compact $G \subset \mathrm{Aut}(\mathcal{S}, g)$ one can define the 3-*Sasakian moment map*

$$\mu_{\mathcal{S}} : \mathcal{S} \to \mathfrak{g}^* \otimes \mathbb{R}^3, \tag{18}$$

where if $\tilde{X}$ is the vector field on $\mathcal{S}$ induced by $X \in \mathfrak{g}$ we have

$$\langle \mu^a_{\mathcal{S}}, X \rangle = \frac{1}{2}\eta^a(\tilde{X}), \quad a = 1, 2, 3 \ \text{ for } X \in \mathfrak{g}. \tag{19}$$

There is a quotient similar to the Marsden–Weinstein quotient of symplectic manifolds [12]. If a connected compact $G \subset \mathrm{Aut}(\mathcal{S}, g)$ acts freely (locally freely) on $\mu_{\mathcal{S}}^{-1}(0)$,then

$$\mathcal{S}/\!\!/G = \mu_{\mathcal{S}}^{-1}(0)/G$$

has the structure of a 3-Sasakian manifold (orbifold).

Consider the unit sphere $S^{4n-1} \subset \mathbb{H}^n$ with the round metric $g$ and the standard 3-Sasakian structure induced by the right action of Sp(1). Then $\mathrm{Aut}(S^{4n-1}, g) = \mathrm{Sp}(n)$ acting by the standard linear representation on the left. We have the maximal torus $T^n \subset \mathrm{Sp}(n)$ and every representation of a subtorus $T^k$ is conjugate to an inclusion $\iota_\Omega : T^k \to T^n$ which is represented by a *weight matrix* $\Omega = (a^i_j) \in \mathscr{M}_{k,n}(\mathbb{Z})$, an integral $k \times n$ matrix.

Let $\{e_i\}, i = 1, \ldots, k$ be a basis for the dual of the Lie algebra of $T^k$, $\mathfrak{t}^*_k \cong \mathbb{R}^k$. Then the moment map $\mu_\Omega : S^{4n-1} \to \mathfrak{t}^*_k \otimes \mathbb{R}^3$ can be written as $\mu_\Omega = \sum_j \mu^j_\Omega e_j$ where in terms of complex coordinates $z_l + w_l j$ on $\mathbb{H}^n$ we have

$$\mu^j_\Omega(\mathbf{z}, \mathbf{w}) = i \sum_l a^j_l (|z_l|^2 - |w_l|^2) + 2k \sum_l a^j_l \bar{w}_l z_l. \tag{20}$$

We assume $\mathrm{rank}(\Omega) = k$ otherwise we just have an action of a subtorus of $T^k$. Denote by

$$\Delta_{\alpha_1,\ldots,\alpha_k} = \det \begin{pmatrix} a^1_{\alpha_1} & \cdots & a^1_{\alpha_k} \\ \vdots & & \vdots \\ a^k_{\alpha_1} & \cdots & a^k_{\alpha_k} \end{pmatrix} \tag{21}$$

the $\binom{n}{k}$ $k \times k$ minor determinants of $\Omega$.

**Definition 1.10.** Let $\Omega \in \mathscr{M}_{k,n}(\mathbb{Z})$ be a weight matrix.

(i) $\Omega$ is *nondegenerate* if $\Delta_{\alpha_1,\ldots,\alpha_k} \neq 0$, for all $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$.

(ii) Let $\Omega$ be nondegenerate, and let $d$ be the gcd of all the $\Delta_{\alpha_1,\ldots,\alpha_k}$, the kth determinantal divisor. Then $\Omega$ is *admissible* if

$$\gcd(\Delta_{\alpha_2,\ldots,\alpha_{k+1}}, \ldots, \Delta_{\alpha_1,\ldots,\hat{\alpha}_t,\ldots,\alpha_{k+1}}, \ldots, \Delta_{\alpha_1,\ldots,\alpha_k}) = d$$

for all length $k+1$ sequences $1 \leq \alpha_1 < \cdots < \alpha_t < \cdots < \alpha_{k+1} \leq n+1$.

The gcd $d_j$ of the jth row of $\Omega$ divides $d$. We may assume that the gcd of each row of $\Omega$ is 1 by merely reparametrizing the coordinates $\tau_j$ on $T^k$. We say that $\Omega$ is in reduced form if $d = 1$.

Choosing a different basis of $\mathfrak{t}_k$ results in an action on $\Omega$ by an element in $\mathrm{GL}(k, \mathbb{Z})$. We also have the normalizer of $T^n$ in $\mathrm{Sp}(n)$, the Weyl group $\mathscr{W}(\mathrm{Sp}(n)) = \Sigma_n \times \mathbb{Z}^n_2$ where $\Sigma_n$ is the permutation group. $\mathscr{W}(\mathrm{Sp}(n))$ acts on $S^{4n-1}$ preserving the 3-Sasakian structure, and it acts on weight matrices by permutations and sign changes of columns. Thus the group $\mathrm{GL}(k, \mathbb{Z}) \times \mathscr{W}(\mathrm{Sp}(n))$ acts on $\mathscr{M}_{k,n}(\mathbb{Z})$, with the quotient only depending on the equivalence class.

**Theorem 1.11** ([14]). *Let $\Omega \in \mathscr{M}_{k,n}(\mathbb{Z})$ be reduced.*

(i) *If $\Omega$ is nondegenerate, then $\mathcal{S}_\Omega$ is an orbifold.*

(ii) *Supposing $\Omega$ is nondegenerate, $\mathcal{S}_\Omega$ is smooth if and only if $\Omega$ is admissible.*

The quotient $\mathcal{S}_\Omega$ is toric, because its automorphism group contains $T^{n-k} \cong T^n/\iota_\Omega(T^k)$.

We are primarily interested in 7-dimensional toric quotients. In this case there are infinite families of distinct quotients. We may take matrices of the form

$$\Omega = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 & b_1 \\ 0 & 1 & \cdots & 0 & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_k & b_k \end{pmatrix}. \tag{22}$$

**Proposition 1.12** ([14])**.** *Let $\Omega \in \mathscr{M}_{k,k+2}(\mathbb{Z})$ be as above. Then $\Omega$ is admissible if and only if $a_i, b_j, i, j = 1, \ldots, k$ are all nonzero,* $\gcd(a_i, b_i) = 1$ *for $i = 1, \ldots, k$, and we do not have $a_i = a_j$ and $b_i = b_j$, or $a_i = -a_j$ and $b_i = -b_j$ for some $i \neq j$.*

Proposition 1.12 shows that for $n = k+2$ there are infinitely many reduced admissible weight matrices. One can, for example, choose $a_i, b_j, i, j = 1, \ldots k$ be all pairwise relatively prime. We will make use of the cohomology computation of R. Hepworth [31] to show that we have infinitely many smooth 3-Sasakian 7-manifolds of each second Betti number $b_2 \geq 1$. Let $\Delta_{p,q}$ denote the $k \times k$ minor determinant of $\Omega$ obtained by deleting the $p^{th}$ and $q^{th}$ columns.

**Theorem 1.13** ([14, 31])**.** *Let $\Omega \in \mathscr{M}_{k,k+2}(\mathbb{Z})$ be a reduced admissible weight matrix. Then $\pi_1(\mathcal{S}_\Omega) = e$. And the cohomology of $\mathcal{S}_\Omega$ is*

| $p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $H^p$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}^k$ | 0 | $G_\Omega$ | $\mathbb{Z}^k$ | 0 | $\mathbb{Z}$ |

*where $G_\Omega$ is a torsion group of order*

$$\sum |\Delta_{s_1,t_1}| \cdots |\Delta_{s_{k+1},t_{k+1}}|$$

*with the summand with index $s_1, t_1, \ldots, s_{k+1}, t_{k+1}$ included if and only if the graph on the vertices $\{1, \ldots, k+2\}$ with edges $\{s_i, t_i\}$ is a tree.*

If we consider weight matrices as in Proposition 1.12 then the order of $G_\Omega$ is greater than $|a_1 \cdots a_k| + |b_1 \cdots b_k|$. We have the following.

**Corollary 1.14** ([14, 31])**.** *There are smooth toric 3-Sasakian 7-manifolds with second Betti number $b_2 = k$ for all $k \geq 0$. Furthermore, there are infinitely many possible homotopy types of examples $\mathcal{S}_\Omega$ for each $k > 0$.*

Note that the reduction procedure can be done on any of the four spaces in Figure 1. In particular, we have the ASD Einstein orbifold $\mathcal{M}_\Omega = \mathcal{S}_\Omega / \mathrm{Sp}(1)$, which is a quaternic-Kähler quotient [26] of $\mathbb{HP}^{n-1}$ by the torus $T^k$. An ASD Einstein orbifold $\mathcal{M}$ is *toric* if it has an effective isometric action of $T^2$.

Recall the orbifold $\mathcal{M}_\Omega$ has an action of $T^2 \cong T^{k+2}/\iota_\Omega(T^k)$, and can be characterized as in [42, 30] by its orbit space and stabilizer groups. The stabilizers were determined in [13, 18]. The orbit space is $Q_\Omega := \mathcal{M}_\Omega / T^2$. Then

$Q_\Omega$ is a polygon with $k+2$ edges $C_1, C_2, \ldots, C_{k+2}$, labeled in cyclic order with the interior of $C_i$ being orbits with stabilizer $G_i$, where $G_i \subset T^2$, $i = 1, \ldots, k+2$ are $S^1$ subgroups. Choose an explicit surjective homomorphism $\Phi : \mathbb{Z}^{k+2} \to \mathbb{Z}^2$ annihilating the rows of $\Omega$. So

$$\Phi = \begin{pmatrix} b_1 & b_2 & \cdots & b_{k+2} \\ c_1 & c_2 & \cdots & c_{k+2} \end{pmatrix}. \tag{23}$$

It will be helpful to normalize $\Phi$. After acting on the columns of $\Phi$ by $\mathscr{W}(\mathrm{Sp}(k+2))$ and on the right by $\mathrm{GL}(2,\mathbb{Z})$ we may assume that $b_i > 0$ for $i = 1, \ldots, k+2$ and $c_1/b_1 < \cdots < c_i/b_i < \cdots < c_{k+2}/b_{k+2}$. Then the stabilizer groups $G_i \subset T^2$ are characterized by $(m_i, n_i) \in \mathbb{Z}^2$ where

$$(m_i, n_i) = \sum_{l=1}^{i}(b_l, c_l) - \sum_{l=i+1}^{k+2}(b_l, c_l), \quad i = 1, \cdots k+2. \tag{24}$$

It is convenient to take $(m_0, n_0) = -(m_{k+2}, n_{k+2})$.

## 2. Kähler–Einstein metric on symmetric Fano orbifolds

We prove the existence of the Kähler–Einstein metric on symmetric toric Fano orbifolds in this section. The existence of a Kähler–Einstein metric on a toric Fano manifold with vanishing Futaki invariant was proved by X. Wang and X. Zhu [49]. More generally, they proved the existence of a Kähler–Ricci soliton which is Kähler–Einstein if the Futaki invariant vanishes. Then A. Futaki, H. Ono, and G. Wang [25] proved an extension of that result, namely that any toric Sasaki manifold, which satisfies the necessary positivity condition, admits a Kähler–Ricci soliton which is Sasaki–Einstein if the transverse Futaki invariant vanishes. This latter result includes the existence result proved here. But the proof given here, as the proof in [4] for symmetric toric Fano manifolds, shows that the invariant of G. Tian [46], extended by J.-P. Demailly and J. Kollár [20] to orbifolds, satisfies $\alpha_G(X) \geq 1$, which in this case is an invariant of $X$ as a Fano orbifold.

**2.1. Symmetric Fano orbifolds.** Let $N \cong \mathbb{Z}^r$ be the free $\mathbb{Z}$-module of rank r and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual. We denote $N_\mathbb{Q} = N \otimes \mathbb{Q}$ and $M_\mathbb{Q} = M \otimes \mathbb{Q}$ with the natural pairing

$$\langle\ ,\ \rangle : M_\mathbb{Q} \times N_\mathbb{Q} \to \mathbb{Q}.$$

Similarly we denote $N_\mathbb{R} = N \otimes \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{R}$.

Let $T_\mathbb{C} := N \otimes_\mathbb{Z} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ be the algebraic torus. Each $m \in M$ defines a character $\chi^m : T_\mathbb{C} \to \mathbb{C}^*$ and each $n \in N$ defines a one-parameter subgroup $\lambda_n : \mathbb{C}^* \to T_\mathbb{C}$. In fact, this gives an isomorphism between $M$ (resp. $N$) and the multiplicative group $\mathrm{Hom}_{\mathrm{alg.}}(T_\mathbb{C}, \mathbb{C}^*)$ (resp. $\mathrm{Hom}_{\mathrm{alg.}}(\mathbb{C}^*, T_\mathbb{C})$).

An n-dimensional toric variety $X$ has $T_\mathbb{C} \subseteq \mathrm{Aut}(X)$ with an open dense orbit $U \subset X$. Then $X$ is defined by a fan $\Delta$ in $N_\mathbb{Q}$. We denote this $X_\Delta$. See [24] or [38] for background on toric varieties. We denote by $\Delta(i)$ the set of $i$-dimensional cones in $\Delta$.

Recall that every element $\rho \in \Delta(1)$ is generated by a unique primitive element of $N$. We will consider nonprimitive generators to encode an orbifold structure.

**Definition 2.1.** We will denote by $\Delta^*$ an *augmented fan* by which we mean a fan $\Delta$ with elements $n(\rho) \in N \cap \rho$ for every $\rho \in \Delta(1)$.

**Proposition 2.2.** *For a complete simplicial augmented fan $\Delta^*$ we have a natural orbifold structure compatible with the action of $T_{\mathbb{C}}$ on $X_\Delta$. We denote $X_\Delta$ with this orbifold structure by $X_{\Delta^*}$.*

**Proof.** Let $\sigma \in \Delta^*(n)$ have generators $p_1, p_2, \dots, p_n$ as in the definition. Let $N' \subseteq N$ be the sublattice $N' = \mathbb{Z}\{p_1, p_2, \dots, p_n\}$, and $\sigma'$ the equivalent cone in $N'$. Denote by $M'$ the dual lattice of $N'$ and $T'_{\mathbb{C}}$ the torus. Then $U_{\sigma'} \cong \mathbb{C}^n$. It is easy to see that

$$N/N' = \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*).$$

And $N/N'$ is the kernel of the homomorphism

$$T'_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{C}^*) \to T_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*).$$

Let $\Gamma = N/N'$. An element $t \in \Gamma$ is a homomorphism $t : M' \to \mathbb{C}^*$ equal to 1 on $M$. The regular functions on $U_{\sigma'}$ consist of $\mathbb{C}$-linear combinations of $x^m$ for $m \in \sigma'^\vee \cap M'$. And $t \cdot x^m = t(m)x^m$. Thus the invariant functions are the $\mathbb{C}$-linear combinations of $x^m$ for $m \in \sigma^\vee \cap M$, the regular functions of $U_\sigma$. Thus $U_{\sigma'}/\Gamma = U_\sigma$. And the charts are easily seen to be compatible on intersections. □

Conversely one can prove that this definition gives all structures of interest.

**Proposition 2.3.** *Let $\Delta$ be a complete simplicial fan. Suppose for simplicity that the local uniformizing groups are abelian. Then every orbifold structure on $X_\Delta$ compatible with the action of $T_{\mathbb{C}}$ arises from an augmented fan $\Delta^*$.*

Note that these orbifold structures are not well formed, i.e., have complex codimension one singular sets. That is, if some $n(\rho) = a_\rho p_\rho$, $a_\rho \in \mathbb{N}_{>1}$, is not primitive, then the divisor $D_\rho$ has a cone angle of $2\pi/a_\rho$. This has no significance for the complex structure, but compatible metrics will have this cone singularity.

We modify the usual definition of a support function to characterize orbifold line bundles on $X_{\Delta^*}$. We will assumed from now on that the fan $\Delta$ is simplicial and complete.

**Definition 2.4.** A real function $h : N_{\mathbb{R}} \to \mathbb{R}$ is a $\Delta^*$ *-linear support function* if for each $\sigma \in \Delta^*$ with given $\mathbb{Q}$-generators $p_1, \dots, p_r$ in $N$, there is an $l_\sigma \in M_{\mathbb{Q}}$ with $h(s) = \langle l_\sigma, s\rangle$ and $l_\sigma$ is $\mathbb{Z}$-valued on the sublattice $\mathbb{Z}\{p_1, \dots, p_r\}$. And we require that $\langle l_\sigma, s\rangle = \langle l_\tau, s\rangle$ whenever $s \in \sigma \cap \tau$. The additive group of $\Delta^*$-linear support functions will be denoted by $\mathrm{SF}(\Delta^*)$.

Note that $h \in \mathrm{SF}(\Delta^*)$ is completely determined by the integers $h(n(\rho))$ for all $\rho \in \Delta(1)$. And conversely, an assignment of an integer to $h(n(\rho))$ for all $\rho \in \Delta(1)$ defines $h$. Thus

$$\mathrm{SF}(\Delta^*) \cong \mathbb{Z}^{\Delta(1)}.$$

**Definition 2.5.** Let $\Delta^*$ be a complete augmented fan. For $h \in \mathrm{SF}(\Delta^*)$,

$$\Sigma_h := \{m \in M_{\mathbb{R}} : \langle m, n\rangle \geq h(n), \text{ for all } n \in N_{\mathbb{R}}\},$$

is a, possibly empty, convex polytope in $M_{\mathbb{R}}$.

Recall that a certain subset of $\mathbb{Q}$-Weil divisors correspond to orbifold line bundles.

**Definition 2.6.** A *Baily divisor* is a $\mathbb{Q}$-Weil divisor $D \in \mathrm{Weil}(X) \otimes \mathbb{Q}$ whose inverse image $D_{\tilde{U}} \in \mathrm{Weil}(\tilde{U})$ in every local uniformizing chart $\pi : \tilde{U} \to U$ is Cartier. The additive group of Baily divisors is denoted $\mathrm{Div}^{\mathrm{orb}}(X)$.

A Baily divisor $D$ defines a holomorphic orbifold line bundle

$$[D] \in \mathrm{Pic}^{\mathrm{orb}}(X)$$

in a way completely analogous to Cartier divisors. We denote the Baily divisors invariant under $T_{\mathbb{C}}$ by $\mathrm{Div}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$. We denote the group of isomorphism equivariant orbifold line bundles by $\mathrm{Pic}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$. Then likewise we have $[D] \in \mathrm{Pic}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$ whenever $D \in \mathrm{Div}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$.

A straight forward generalization of [38, Prop. 2.1] to this situation gives the following.

**Proposition 2.7.** *Let $X = X_{\Delta^*}$ be compact with the standard orbifold structure, i.e., $\Delta^*$ is simplicial and complete.*

(i) *There is an isomorphism $\mathrm{SF}(\Delta^*) \cong \mathrm{Div}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$ obtained by sending $h \in \mathrm{SF}(\Delta^*)$ to*

$$D_h := -\sum_{\rho \in \Delta(1)} h(n(\rho)) D_\rho,$$

*where $D_\rho$ is the divisor of $X$ associated to $\rho \in \Delta(1)$.*

(ii) *There is a natural homomorphism $\mathrm{SF}(\Delta^*) \to \mathrm{Pic}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$ which associates an equivariant orbifold line bundle $\mathbf{L}_h$ to each $h \in \mathrm{SF}(\Delta^*)$.*

(iii) *Suppose $h \in \mathrm{SF}(\Delta^*)$ and $m \in M$ satisfies*

$$\langle m, n\rangle \geq h(n) \text{ for all } n \in N_{\mathbb{R}},$$

*then $m$ defines a section $\psi : X \to \mathbf{L}_h$ which has the equivariance property $\psi(tx) = \chi^m(t)(t\psi(x))$.*

(iv) *The set of sections $H^0(X, \mathcal{O}(\mathbf{L}_h))$ is the finite dimensional $\mathbb{C}$-vector space with basis $\{x^m : m \in \Sigma_h \cap M\}$.*

(v) *Every Baily divisor is linearly equivalent to a $T_{\mathbb{C}}$-invariant Baily divisor. Thus for $D \in \mathrm{Pic}^{\mathrm{orb}}(X)$, $[D] \cong [D_h]$ for some $h \in \mathrm{SF}(\Delta^*)$.*

(vi) *If* $\mathbf{L}$ *is any holomorphic orbifold line bundle, then* $\mathbf{L} \cong \mathbf{L}_h$ *for some* $h \in \mathrm{SF}(\Delta^*)$. *The homomorphism in part* (i). *induces an isomorphism* $\mathrm{SF}(\Delta^*) \cong {\mathrm{Pic}^{\mathrm{orb}}}_{T_{\mathbb{C}}}(X)$ *and we have the exact sequence*

$$0 \to M \to \mathrm{SF}(\Delta^*) \to \mathrm{Pic}^{\mathrm{orb}}(X) \to 1.$$

**Remark 2.8.** The notation is a bit deceptive that in (i) it appears that $D_h$ is a $\mathbb{Z}$-Weil divisor. But they are written with their coefficients in the uniformizing chart, and the components in ramification divisors of the chart are generally fractional.

For $X = X_{\Delta^*}$ there is a unique $k \in \mathrm{SF}(\Delta^*)$ such that $k(n(\rho)) = 1$ for all $\rho \in \Delta(1)$. The corresponding Baily divisor

$$D_k := -\sum_{\rho \in \Delta(1)} D_\rho \tag{25}$$

is the *orbifold canonical divisor*. The corresponding orbifold line bundle is $\mathbf{K}_X$, the orbifold bundle of holomorphic n-forms. This will in general be different from the canonical sheaf in the algebraic geometric sense.

**Definition 2.9.** Consider support functions as above but which are only required to be $\mathbb{Q}$-valued on $N_{\mathbb{Q}}$, denoted $\mathrm{SF}(\Delta, \mathbb{Q})$. $h$ is *strictly upper convex* if $h(n + n') \geq h(n) + h(n')$ for all $n, n' \in N_{\mathbb{Q}}$ and for any two $\sigma, \sigma' \in \Delta(n)$, $l_\sigma$ and $l_{\sigma'}$ are different linear functions.

Given a strictly upper convex support function $h$, the polytope $\Sigma_h$ is the convex hull in $M_{\mathbb{R}}$ of the vertices $\{l_\sigma : \sigma \in \Delta(n)\}$. Each $\rho \in \Delta(1)$ defines a facet by

$$\langle m, n(\rho) \rangle \geq h(n(\rho)).$$

If $n(\rho) = a_\rho n'$ with $n' \in N$ primitive and $a_\rho \in \mathbb{N}$ we may label the face with $a_\rho$ to get the labeled polytope $\Sigma_h^*$ which encodes the orbifold structure. Conversely, from a rational convex polytope $\Sigma^*$ we associate a fan $\Delta^*$ and a support function $h$.

**Proposition 2.10** ([38, 24])**.** *There is a one-to-one correspondence between the set of pairs* $(\Delta^*, h)$ *with* $h \in \mathrm{SF}(\Delta, \mathbb{Q})$ *strictly upper convex, and rational convex marked polytopes* $\Sigma_h^*$.

**Definition 2.11.** Let $X = X_{\Delta^*}$ be a compact toric orbifold. We say that $X$ is *Fano* if $-k \in \mathrm{SF}(\Delta^*)$, which defines the anti-canonical orbifold line bundle $\mathbf{K}_X^{-1}$, is strictly upper convex.

These toric variety aren't necessarily Fano in the usual sense, since $\mathbf{K}_X^{-1}$ is the *orbifold* anti-canonical class. This condition is equivalent to

$$\{n \in N_{\mathbb{R}} : k(n) \leq 1\} \subset N_{\mathbb{R}}$$

being a convex polytope with vertices $n(\rho), \rho \in \Delta(1)$. We will use $\Delta^*$ to denote both the augmented fan and this polytope in this case.

**2.2. Symmetric toric varieties.** Let $X_\Delta$ be an $n$-dimensional toric variety. Let $\mathcal{N}(T_{\mathbb{C}}) \subset \mathrm{Aut}(X)$ be the normalizer of $T_{\mathbb{C}}$. Then $\mathcal{W}(X) := \mathcal{N}(T_{\mathbb{C}})/T_{\mathbb{C}}$ is isomorphic to the finite group of all symmetries of $\Delta$, i.e., the subgroup of $\mathrm{GL}(n,\mathbb{Z})$ of all $\gamma \in \mathrm{GL}(n,\mathbb{Z})$ with $\gamma(\Delta) = \Delta$. Then we have the exact sequence.

$$1 \to T_{\mathbb{C}} \to \mathcal{N}(T_{\mathbb{C}}) \to \mathcal{W}(X) \to 1. \tag{26}$$

Choosing a point $x \in X$ in the open orbit, defines an inclusion $T_{\mathbb{C}} \subset X$. This also provides a splitting of (26). Let $\mathcal{W}_0(X) \subseteq \mathcal{W}(X)$ be the subgroup which are also automorphisms of $\Delta^*$; $\gamma \in \mathcal{W}_0(X)$ is an element of $\mathcal{N}(T_{\mathbb{C}}) \subset \mathrm{Aut}(X)$ which preserves the orbifold structure. Let $G \subset \mathcal{N}(T_{\mathbb{C}})$ be the compact subgroup generated by $T^n$, the maximal compact subgroup of $T_{\mathbb{C}}$, and $\mathcal{W}_0(X)$. Then we have the, split, exact sequence

$$1 \to T^n \to G \to \mathcal{W}_0(X) \to 1. \tag{27}$$

**Definition 2.12.** A *symmetric Fano toric orbifold* $X$ is a Fano toric orbifold with $\mathcal{W}_0$ acting on $N$ with the origin as the only fixed point. Such a variety and its orbifold structure is characterized by the convex polytope $\Delta^*$ invariant under $\mathcal{W}_0$. We call a toric orbifold *special symmetric* if $\mathcal{W}_0(X)$ contains the involution $\sigma : N \to N$, where $\sigma(n) = -n$.

Conversely, given an integral convex polytope $\Delta^*$, inducing a simplicial fan $\Delta$, invariant under a subgroup $\mathcal{W}_0 \subset \mathrm{GL}(n,\mathbb{Z})$ fixing only the origin, we have a symmetric Fano toric orbifold $X_{\Delta^*}$.

**Definition 2.13.** The *index* of a Fano orbifold $X$ is the largest positive integer $m$ such that there is a holomorphic $V$-bundle $\mathbf{L}$ with $\mathbf{L}^m \cong \mathbf{K}_X^{-1}$. The index of $X$ is denoted $\mathrm{Ind}(X)$.

Note that $c_1(X) \in H^2_{\mathrm{orb}}(X,\mathbb{Z})$, and $\mathrm{Ind}(X)$ is the greatest positive integer $m$ such that $\frac{1}{m}c_1(X) \in H^2_{\mathrm{orb}}(X,\mathbb{Z})$.

**Proposition 2.14.** *Let $X_{\Delta^*}$ be a special symmetric toric Fano orbifold. Then* $\mathrm{Ind}(X) = 1$ *or* 2*.*

**Proof.** We have $\mathbf{K}^{-1} \cong \mathbf{L}_{-k}$ with $-k \in \mathrm{SF}(\Delta^*)$ where $-k(n_\rho) = -1$ for all $\rho \in \Delta(1)$. Suppose we have $\mathbf{L}^m \cong \mathbf{K}^{-1}$. By Proposition 2.7 there is an $h \in \mathrm{SF}(\Delta^*)$ and $f \in M$ so that $mh = -k + f$. For some $\rho \in \Delta(1)$,

$$\begin{aligned} mh(n_\rho) &= -1 + f(n_\rho) \\ mh(-n_\rho) &= -1 - f(n_\rho). \end{aligned}$$

Thus $m(h(n_\rho) + h(-n_\rho)) = -2$, and $m = 1$ or 2. □

In the in the subsequent sections we will be interested in special symmetric toric Fano surfaces. Figure 2 gives the polytopes $\Delta^*$ for the two smooth such examples.

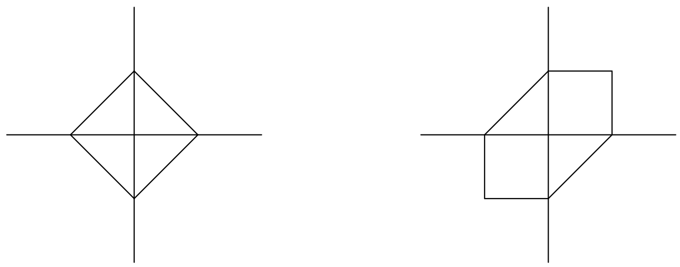

FIGURE 2. Smooth examples

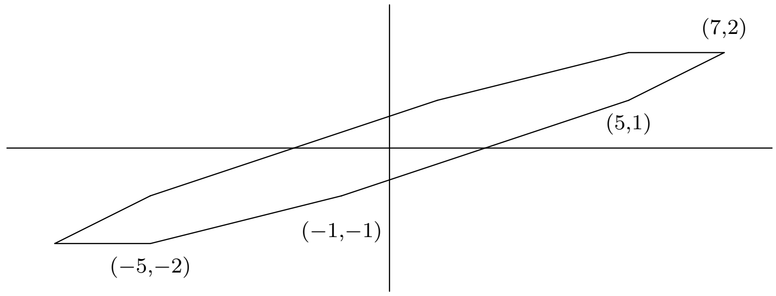


FIGURE 3. Example with 8 point singular set and $\mathcal{W}_0 = \mathbb{Z}_2$

**2.3. Kähler–Einstein metric.** Any compact toric orbifold associated to a polytope admits a Kähler metric. In particular, we need a Kähler metric with Kähler form $\omega$ satisfying $[\omega] \in 2\pi c_1^{\mathrm{orb}}(X) = -c_1(\mathbf{K}_X)$. The Hamiltonian reduction procedure of [28, 29] and [15] provides an explicit metric on the toric orbifold associated to the marked polytope $\Sigma_h^*$. Let $X_{\Sigma_{-k}^*}$ be Fano, it will follow that this metric will satisfy $[\omega] \in 2\pi c_1^{\mathrm{orb}}(X)$.

Let $\Sigma^*$ be a convex polytope in $M_{\mathbb{R}} \cong \mathbb{R}^{n*}$ defined by the inequalities

$$\langle x, u_i \rangle \geq \lambda_i, \quad i = 1, \dots, d, \tag{28}$$

where $u_i \in N \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$. If $\Sigma_h^*$ is associated to $(\Delta^*, h)$, then the $u_i$ and $\lambda_i$ are the set of pairs $n(\rho)$ and $h(n(\rho))$ for $\rho \in \Delta(1)$. We allow the $\lambda_i$ to be real but require any set $u_{i_1}, \dots, u_{i_n}$ corresponding to a vertex to form a $\mathbb{Q}$-basis of $N_{\mathbb{Q}}$.

Let $(e_1, \dots, e_d)$ be the standard basis of $\mathbb{R}^d$ and $\beta : \mathbb{R}^d \to \mathbb{R}^n$ be the map which takes $e_i$ to $u_i$. Let $\mathfrak{n}$ be the kernel of $\beta$, so we have the exact sequence

$$0 \to \mathfrak{n} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \to 0, \tag{29}$$

and the dual exact sequence

$$0 \to \mathbb{R}^{n*} \xrightarrow{\beta^*} \mathbb{R}^{d*} \xrightarrow{\iota^*} \mathfrak{n}^* \to 0. \tag{30}$$

Since (29) induces an exact sequence of lattices, we have an exact sequence

$$1 \to N \to T^d \to T^n \to 1, \tag{31}$$

where the connected component of the identity of $N$ is an $(d-n)$-dimensional torus. The standard representation of $T^d$ on $\mathbb{C}^d$ preserves the Kähler form

$$\frac{i}{2}\sum_{k=1}^{d} dz_k \wedge d\bar{z}_k, \tag{32}$$

and is Hamiltonian with moment map

$$\mu(z) = \frac{1}{2}\sum_{k=1}^{d} |z_k|^2 e_k + c, \tag{33}$$

unique up to a constant $c$. We will set $c = \sum_{k=1}^{d} \lambda_k e_k$. Restricting to $\mathfrak{n}^*$ we get the moment map for the action of $N$ on $\mathbb{C}^d$

$$\mu_N(z) = \frac{1}{2}\sum_{k=1}^{d} |z_k|^2 \alpha_k + \lambda, \tag{34}$$

with $\alpha_k = \iota^* e_k$ and $\lambda = \sum \lambda_k \alpha_k$.

We have the Marsden–Weinstein quotient

$$X_{\Sigma^*} = \mu_N^{-1}(0)/N$$

with a canonical metric with Kähler form $\omega_0$. We have an action of $T^n = T^d/N$ on $X_{\Sigma^*}$ which is Hamiltonian for $\omega$. The map $\nu$ is $T^d$ invariant, and it descends to a map, which we also call $\nu$,

$$\nu : X_{\Sigma^*} \to \mathbb{R}^{n*}, \tag{35}$$

which is the moment map for this action. The above comments show that $\mathrm{Im}(\nu) = \Sigma^*$. The action $T^n$ extends to the complex torus $T^n_{\mathbb{C}}$ and one can show that as an analytic variety and orbifold $X_{\Sigma^*}$ is the toric variety constructed from $\Sigma^*$ in the previous section.

It follows from results of [28, 29] that

$$[\omega_0] = -2\pi \sum_{i=1}^{d} \lambda_i c_i,$$

where $c_i \in H^(X, \mathbb{R})$ is dual to the divisor $D_i \subset X$ associated with the face $\langle x, u_i\rangle = \lambda_i$ of $\Sigma^*$. In particular, if $X_{\Sigma^*}$ is Fano, then $\lambda_i = -1,\ i = 1, \dots, d$ and

$$[\omega_0] = 2\pi \sum_{i=1}^{d} c_i = 2\pi c_1^{\mathrm{orb}}(X).$$

From now on we assume that $X_{\Sigma^*}$ is symmetric and Fano, and we have a metric $g_0$ invariant under the compact group $G \subset \mathrm{Aut}(X)$ with Kähler form $\omega_0$ representing $2\pi c_1^{\mathrm{orb}}(X)$. Finding a Kähler–Einstein metric on $X_{\Sigma^*}$ is equivalent to solving the complex Monge–Ampère for $\phi \in C^\infty(X)$:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n = \omega_0^n e^{f - t\phi}, \quad t \in [0,1], \tag{36}$$

where $f \in C^\infty(X)$ is defined by

$$\sqrt{-1}\partial\bar{\partial} f = \mathrm{Ricci}(\omega_0) - \omega_0 \quad \text{and} \quad \int_X e^f \, d\mu_{g_0} = \mathrm{Vol}_{g_0}(X).$$

If $\phi$ is a solution to (36) for $t = 1$, then

$$\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$$

is Kähler–Einstein. It is well-known that a solution to (36) exists for $t \in [0, \epsilon)$ for $\epsilon$ small, and the existence of a solution at $t = 1$ is equivalent to an *a priori* $C^0$ estimate on $\phi$.

We recall the definition of the invariant $\alpha_G(X)$ introduced by G. Tian [46]. Define

$$P_G(X, g_0) := \Big\{ \phi \in C^2(X)^G : \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi \geq 0 \text{ and } \sup_X \phi = 0 \Big\}.$$

The *Tian invariant* $\alpha_G(X)$ is the supremum of $\alpha > 0$ such that

$$\int_X e^{-\alpha\phi} d\mu_{g_0} \leq C(\alpha), \quad \forall\, \phi \in P_G(X, g_0),$$

where $C(\alpha)$ depends only on $\alpha, X$ and $g_0$.

G. Tian proved the following sufficient condition for an *a priori* $C^0$ estimate on (36). It was shown to also suffice for orbifolds in [20].

**Theorem 2.15.** *Let $X$ be a Fano orbifold and $G \subset \mathrm{Aut}(X)$ a compact subgroup such that*

$$\alpha_G(X) > \frac{n}{n+1},$$

*then $X$ admits a Kähler–Einstein metric.*

Choosing a point $x_0 \in U \subset X$ gives identifications $\mathcal{W}_0(X) \subset \mathrm{Aut}(X)$, $U \cong T_{\mathbb{C}}$, and $U/T \cong N_{\mathbb{R}}$, which identifies $Tx_0$ with $0 \in N_R$. Thus $\mathcal{W}_0(X)$ acts linearly on $N_{\mathbb{R}}$. And if we choose an integral basis $e_1, \ldots, e_n$ of $N$, then we have identifications $N_{\mathbb{R}} \cong \mathbb{R}^n$, $M_R \cong \mathbb{R}^n$, and $T_{\mathbb{C}} \cong (\mathbb{C}^*)^n$. And we introduce logarithmic coordinates $x_i = \log|t_i|^2$ on $N_{\mathbb{R}}$, where $t_1, \ldots, t_n$ are the usual holomorphic coordinates on $(\mathbb{C}^*)^n$. Thurs $t_i = e^{\frac{1}{2}x_i + \sqrt{-1}\theta_i}$, where $0 \leq \theta_i \leq 2\pi$. We will denote the dual coordinates on $M_{\mathbb{R}}$ by $y_1, \ldots, y_n$. We define $l_k(y) = \langle u_k, y\rangle - \lambda_k,\ k = 1, \ldots, d$. So $\Sigma$ is defined by $\cap_{k=1}^d \{l_k \geq 0\}$.

Since the action of $T$ on $U$ is Hamiltonian for $\omega_0$, the orbits of $T$ are isotropic and $\omega_0|_U$ is exact. Furthermore, since $H^{0,k}(U) = H^{0,k}(U)^T = 0$, we easily get the following.

**Lemma 2.16.** *The Kähler form $\omega_0$ restricted to $U$ has $T$-invariant potential function. That is, there is a $F \in C^\infty(N_{\mathbb{R}})$ with*

$$\omega_0|_U = \sqrt{-1}\partial\bar{\partial}F.$$

It was observed in [28] that up to a constant the moment map (35) is

$$\nu : N_{\mathbb{R}} \to M_{\mathbb{R}},$$

$$\nu(x_1, \dots, x_n) = \Big(\frac{\partial F}{\partial x_1}(x), \cdots, \frac{\partial F}{\partial x_n}(x)\Big). \tag{37}$$

By replacing $F$ with $F + \sum_k c_k x_k$ if necessary, we have that (37) coincides with (35) restricted to $U$. Therefore, it is a diffeomorphism of $N_{\mathbb{R}}$ onto the interior of $\Sigma$.

It was shown in [28] that the symplectic potential $G$ of the metric $\omega_0$ is

$$G = \frac{1}{2}\sum_{k=1}^{d} l_k(y)\log l_k(y), \tag{38}$$

where it was also shown that $F$ and $G$ are related by the Legendre transform. As a consequence we get

$$F = \nu^*\left(\frac{1}{2}\sum_{k=1}^{d} \lambda_k \log l_k + l_\infty\right), \tag{39}$$

where $l_\infty(y) = \langle u_\infty, y\rangle,\ u_\infty = \sum_{k=1}^d u_k$. It is easy to see that the symmetric condition on $X_{\Sigma^*}$ implies $u_\infty = 0$.

Let $\sigma_j,\ j = 1, \dots, e$ be the vertices of $\Sigma$. Thus for each element of $\Delta^*(n)$ defined by $u_{j_1}, \dots, u_{j_n}$ one has that $\sigma_j$ is the unique linear function with $\sigma_j(u_{j_i}) = -1$, $i = 1, \dots, n$. Recall that $\lambda_i = -1$, $i = 1, \dots, d$. We define the piecewise linear function on $N_{\mathbb{R}}$

$$\bar{w}(x) := \sup_{j=1,\dots,e} \langle \sigma_j, x\rangle. \tag{40}$$

**Lemma 2.17.** *There exists a constant $C > 0$, depending only on $\Sigma^*$, so that*

$$|F - \bar{w}| \le C.$$

**Proof.** We prove this on momentum coordinates on $M_{\mathbb{R}}$. The moment map is inverted by $x_i = \frac{\partial G}{\partial y_i}$ for $i = 1, \dots, n$. And one computes

$$\begin{aligned}\frac{\partial G}{\partial y}(y) &= \frac{1}{2}\sum_{k=1}^{d} u_k \log l_k(y) + u_\infty \\ &= \frac{1}{2}\sum_{k=1}^{d} u_k \log l_k(y).\end{aligned}$$

Thus on the interior of $\Sigma$,

$$\bar{w}(y) = \sup_{j=1,\dots,e} \frac{1}{2}\sum_{k=1}^{d} \langle \sigma_j, u_k\rangle \log l_k(y). \tag{41}$$

Fix a $j \in \{1, \dots, e\}$, then

$$F(y) - \frac{1}{2}\sum_{k=1}^{d} \langle \sigma_j, u_k \rangle \log l_k(y) = \frac{1}{2}\sum_{k=1}^{d} (-1 - \langle \sigma_j, u_k \rangle) \log l_k(y) \\ \geq C_j,$$

for some constant $C_j$, because each term $(-1 - \langle \sigma_j, u_k \rangle) \log l_k(y)$ is bounded below. Recall that $\langle \sigma_j, u_k \rangle \geq -1, \ \forall k = 1, \dots, d$.

Taking the infimum $C$ of the $C_j, \ j = 1, \dots, e$, we get $F - \bar{w} \geq C$.

To prove the inequality $C' \geq F - \bar{w}$ we define subsets of $\Sigma$. Define $V_i = \{y \mid l_i(y) \leq \epsilon\} \cap \Sigma$, where $\epsilon > 0$ is chosen small enough that the polytope $\cap_{k=1}^{d} \{y \mid l_i(y) \geq \epsilon\} \subset \Sigma$ has the same faces as $\Sigma$. Recall that a face of $\Sigma$ is given by a multi-index $i_1, \dots, i_\ell$ with $l_{i_1} = \cdots = l_{i_\ell} = 0$. For each face define $V_{i_1 \cdots i_\ell} = \cap_{k=1}^{\ell} V_{i_k}$.

For each face of $\Sigma$ we define a subset $W_{i_1, \dots, i_\ell}$ as follows.

$$W_0 = \Sigma - \bigcup_{j=1}^{d} V_j$$

$$W_i = V_i - V_i \cap \left( \bigcup_{j \neq i} V_j \right)$$

$$W_{i_1 i_2} = V_{i_1 i_2} - V_{i_1 i_2} \cap \left( \bigcup_{j \neq i_1, i_2} V_j \right)$$

$$\cdots$$

$$W_{i_1 i_2 \cdots i_\ell} = V_{i_1 i_2 \cdots i_\ell} - V_{i_1 i_2 \cdots i_k} \cap \left( \bigcup_{j \neq i_1, i_2, \dots, i_\ell} V_j \right)$$

$$\cdots$$

$$W_{i_1 \cdots i_n} = V_{i_1 \cdots i_n}.$$

$$F(y) - \bar{w}(y) = \frac{1}{2}\sum_{k=1}^{d} - \log l_k(y) - \sup_{j=1,\dots,e} \left[ \frac{1}{2}\sum_{k=1}^{d} \langle \sigma_j, u_k \rangle \log l_k(y) \right] \\ \leq C_0,$$

on $W_0$ for some $C_0$, because it is continuous and $\bar{W}_0$ is compact.

For $W_i$ choose a $\sigma_j$ with $\sigma_j(u_i) = -1$. Then

$$\begin{aligned} F(y) - \bar{w}(y) &\leq \frac{1}{2}\sum_{k=1}^{d} - \log l_k(y) - \frac{1}{2}\sum_{k=1}^{d}\langle \sigma_j, u_k\rangle \log l_k(y) \\ &= \frac{1}{2}\sum_{\substack{k=1\\ k\neq i}}^{d} - \log l_k(y) - \frac{1}{2}\sum_{\substack{k=1\\ k\neq i}}^{d}\langle \sigma_j, u_k\rangle \log l_k(y) \\ &\leq C_i, \end{aligned}$$

for some constant $C_i$, because the remaining terms are continuous on $\bar{W}_i$. In general, for the set $W_{i_1\cdots i_\ell}$ choose $\sigma_j$ so that $\sigma_j(u_{i_k}) = -1,\ k = 1, \ldots, \ell$. Then as before

$$\begin{aligned} F(y) - \bar{w}(y) &\leq \frac{1}{2}\sum_{\substack{k=1\\ k\neq i_1,\ldots,i_\ell}}^{d} - \log l_k(y) - \frac{1}{2}\sum_{\substack{k=1\\ k\neq i_1,\ldots,i_\ell}}^{d}\langle \sigma_j, u_k\rangle \log l_k(y) \\ &\leq C_{i_1\cdots i_\ell}, \end{aligned}$$

where the constant $C_{i_1\cdots i_\ell}$ exists because all the expression is continuous on $\bar{W}_{i_1\cdots i_\ell}$.

Letting $C'$ be the supremum of the $C_{i_1\cdots i_\ell}$, we have $C \leq F - \bar{w} \leq C'$. □

Given a $G$-invariant $\phi \in C^\infty(X)$, we will denote its descent to a $\mathcal{W}_0$-invariant smooth function on $N_\mathbb{R}$ by $\tilde{\phi}$. We define

$$P_G(N_\mathbb{R}, F) = \left\{\tilde{\phi} \in C^2(N_\mathbb{R})^{\mathcal{W}_0} : \frac{\partial^2(F + \tilde{\phi})}{\partial x_i \partial x_j} \geq 0,\ \sup_{N_\mathbb{R}} \tilde{\phi} = 0,\ \text{and } |\tilde{\phi}| \text{ is bounded on } N_\mathbb{R}\right\}.$$

The following proposition was proved in [4].

**Proposition 2.18.** *Let $X$ be a toric Fano orbifold with $G \subset \mathrm{Aut}(X)$ as above. Let $dx$ be the volume form on $N_\mathbb{R} \cong \mathbb{R}^n$ corresponding to the Haar measure normalized by the lattice $N \subset N_\mathbb{R}$. Let $\tilde{\alpha}_G(X)$ be the supremum of all $\alpha > 0$ such that*

$$\int_{N_\mathbb{R}} e^{-\alpha\tilde{\phi} - F}\, dx \leq \tilde{C}(\alpha), \quad \forall \tilde{\phi} \in P_G(N_\mathbb{R}, F).$$

*Then*

$$\tilde{\alpha}_G(X) \leq \alpha_G(X).$$

The proof in [4] works here, so we omit it. It follows easily from the following observation. As in the smooth case, we have that

$$e^{-F}\frac{dt_1 \wedge d\bar{t}_1 \wedge \ldots \wedge dt_n \wedge d\bar{t}_n}{|t_1|^2 \cdots |t_n|^n} = e^{-F} dx_1 \wedge \ldots \wedge dx_n \wedge d\theta_1 \wedge \ldots \wedge \theta_n$$

can be extended to a nonvanishing volume form on $X$. Therefore it is related to the volume form of $g_0$ by

$$e^h d\mu_{g_0} = e^{-F} dx_1 \wedge \ldots \wedge dx_n \wedge d\theta_1 \wedge \ldots \wedge \theta_n$$

for $h \in C^\infty(X)$, where $h$ differs from $f$ defined after (36) by a constant.

**Lemma 2.19.** *Let $\lambda > 0$. Then $\int_{N_{\mathbb{R}}} e^{-\lambda F}\, dx \le C(\lambda)$.*

**Proof.** By Lemma 2.17 we have

$$\int e^{-\lambda F}\, dx \le \int e^{-\lambda C - \lambda \bar{w}}\, dx = e^{-\lambda C} \int e^{-\lambda \bar{w}}\, dx. \tag{42}$$

Let $\tau \in \Delta^*(n)$ be spanned by $u_{i_1}, \ldots, u_{i_n}$. Then restricted to the cone $-\tau = \mathbb{R}_{\ge 0}\{-u_{i_1}, \ldots, -u_{i_n}\}$ we have $\bar{w} = \sigma$, where $\sigma$ is the linear function with $\sigma(-u_{i_k}) = 1$. Therefore

$$\begin{aligned} e^{-\lambda C} \int_{-\tau} e^{-\lambda \bar{w}}\, dx &= \frac{e^{-\lambda C}}{|\Gamma_\tau|} \int_{\mathbb{R}^n_{\ge 0}} e^{-\lambda(x_1 + \cdots + x_n)}\, dx_1 \cdots dx_n \\ &= \frac{e^{-\lambda C}}{|\Gamma_\tau|} \prod_{i=1}^n \Big( \int_{\mathbb{R}_{\ge 0}} e^{-\lambda x_i}\, dx_i \Big) \\ &= \frac{e^{-\lambda C}}{|\Gamma_\tau|} \frac{1}{\lambda^n}, \end{aligned}$$

where $|\Gamma_\tau|$ is the order of the orbifold group $\Gamma_\tau$ associated to $\tau$. Combining with (42) completes the proof since $N_{\mathbb{R}} = \cup_{\tau \in \Delta} - \tau$. □

**Lemma 2.20.** *There exists a constant $C$ so that for any $\tilde{\phi} \in P_G(N_{\mathbb{R}}, F)$ we have*

$$F(x) + \tilde{\phi} \ge C, \quad \forall x \in N_{\mathbb{R}}.$$

**Proof.** Given an arbitrary $\tilde{\phi} \in P_G(N_{\mathbb{R}}, F)$ we consider the moment map

$$\nu_{F+\tilde{\phi}} : N_{\mathbb{R}} \to M_{\mathbb{R}},$$

$$\nu_{F+\tilde{\phi}}(x) := \Big( \frac{\partial (F + \tilde{\phi})}{\partial x_1}(x), \cdots, \frac{\partial (F + \tilde{\phi})}{\partial x_n}(x) \Big).$$

We will first show that $\nu_{F+\tilde{\phi}}(N_{\mathbb{R}}) \subset \Sigma$. Let $y_0 = \nu_{F+\tilde{\phi}}(x_0)$. By the convexity of $F + \tilde{\phi}$,

$$F(x) + \tilde{\phi}(x) \ge \langle y_0, x - x_0 \rangle + F(x_0) + \tilde{\phi}(x_0).$$

Thus $F(x) + \tilde{\phi}(x) - \langle y_0, x \rangle$ has a global minimum at $x_0$. By Lemma 2.17 and the fact that $\tilde{\phi}$ is globally bounded, $\bar{w} - \langle y_0, x \rangle \ge c$ for some constant $c$. Since this is a piecewise linear function, we have

$$\bar{w} - \langle y_0, x \rangle \ge 0,$$

and this implies that $y_0 \in \Sigma$.

Since $\sup_{N_{\mathbb{R}}} \tilde{\phi} = 0$, we choose a sequence $\{p_k\}$ in $N_{\mathbb{R}}$ so that $-1/k \leq \tilde{\phi}(p_k) \leq 0$. Set $q_k = \nu_{F+\tilde{\phi}}(p_k)$. Since $\Sigma$ is compact by passing to a subsequence if necessary, we may assume that

$$\lim_k q_k = q \in \Sigma.$$

The convexity of $F + \tilde{\phi}$ implies that

$$F(x) + \tilde{\phi}(x) - \langle q_k, x\rangle \geq F(p_k) + \tilde{\phi}(p_k) - \langle q_k, p_k\rangle.$$

By Lemma 2.17 there is a constant $C$ so that

$$F(p_k) + C \geq \bar{w}(p_k) \geq \langle q_k, p_k\rangle,$$

where the second inequality holds because $q_k \in \Sigma$. Therefore

$$F(x) + \tilde{\phi}(x) - \langle q_k, x\rangle \geq -C - \frac{1}{k},$$

and taking $k \to \infty$

$$F(x) + \tilde{\phi}(x) - \langle q, x\rangle \geq -C. \tag{43}$$

Since $\mathcal{W}_0$ is a finite group and $F$, $\tilde{\phi}$ are $\mathcal{W}_0$-invariant, one can average (43) to get

$$F(x) + \tilde{\phi}(x) - \langle \bar{q}, x\rangle \geq -C. \tag{44}$$

Here $\bar{q} = \frac{1}{|\mathcal{W}_0|}\sum_{g\in\mathcal{W}_0} g^* q$ is $\mathcal{W}_0$-invariant, and therefore $\bar{q} = 0$. □

We can now prove the main theorem of the section.

**Theorem 2.21.** *Let $X_{\Sigma^*}$ be a symmetric toric Fano orbifold with $G \subset$ $\mathrm{Aut}(X)$ as above, then $\alpha_G(X) \geq 1$. Therefore, $X$ admits a $G$-invariant Kähler–Einstein metric.*

**Proof.** Let $0 < \alpha < 1$ and $\tilde{\phi} \in P_G(N_{\mathbb{R}}, F)$, then

$$\begin{aligned}\int_{N_{\mathbb{R}}} e^{-\alpha\tilde{\phi}-F}\,dx &= \int_{N_{\mathbb{R}}} e^{-\alpha(\tilde{\phi}+F)}\, e^{(\alpha-1)F}\,dx\\ &\leq e^{-\alpha C}\int_{N_{\mathbb{R}}} e^{(\alpha-1)F}\,dx \qquad \text{(Lemma 2.20)}\\ &\leq e^{-\alpha C} C(1-\alpha), \qquad \text{(Lemma 2.19).}\end{aligned}$$

Thus $\tilde{\alpha}_G \geq 1$, and the theorem follows from Proposition 2.18. □

## 3. Corresponding Sasaki–Einstein spaces and embeddings

In this section we prove the correspondence in (1). First we obtain the toric surface $X$ and Sasaki–Einstein space $M$ from $\mathcal{S}$ only using toric geometry. It is an elementary result of the toric geometry of a toric ASD Einstein space $\mathcal{M}$ that there is a toric Fano orbifold surface $X_{\delta^*}$ associated to it. The Sasaki–Einstein space $M$ is not necessarily smooth. In the following section we prove the embeddings in (1) from which it follows that $M$ is smooth precisely when the 3-Sasaki space $\mathcal{S}$ associated to $\mathcal{M}$ is.

**3.1. Toric surfaces and ASD Einstein orbifolds.** We will consider toric anti-self-dual Einstein orbifolds $\mathcal{M}$ in greater detail. By the previous Section 1.3 quaternion-Kähler reduction gives us infinitely many examples. By reducing $\mathbb{HP}^{k+1}$ by a subtorus $T^k \subset \mathrm{Sp}(k+2)$ defined by an admissible matrix $\Omega$ we get a toric ASD Einstein orbifold $\mathcal{M}_\Omega$ with $b_2(\mathcal{M}) = k$. The orbifold $\mathcal{M}$ is characterized by a polygon $Q_\Omega = \mathcal{M}/T^2$ with $k+2$ edges labeled in cyclic order with $(m_0, n_0), (m_1, n_1), \ldots, (m_{k+2}, n_{k+2})$ in $\mathbb{Z}^2$ with $(m_0, n_0) = -(m_{k+2}, n_{k+2})$. These vectors satisfy the following:

(a) The sequence $m_i,\ i = 0, \ldots k+2$ is strictly increasing.
(b) The sequence $(n_i - n_{i-1})/(m_i - m_{i-1}),\ i = 1, \ldots k+2$ is strictly increasing.

We will make use of the following classification result of D. Calderbank and M. Singer [18].

**Theorem 3.1.** *Let $\mathcal{M}$ be a compact toric 4-orbifold with $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$ and $k = b_2(\mathcal{M})$. Then the following are equivalent.*

(i) *One can arrange that the isotropy data of $\mathcal{M}$ satisfy* (a) *and* (b) *above by cyclic permutations, changing signs, and the action of* $\mathrm{GL}(2, \mathbb{Z})$.
(ii) $\mathcal{M}$ *admits a toric* ASD *Einstein metric unique up to homothety and equivariant diffeomorphism. Furthermore, $(\mathcal{M}, g)$ is isometric to the quaternionic Kähler reduction of $\mathbb{HP}^{k+1}$ by a torus $T^k \subset \mathrm{Sp}(k+2)$.*

It is well-known that the only possible smooth compact ASD Einstein spaces with positive scalar curvature are $S^4$ and $\overline{\mathbb{CP}}^2$ with the round and Fubini–Study metrics [32, 22]. Note that the stabilizer vectors $v_0 = (m_0, n_0)$, $v_1 = (m_1, n_1), \ldots, v_{k+2} = (m_{k+2}, n_{k+2})$ form half a convex polygon with edges of increasing slope.

**Theorem 3.2.** *There is a one to one correspondence between compact toric anti-self-dual Einstein orbifolds $\mathcal{M}$ with $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$ and special symmetric toric Fano orbifold surfaces $X$ with $\pi_1^{\mathrm{orb}}(X) = e$. By Theorem* 2.21 *$X$ has a Kähler–Einstein metric of positive scalar curvature. Under the correspondence if $b_2(\mathcal{M}) = k$, then $b_2(X) = 2k+2$.*

**Proof.** Suppose $\mathcal{M}$ has isotropy data $v_0, v_1, \ldots, v_{k+2}$. Then it is immediate that $v_0, v_1, \ldots, v_{k+2}, -v_1, -v_2, \ldots, -v_{k+1}$ are the vertices of a convex polygon in $N_{\mathbb{R}} = \mathbb{R}^2$, which defines an augmented fan $\Delta^*$ defining $X$. The symmetry of $X$ is clear.

Suppose $X$ is a special symmetric toric Fano surface. Then $X$ is characterized by a convex polygon $\Delta^*$ with vertices $v_0, v_1, \ldots, v_{2k+4}$ with $v_{2k+4} = v_0$. Choose a primitive $p = (u, w) \in \mathbb{Z} \times \mathbb{Z}, w > 0$ which is not proportional to any $v_i - v_{i-1}, i = 1, \ldots, k+2$. Choose $s, t \in \mathbb{Z}$ with $su + tw = 1$. Then let $v_i', i = 0, \dots, 2k+4$ be the images of the $v_i$ under $\begin{bmatrix} w & -u \\ s & t \end{bmatrix}$ There is a $v_j' = (m_j', n_j')$ with $m_j'$ smallest. And $v_j', v_{j+1}', \dots, v_{j+k+2}'$, where the subscripts are mod $2k+4$, satisfy (a) and (b). Such a toric orbifold is simply connected if and only if the isotropy data span $\mathbb{Z} \times \mathbb{Z}$. One can show that the correspondence does not depend on the particular isotropy data. □

In the next section we will prove a more useful geometric correspondence between toric ASD Einstein orbifolds and symmetric toric Kähler–Einstein surfaces.

**Example 3.3.** Consider the admissible weight matrix

$$\Omega = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Then the 3-Sasakian space $\mathcal{S}_\Omega$ is smooth and $b_2(\mathcal{S}_\Omega) = b_2(\mathcal{M}_\Omega) = 2$. And the anti-self-dual orbifold $\mathcal{M}_\Omega$ has isotropy data

$$v_0 = (-7,-2), (-5,-2), (-1,-1), (5,1), (7,2) = v_4.$$

The singular set of $\mathcal{M}$ consists of two points with stabilizer group $\mathbb{Z}_3$ and two with $\mathbb{Z}_4$. The associated toric Kähler–Einstein surface is that in Figure 3.

**Proposition 3.4.** *Let $X$ be the symmetric toric Fano surface associated to the* ASD *Einstein orbifold $\mathcal{M}$. Then* $\mathrm{Ind}(X) = 2$ *if and only if* $w_2(\mathcal{M}) = 0$. *In other words,* $\mathbf{K}_X^{-1}$ *has a square root if and only if the contact line bundle on* $\mathcal{Z}$, $\mathbf{L}$, *does.*

Recall that $w_2(\mathcal{M})$ is equal to the Marchiafava–Romani class $\varepsilon$. Thus the vanishing of $w_2(\mathcal{M})$ is equivalent to the existence of a square root $\mathbf{L}^{\frac{1}{2}}$ of the contact line bundle $\mathbf{L}$ on $\mathcal{Z}$.

**Proof.** Suppose $\mathrm{Ind}(X) = 2$ which is equivalent to $w_2(X) = 0$, where $w_2$ denotes the orbifold Seifel-Whitney class. Recall that the orbit space of $\mathcal{M}$ is a $k+2$-gon $W$ with labeled edges $C_1, \dots, C_{k+2}$. Since $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$, there exists an edge $C_i$ for which the orbifold uniformizing group $\Gamma$ has odd order. Let $U$ be a tubular neighborhood of an orbit in $C_i$. So $U \cong S^1 \times I \times D/\Gamma$, where $I$ is an open interval and $D$ is a 2-disk. And let $V$ be a neighborhood homotopically equivalent to $\mathcal{M} \setminus U$ with $U \cup V = \mathcal{M}$. Consider the exact homology sequence in $\mathbb{Z}_2$-coefficients,

$$(45) \quad \cdots \to H_2(BU) \oplus H_2(BV) \to H_2(B\mathcal{M}) \to H_1(B(U \cap V)) \\ \to H_1(BU) \oplus H_1(BV) \to 0.$$

We have $BU \cong S^1 \times I \times EO(4)/\Gamma$. Since $EO(4)$ is contractible,

$$H_*(EO(4)/\Gamma, A) = H_*(\Gamma, A)$$

for any abelian group $A$. In particular, $H^n(\Gamma, \mathbb{Z}_2) = 0$ for all $n > 0$, since $|\Gamma|$ is odd. Thus $H_2(BU, \mathbb{Z}_2) = 0$ and $H_1(BU, \mathbb{Z}_2) = \mathbb{Z}_2$. Similarly, it not hard to show that $H_1(B(U\cap V),\mathbb{Z}_2)=\mathbb{Z}_2$. From the exact sequence (45) the inclusion $j: V\to \mathcal{M}$ induces a surjection $j_*: H_2(BV,\mathbb{Z}_2)\to H_2(B\mathcal{M},\mathbb{Z}_2)$. Considering the orbit spaces one sees that there is a smooth embedding $\iota: V\to X$. The tangent V-bundle $T\mathcal{M}$ lifts to a genuine vector bundle on $B\mathcal{M}$ which will also be denoted $T$. Then

$$w_2(\mathcal{M})=w_2(T\mathcal{M})\in H^2(B\mathcal{M},\mathbb{Z}_2)=\mathrm{Hom}(H_2(B\mathcal{M},\mathbb{Z}_2),\mathbb{Z}_2).$$

Let $\alpha\in H_2(B\mathcal{M},\mathbb{Z}_2)$. Then there exists a $\beta\in H_2(BV,\mathbb{Z}_2)$ with $j_*\beta=\alpha$. Then

$$w_2(T\mathcal{M})(\alpha)=w_2(TV)(\beta)=w_2(TX)(\iota_*\beta)=0.$$

Thus $w_2(\mathcal{M})=0$.

The converse statement will follow from the main result of the next section. □

**3.2. Twistor space and divisors.** We will consider the twistor space $\mathcal{Z}$ of an ASD positive scalar curvature Einstein orbifold $\mathcal{M}$. For now suppose $(\mathcal{M},[g])$ is an anti-self-dual, i.e., $W_g^+\equiv 0$, conformal orbifold. There exists a complex three dimensional orbifold $\mathcal{Z}$ with the following properties:

(a) There is a $C^\infty$ orbifold bundle $\varpi:\mathcal{Z}\to\mathcal{M}$.
(b) The general fiber of $P_x=\varpi^{-1}(x)$, $x\in\mathcal{M}$ is a projective line $\mathbb{CP}^1$ with normal bundle $N\cong\mathcal{O}(1)\oplus\mathcal{O}(1)$, which holds over singular fibers with $N$ an orbifold bundle.
(c) There exists an anti-holomorphic involution $\varsigma$ of $\mathcal{Z}$ leaving the fibers $P_x$ invariant.

Let $T$ be an oriented real 4-dimensional vector space with inner product $g$. Let $C(T)$ be set of orthogonal complex structures inducing the orientation, i.e., if $r,s\in T$ is a complex basis then $r, Jr, s, Js$ defines the orientation. One has $C(T)=S^2\subset\Lambda^2_+(T)$, where $S^2$ is the sphere of radius $\sqrt{2}$. Now take $T$ to be $\mathbb{H}$. Recall that $\mathrm{Sp}(1)$ is the group of unit quaternions. Let

$$\mathrm{Sp}(1)_+\times\mathrm{Sp}(1)_- \tag{46}$$

act on $\mathbb{H}$ by

$$w\to gwg'^{-1},\ \text{for } w\in\mathbb{H} \text{ and } (g,g')\in\mathrm{Sp}(1)_+\times\mathrm{Sp}(1)_-. \tag{47}$$

Then we have

$$\mathrm{Sp}(1)_+\times_{\mathbb{Z}_2}\mathrm{Sp}(1)_-\cong\mathrm{SO}(4), \tag{48}$$

where $\mathbb{Z}_2$ is generated by $(-1,-1)$. Let

$$\begin{aligned} C&=\{ai+bj+ck: a^2+b^2+c^2=1, a,b,c\in\mathbb{R}\}\\ &=\{g\in\mathrm{Sp}(1)_+: g^2=-1\}\cong S^2. \end{aligned} \tag{49}$$

Then $g\in C$ defines an orthogonal complex structure by

$$w\to gw,\ \text{for } w\in\mathbb{H},$$

giving an identification $C = C(\mathbb{H})$. Let $V_+ = \mathbb{H}$ considered as a representation of $\mathrm{Sp}(1)_+$ and a right $\mathbb{C}$-vector space. Define $\pi : V_+ \setminus \{0\} \to C$ by $\pi(h) = -hih^{-1}$. Then the fiber of $\pi$ over $hih^{-1}$ is $h\mathbb{C}$. Then $\pi$ is equivariant if $\mathrm{Sp}(1)_+$ acts on $C$ by $q \to gqg^{-1}, g \in \mathrm{Sp}(1)_+$. We have a the identification

$$C = V_+ \setminus \{0\}/\mathbb{C}^* = \mathbb{P}(V_+). \tag{50}$$

Fix a Riemannian metric $g$ in $[g]$. Let $\phi : \tilde{U} \to U \subset \mathcal{M}$ be a local uniformizing chart with group $\Gamma$. Let $F_{\tilde{U}}$ be the bundle of oriented orthonormal frames on $\tilde{U}$. Then

$$F_{\tilde{U}} \times_{\mathrm{SO}(4)} \mathbb{P}(V_+) = F_{\tilde{U}} \times_{\mathrm{SO}(4)} C \tag{51}$$

defines a local uniformizing chart for $\mathcal{Z}$ mapping to

$$F_{\tilde{U}} \times_{\mathrm{SO}(4)} \mathbb{P}(V_+)/\Gamma = F_{\tilde{U}}/\Gamma \times_{\mathrm{SO}(4)} \mathbb{P}(V_+).$$

Right multiplication by $j$ on $V_+ = \mathbb{H}$ defines the anti-holomorphic involution $\varsigma$ which is fixed point free on (51). We will denote a neighborhood as in (51) by $\tilde{U}_{\mathcal{Z}}$.

An almost complex structure is defined as follows. At a point $z \in \tilde{U}_{\mathcal{Z}}$ the Levi-Civita connection defines a horizontal subspace $H_z$ of the real tangent space $T_z$ and we have a splitting

$$T_z = H_z \oplus T_z P_x = T_x \oplus T_z P_x, \tag{52}$$

where $\varpi(z) = x$ and $T_x$ is the real tangent space of $\tilde{U}$. Let $J_z$ be the complex structure on $T_x$ given by $z \in P_x = C(T_x)$, and let $J'_z$ be the complex structure on $T_z P_x$ arising from the natural complex structure on $P_x$. Then the almost complex structure on $T_z$ is the direct sum of $J_z$ and $J'_z$. This defines a natural almost complex structure on $Z_{\tilde{U}}$ which is invariant under $\Gamma$. We get an almost complex structure on $\mathcal{Z}$ which is integrable precisely when $W^+_g \equiv 0$ [2].

Assume that $\mathcal{M}$ is ASD Einstein with nonzero scalar curvature. Then $\mathcal{Z}$ has a complex contact structure $D \subset T^{1,0}\mathcal{Z}$ with holomorphic contact form $\theta \in \Gamma(\Lambda^{1,0}\mathcal{Z} \otimes \mathbf{L})$ where $\mathbf{L} = T^{1,0}\mathcal{Z}/D$.

The group of isometries $\mathrm{Isom}(\mathcal{M})$ lifts to an action on $\mathcal{Z}$ by real holomorphic transformations. Real means commuting with $\varsigma$. This extends to a holomorphic action of the complexification $\mathrm{Isom}(\mathcal{M})_{\mathbb{C}}$. For $X \in \mathfrak{Isom}(\mathcal{M}) \otimes \mathbb{C}$, the Lie algebra of $\mathrm{Isom}(\mathcal{M})_{\mathbb{C}}$, we will also denote by $X$ the holomorphic vector field induced on $\mathcal{Z}$. Then $\theta(X) \in H^0(\mathcal{Z}, \mathcal{O}(\mathbf{L}))$. By a well-known twistor correspondence the map $X \to \theta(X)$ defines an isomorphism

$$\mathfrak{Isom}(\mathcal{M}) \otimes \mathbb{C} \cong H^0(\mathcal{Z}, \mathcal{O}(\mathbf{L})), \tag{53}$$

which maps real vector fields to real sections of $\mathbf{L}$.

Suppose for now on that $\mathcal{M}$ is a toric ASD Einstein orbifold with twistor space $\mathcal{Z}$. We will assume that $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$ which can always be arranged by taking the orbifold cover. Then as above $T^2$ acts on $\mathcal{Z}$ by holomorphic transformations. And the action extends to $T^2_{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^*$, which in this case is an algebraic action. Let $\mathfrak{t}$ be the Lie algebra of $T^2$ with $\mathfrak{t}_{\mathbb{C}}$ the Lie algebra of $T^2_{\mathbb{C}}$. Then we have from (53) the pencil

$$P = \mathbb{P}(\mathfrak{t}_{\mathbb{C}}) \subseteq |L|, \tag{54}$$

where for $t \in P$ we denote $X_t = (\theta(t))$ the divisor of the section $\theta(t) \in H^0(\mathcal{Z}, \mathcal{O}(\mathbf{L}))$. Note that $P$ has an equator of real divisors. Also, since $T^2_{\mathbb{C}}$ is abelian, every $X_t, t \in P$ is $T^2_{\mathbb{C}}$ invariant.

Consider again the $T^2$-action on $\mathcal{M}$. Let $K_x$ denote the stabilizer of $x \in \mathcal{M}$. Recall the set with nontrivial stabilizers of the $T^2$-action on $\mathcal{M}$ is $B = \bigcup_{i=1}^{k+2} B_i$ where $B_i$ is topologically a 2-sphere. Denote $x_i = B_i \cap B_{i+1}$, $B'_i = B_i \setminus \{x_i, x_{i-1}\}$ and $B' = \bigcup_{i=1}^{k+2} B'_i$. And denote the stabilizer of $B'_i = B_i \setminus \{x_i, x_{i-1}\}$ by $K_i = S^1(m_i, n_i)$. The stabilizer of $x_i$ is $K = T^2$. We will first determine the singular set $\Sigma \subset \mathcal{Z}$ for the $T^2$-action on $\mathcal{Z}$.

**Lemma 3.5.** *For $x \in B$ there exists on $P_x$ precisely two fixed points $z^+, z^-$ for the action of $K_x$ which are $\varsigma$ conjugate. For $x \in B'$, the stabilizer group in $T^2$ of any other $z \in P_x$ is trivial.*

**Proof.** Let $\phi : \tilde{U} \to U$ be a uniformizing chart centered at $x$ with group $\gamma$. We may assume that $\tilde{K}_x$ acts on $\tilde{U}$ with $\gamma \subset \tilde{K}_x$ and $\tilde{K}_x/\gamma = K_x$. Then the uniformized tangent space splits

$$T_{\tilde{x}} = T_1 \oplus T_2. \tag{55}$$

When $x \in B'$ we take $T_1$ to be the space on which $\tilde{K}_i$ acts trivially and $T_2$ on which $\tilde{K}_i$ act faithfully. When $x = x_i$, $\tilde{K}_x = \tilde{K}_i \times \tilde{K}_{i+1}$ assume $\tilde{K}_i$ acts faithfully on $T_1$ and trivially on $T_2$, and $\tilde{K}_{i+1}$ trivially on $T_1$ and faithfully on $T_2$.

We determine the action of $\tilde{K}_x$ on $z \in \tilde{P}_x$. Identify (55) with $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, considered as a right $\mathbb{C}$-vector space. The action of $\tilde{K}_x = S^1(t)$ in the first case is

$$(x, y) \to (x, ty),$$

and the action of $\tilde{K}_x = S^1(s) \times S^1(t)$ in the second is

$$(x, y) \to (sx, ty).$$

If $(u, v) \in S^1 \times S^1 \subset \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$, then the action of $(u, v)$ on $T_x$ is

$$(x, y) \to (uv^{-1}x, (uv)^{-1}y).$$

In the first case the action of $\tilde{K}_i$ is realized by the subgroup $\{(u, u)\}$ with $t = u^{-1}$. Considering the representation of $\mathrm{Sp}(1)_+$ on $V_+ = \mathbb{H}$, $u$ acts by

$$(w, z) \to (uw, u^{-1}z).$$

One sees that the only fixed points on $\tilde{P}_x = \mathbb{P}(V_+)$ are $[1 : 0]$ and $[0 : 1]$. It is easy to see that $\tilde{K}_i$ acts freely on every other point of $\tilde{P}_x$. This also proves the statement for $x = x_i$. $\square$

Denote the two $K_x$ fixed points on $P_x$ for $x = x_i$ by $z_i^{\pm}$. We will denote $P_i := P_{x_i}, i = 1, \dots, k+2$. The next result is an easy consequence of the last lemma.

**Lemma 3.6.** *There exist two irreducible rational curves $C_i^{\pm}, i = 1, \dots, k+2$ mapped diffeomorphically to $B_i$ by $\varpi$. Furthermore, $\varsigma(C_i^{\pm}) = C_i^{\mp}$.*

The singular set for the $T^2$-action on $\mathcal{Z}$ is the union of rational curves

$$\Sigma = \Big(\cup_{i=1}^{k+2} P_i\Big) \bigcup \Big(\cup_{i+1}^{k+2} C_i^{+} \cup C_i^{-}\Big). \tag{56}$$

The fixed points for $T^2$ are $z_i^{\pm}, i = 1, \dots, k+2$. And the stabilizer group of $C_i'^{\pm} = C_i^{\pm} \setminus \{z_i^{\pm}, z_{i-1}^{\pm}\}$ is $K_i$. If $S_{\mathcal{Z}}$ is the orbifold singular set, then $S_{\mathcal{Z}} \subset \Sigma$. In this case $S_{\mathcal{Z}} = \mathrm{Sing}(\mathcal{Z})$, the singular set of $\mathcal{Z}$ as an analytic variety.

We will denote the union of the curves $C_i^{\pm}$ by

$$C = \bigcup_{i=1}^{k+2} (C_i^{+} \cup C_i^{-}).$$

Then either $C$ is a connected cycle, or it consists of two $\varsigma$-conjugate cycles. It will turn out that $C$ is always connected. Thus it may be more convenient to denote its components by $C_i, i = 1, \dots, 2n$, where $n = k+2$, and the points $z_i^{\pm}$ by $z_i$ and $z_{i+n}$ such that

$$z_i = C_i \cap C_{i+1}, i = 1, \dots, 2n,$$

where we take the index to be mod $2n$.

We now consider the action of $T^2_{\mathbb{C}}$ on $\mathcal{Z}$. The stabilizer group of $z \in \mathcal{Z}$ in $T^2_{\mathbb{C}}$ will be denoted $G_z$. Let $G_i \subset T^2_{\mathbb{C}}$ be the complexification of $K_i$.

**Lemma 3.7.** *For $z \in C_i', i = 1, \dots, 2n$, the stabilizer group $G_z$ coincides with $G_i$.*

**Proof.** We have $G_i \subset G_z$ with $\dim G_i = 1$. Suppose $G_i \neq G_z$ then $G_z/G_i$ is a discrete subgroup of $T^2_{\mathbb{C}}/G_i \cong \mathbb{C}^*$. It is easy to see that $G_z/G_i$ is an infinite cyclic subgroup of $T^2_{\mathbb{C}}/G_i$. Then the orbit of $z$, $C_i' \cong T^2_{\mathbb{C}}/G_z$ must be a one dimensional complex torus, which is a contradiction. □

Recall that a parametrization of a stabilizer group $K_i = S^1(m_i, n_i)$, of $B_i$, $i = 1, \dots, n$, is only fixed up to sign. This amounts to a choice of orientation of $B_i$. In view of Proposition 3.9 for the stabilizer group $G_i$ of $C_i'$, $i = 1, \dots, 2n$, there is a fixed parametrization $\rho_i : \mathbb{C}^* \to T^2_{\mathbb{C}}$. One picks one of two possibilities by the rule: For $z$ in a sufficiently small neighborhood of a point of $C_i'$ one has

$$\lim_{t \to 0} \rho_i(t) z \in C_i.$$

**Lemma 3.8.** *We have $\rho_i = -\rho_{i+n}$ for $i = 1, \dots, n$, where we consider the $\rho_i$ to be elements of the $\mathbb{Z}^2$ lattice of one parameter subgroups of $T^2_{\mathbb{C}}$.*

**Proof.** Let $x \in B'_i$. And consider the action of $G_i$ on the twistor line $P_i$ as described in the proof of Lemma 3.5. If $z \in P_i$, then $\lim_{t\to 0} \rho_i(z) = z_+ \in C_i$ implies $\lim_{t\to 0} \rho_i^{-1}(z) = z_+ \in C_{i+n}$. $\square$

We now consider the isotropy representations of $G_z$. The proof of the following is straight forward.

**Proposition 3.9.** *Let $z \in C$ with $\varpi(z) = x$. And let $\phi : \tilde{U} \to U$ be a $\tilde{K}_x$-invariant local uniformizing chart with group $\gamma \subset \tilde{K}_x$. Also $\tilde{G}_i$ denotes the complexification of $\tilde{K}_x$.*

(i) *Let $z \in C'_i$, $i = 1, \dots, 2n$. Then there are $\mathbb{C}$-linear coordinates $(u, v, w)$ on $T_{\tilde{z}}\tilde{U}_{\mathbb{Z}}$ and an identification $\tilde{G}_z \cong \mathbb{C}^*(t)$ so that $\tilde{G}_z$ acts by*
$$(u, v, w) \to (u, tv, tw).$$
*And the subspace $v = w = 0$ maps to the tangent space of $C'_i$ at $z$.*

(ii) *Let $z = z_i$ for $i = 1, \dots, 2n$. Then there are $\mathbb{C}$-linear coordinates $(u, v, w)$ on $T_{\tilde{z}}\tilde{U}_{\mathbb{Z}}$ and an identification $\tilde{G}_z \cong \mathbb{C}^*(s) \times \mathbb{C}^*(t)$ so that $\tilde{G}_z$ acts by*
$$(u, v, w) \to (stu, sv, tw).$$
*And the uniformized tangent space of $P_i$ (resp. $C_i$, and $C_{i+1}$) at $z$ is the subspace $v = w = 0$ (resp. $u = v = 0$ and $u = w = 0$).*

We will determine the $T^2_{\mathbb{C}}$-action in a neighborhood of $C$. Let $z \in C$, and let $\tilde{U}_{\mathbb{Z}}$ be a $\tilde{K}_z$-invariant uniformizing neighborhood as above with local group $\gamma \subset \tilde{K}_z$. Then there is

(i) a $\tilde{K}_z$-invariant neighborhood $W$ of the origin in $T_{\tilde{z}}\tilde{U}_{\mathbb{Z}}$,
(ii) a $\tilde{K}_z$-invariant neighborhood $V$ of $\tilde{z}$ in $\tilde{U}_{\mathbb{Z}}$, and
(iii) a $\tilde{K}_z$-invariant biholomorphism $\varphi : W \to V$, i.e.,

$$\varphi(gx) = g\varphi(x), \text{ for } x \in W, g \in \tilde{K}_z. \tag{57}$$

This is well-known; see for example [8].

This linear action extend locally to $\tilde{G}_z$, where $\tilde{G}_z$ is the complexification of $\tilde{K}_z$. Let $W_0 \subset W$ be a connected relatively compact neighborhood of the origin. And define the open set $A = \{(g, w) \in \tilde{G}_z \times W_0 : gw \in W\}$, and let $A_0 \subset A$ be the connected component containing $\tilde{K}_z \times W_0$. Then for any $(g, w) \in A_0$, we have $g\varphi(w) \in V$ and (57).

We now describe the local action of $T^2_{\mathbb{C}}$ around a point $z \in C$. There are two cases, (i) and (ii), distinguished as in Proposition 3.9. In Case (i) $z \in C'_i$ for some $i = 1, \dots, 2n$. And in Case (ii) $z = z_i$ for some $i = 1, \dots, 2n$. we will use Proposition 3.9 and the above remarks to produce a neighborhood $U$ of $z$ as follows.

Case (i). Suppose $z \in C'_i$. There exists an equivariant uniformizing neighborhood $\phi : \tilde{U} \to U$ centered at $z$ with group $\gamma \subset \tilde{K}_i$. One can lift the corresponding one parameter group $\tilde{\rho}_i : \mathbb{C}^*(t) \to \tilde{T}_{\mathbb{C}}$ with image $\tilde{G}_i$. Let

$\tilde{G}' = \mathbb{C}^*(s)$ be a compliment to $\tilde{G}_i$ in $\tilde{T}_{\mathbb{C}}$. There exists coordinates $(u, v, w)$ in $\tilde{U}$ so that

$$\tilde{U} = \{(u, v, w) : |u - 1| < \epsilon, |v| < 1, |w| < 1\}, \quad \epsilon > 0, \tilde{z} = (1, 0, 0). \tag{58}$$

And $v = w = 0$ is the subset mapped to $C$ and $\tilde{G}_i$ acts by

$$(u, v, w) \to (u, tv, tw), \text{ for } |t| \leq 1. \tag{59}$$

The action of $\tilde{G}'$ is given by $(u, v, w) \to (su, v, w)$ for $|su - 1| < \epsilon$.

Case (ii). Suppose $z = z_i$ for some $i = 1, \ldots, 2n$. There exists an equivariant uniformizing neighborhood $\phi : \tilde{U} \to U$ centered at $z$ with group $\gamma \subset \tilde{K}_z = \tilde{T}^2$. And one can lift the one parameter groups to $\tilde{\rho}_i$ and $\tilde{\rho}_{i+1}$ to give an isomorphism

$$\tilde{\rho}_i \times \tilde{\rho}_{i+1} : \mathbb{C}^*(s) \times \mathbb{C}^*(t) \to \tilde{T}^2_{\mathbb{C}},$$

where $\tilde{T}^2_{\mathbb{C}}$ is the complexification of $\tilde{T}^2$. There exists coordinates $(u, v, w)$ in $\tilde{U}$ so that

$$\tilde{U} = \{(u, v, w) : |u| < 1, |v| < 1, |w| < 1\}, \quad \tilde{z} = (0, 0, 0), \tag{60}$$

where the equations $u = v = 0$, $u = w = 0$, and $v = w = 0$ are the equations defining the subsets mapped to $C_i$, $C_{i+1}$, and $P_i$ respectively. And the action of $(s, t) \in \mathbb{C}^*(s) \times \mathbb{C}^*(t)$ is given by

$$(u, v, w) \to (stu, sv, tw), \quad \text{for } |s| \leq 1, |t| \leq 1. \tag{61}$$

We will call such a neighborhood $U$ of a point of $C$ an *admissible neighborhood*, and $\phi : \tilde{U} \to U$ with group $\gamma$ an *admissible uniformizing system.* Let $U$ be an admissible neighborhood. We set

$$U' := U \setminus \Sigma.$$

Denote by $\tilde{U}'$ the preimage of $U'$ in $\tilde{U}$. We will define subsets $\tilde{U}'_{ab}$, $\tilde{U}'_{01}$ and $\tilde{U}''_{01}$ of $\tilde{U}'$.

In case (i), for $(a, b) \neq 0$, define

$$\tilde{U}'_{ab} := \{(u, v, w) \in \tilde{U}' : av = bw\}.$$

In case (ii), for $(a, b)$ with $a \neq 0$, we define

$$\tilde{U}'_{ab} := \{(u, v, w) \in \tilde{U}' : au = bvw\},$$

and the two subsets

$$\tilde{U}'_{01} := \{(u, v, w) \in \tilde{U}' : v = 0\}, \quad \tilde{U}''_{01} := \{(u, v, w) \in \tilde{U}' : w = 0\}.$$

**Lemma 3.10.** *The subsets defined above are connected closed submanifolds of $\tilde{U}'$ and each consists of a single local $\tilde{T}^2_{\mathbb{C}}$-orbit with these being all the orbits. And the closure of each orbit is an analytic submanifold of $\tilde{U}$.*

This follows from the above description of the $\tilde{T}^2_{\mathbb{C}}$-action. Note that $\gamma$ preserves the orbits so this gives a description of the local orbits of $T^2_{\mathbb{C}}$ in $U$. We will denote by $U'_{ab}, U'_{01}$, and $U''_{01}$ the corresponding local orbits in $U$.

We have the local leaf structure of the orbits in an admissible neighborhood. In most cases this gives the global leaf structure.

**Lemma 3.11.** *Let $U$ be an admissible neighborhood. Let $E, F \subset U'$ be separate local leaves not both being of type $U'_{01}$ or $U''_{01}$. Then $E$ and $F$ are not contained in the same $T^2_{\mathbb{C}}$-orbit.*

**Proof.** After acting by an element of $T^2_{\mathbb{C}}$ we may assume $U$ is an admissible neighborhood as in case (i). with coordinates $(u, v, w)$ and $v = w = 0$ defining $C_i \cap U$. Let $z \in E$ and $z' \in F$ both have $u = 1$. There is a $g \in T^2_{\mathbb{C}}$ with $gz = z'$. Let $z_0 = \lim_{t\to 0} \rho_i(t)z = \lim_{t\to 0} \rho_i(t)z'$. Then

$$gz_0 = g\left(\lim_{t\to 0} \rho_i(t)z\right) = \lim_{t\to 0} \rho_i(t)gz = \lim_{t\to 0} \rho_i(t)z' = z_0.$$

So $g \in G_i$, and $g = \rho_i(t_0)$. If $|t_0| \leq 1$, then $g$ preserves the local leaves. If $|t_0| > 0$, the equation $z = g^{-1}z'$ gives a contradiction. □

**Lemma 3.12.** *For any $z \in U'$, an admissible neighborhood, the stabilizer group $G_z$ is the identity.*

**Proof.** If $g \in G_z$, then $g$ fixes the entire $T^2_{\mathbb{C}}$-orbit of $z$. Therefore $g$ fixes the entire set $U'_{ab}$ containing $z$. But the closure of $U'_{ab}$ intersects either $C_i$ or $C_{i+1}$. So $g$ is contained in either $G_i$ or $G_{i+1}$. But from the above description of the action on $U'$, we see that $g = e$. □

**Lemma 3.13.** *Let $z$ be any point of $P'_i = P_i \setminus \{z_i, z_{i+n}\}$. And let $U$ be an admissible neighborhood of $z_i$ or $z_{i+n}$. Then there exists a neighborhood $V$ of $z$ and $g \in T^2_{\mathbb{C}}$ so that $g(V) \subset U$.*

**Proof.** The stabilizer group of $P'_i$ is the image of the one parameter group $\rho_i {\rho_{i+1}}^{-1} : C^*(s) \to T^2_{\mathbb{C}}$. Then the orbit of $z$ by $G_i$ for example is $P'_i$. So a suitable element $g \in G_i$ will work. □

By Lemmas 3.12 and 3.13 there is a small neighborhood $W$ of $\Sigma \subset \mathcal{Z}$, so that if we set $W' := W \setminus \Sigma$, the stabilizer of every point or $W'$ in $T^2_{\mathbb{C}}$ is the identity.

Our goal is to determine the structure of the divisors in the pencil $P$. As before we will consider the one parameter groups $\rho_i \in N = \mathbb{Z} \times \mathbb{Z}$, where $N$ is the lattice of one parameter $\mathbb{C}^*$-subgroups of $T^2_{\mathbb{C}}$. Also, we will identify the Lie algebra $\mathfrak{t}$ of $T^2$ with $N \otimes \mathbb{R}$ and the Lie algebra $\mathfrak{t}_{\mathbb{C}}$ of $T^2_{\mathbb{C}}$ with $N \otimes \mathbb{C}$. Since $\mathbf{L}|_{P_x} = \mathcal{O}(2)$ a divisor $X_t \in P$ intersects a generic twistor line $P_x$ at two points.

**Lemma 3.14.** *For any $X_t \in P$, we have $C \subset X_t$.*

**Proof.** Let $x \in B_i$. Suppose that $z \in P'_x = P_x \setminus \{z^+, z^-\}$ and $z \in X_t$. Then $G_i$ preserves the twistor line $P_x$, and the orbit of $z$ by $G_i$ is $P'_x$. Since $X_t$

is $T^2_{\mathbb{C}}$-invariant $P_x \subset X_t$. Therefore, we either have $P_x \cap X_t = \{z^+, z^-\}$ or $P_x \subset X_t$. □

**Theorem 3.15.** *Let* $\mathcal{M}$ *be a compact* ASD *Einstein orbifold with* $b_2(\mathcal{M}) = k$ *and* $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$. *Let* $n = k + 2$. *Then there are distinct real points* $t_1, t_2, \ldots, t_n \in P$ *so that for* $t \in P \setminus \{t_1, t_2, \ldots, t_n\}$, $X_t \subset \mathcal{Z}$ *is a suborbifold. And* $X_t$ *is a special symmetric toric Fano surface. The anti-canonical cycle of* $X_t$ *is* $C_1, C_2, \ldots, C_{2n}$, *and the corresponding stabilizers are* $\rho_1, \rho_2, \ldots, \rho_{2n}$ *which define the vertices in* $N = \mathbb{Z} \times \mathbb{Z}$ *of* $\Delta^*$ *with* $X_t = X_{\Delta^*}$.

*For* $t_i \in P$, $X_{t_i} = D + \bar{D}$, *where* $D, \bar{D}$ *are irreducible degree one divisors with* $\varsigma(D) = \bar{D}$. $D, \bar{D}$ *are suborbifolds of* $\mathcal{Z}$ *and are toric Fano surfaces. We have* $D \cap \bar{D} = P_i$ *and the elements* $\pm(\rho_1, \ldots, \rho_i, -\rho_i + \rho_{i+1}, \rho_{n+i+1}, \ldots, \rho_{2n})$ *define the augmented fans for* $D$ *and* $\bar{D}$.

**Proof.** Let $z \in W'$, so the stabilizer of $z$ is the identity. Let O be the $T^2_{\mathbb{C}}$-orbit of $z$. Since $T^2_{\mathbb{C}}$ has only one end $\bar{O} \setminus O$ is connected. Since $\bar{O} \cap \Sigma \neq \emptyset$, and the stabilizer of every point of $W'$ is the identity, $\bar{O} \setminus O \subset \Sigma$.

Define elements $t_i \in P$ by $t_i = \rho_{i+1} - \rho_i, i = 1, \ldots, n$. Recall that the stabilizer of $P_i$ is $\rho_i \rho_{i+1}^{-1} : \mathbb{C}^* \to T^2_{\mathbb{C}}$. If $t \in P \setminus \{t_1, t_2, \ldots, t_n\}$, then a vector field induced by $t$ is tangent to, and nonvanishing on, $P_i, i = 1, \ldots, n$. Since the contact structure $D = \ker \theta$ is transverse to the twistor lines, $P_i \cap X_t = \{z^+, z^-\}$. Let $z \in X_t$ be in an admissible neighborhood of $C$. Then the $T^2_{\mathbb{C}}$-orbit O of $z$ satisfies $\bar{O} \setminus O \subset C$. The intersection of O with any admissible neighborhood is a leaf $U'_{ab}$ which has analytic closure. Let $Y = \bar{O}$, then $Y$ is an analytic subvariety.

Suppose $C$ consists of two disjoint cycles with $Y \cap C = \bigcup_{i=1}^n C_i$. Then $Y$ is a degree one divisor, i.e., intersecting a generic twistor line at one point. If $\bar{Y} = \varsigma(Y)$, then $Y \cap \bar{Y} = \bigcup_{i=1}^m P_{x_i}$, a disjoint union of twistor lines with $x_i \notin B, i = 1, \ldots, m$. Since $Y \cap \bar{Y}$ is $T^2_{\mathbb{C}}$-invariant, we must have $Y \cap \bar{Y} = \emptyset$. Thus $Y$ intersects each twistor line at one point. This is impossible. $Y$ defines a, positively oriented, almost complex structure $J$ on $\mathcal{M}$. Then if $c_1 = c_1^{\mathrm{orb}}(\mathcal{M}, J)$, $c_1^2 = 2\chi_{\mathrm{orb}} + 3\tau_{\mathrm{orb}}$ where $\chi_{\mathrm{orb}}$ and $\tau_{\mathrm{orb}}$ are defined by the same Gauss–Bonnet formulae as on smooth 4-manifolds [5]. We have

$$2\chi_{\mathrm{orb}} + 3\tau_{\mathrm{orb}} = \frac{1}{4\pi^2} \int_{\mathcal{M}} \frac{s^2}{24} d\mu > 0.$$

But a familiar Bochner argument shows the intersection form is negative definite. Therefore $C \subset Y$, $Y$ is a degree two divisor, and $X_t = Y$. From the description of the admissible uniformizing systems and the local leaves, we see that $X_t$ is a suborbifold. Since $X_t$ is the closure of an orbit isomorphic to $T^2_{\mathbb{C}}$ it is a toric variety and has the anti-canonical cycle $C$ and stabilizers $\rho_i$ defining $\Delta^*$.

The adjunction for $X = X_t$ formula gives $\mathbf{K}_X \cong \mathbf{K}_{\mathcal{Z}} \otimes [X]|_X = \mathbf{K}_{\mathcal{Z}}^{\frac{1}{2}}|_X$. Thus $\mathbf{K}_X^{-1} > 0$. The orbifold version of the Kodaira embedding theorem [3] implies that $\mathbf{K}_X^{-m}$ is very ample for $m \gg 0$. From basic properties of toric varieties it follows that $-k \in \mathrm{SF}(\Delta^*)$ is strictly upper convex. Thus $X$ is Fano and $\Delta^*$ is a convex polytope. It follows that $t_1, \dots, t_n \subset P$ form a cycle of distinct points.

Suppose $t = t_i, i = 1, \dots, n$. Then $X_t \cap \Sigma = C \cup P_i$. Let $z \in X_t$ be in an admissible neighborhood of type i. with orbit O. Let $D = \bar{O}$. Then $D \setminus O \subset C \cup P_i$. And $P_i \subset D$, for otherwise we would have $D = X_t$ as in the last paragraph. For an admissible neighborhood $U$ of $z_i$ or $z_{i+n}$ O must intersect $U$ in a leaf $U'_{01}$ or $U''_{01}$. This can be seen from Lemma 3.11. We must have either $D \cap \Sigma = C \cup P_i$ or a cycle of the form $C_1, \dots, C_i, P_i, C_{i+n+1}, \dots, C_{2n}$. In the first case $D = X_t$ is irreducible. Since $X_t$ is a real divisor, arguments as in [43] show that $X_t$ must be a suborbifold, i.e., smooth on a uniformizing neighborhood. But $X_t$ has a crossing singularity along $P_i$, a contradiction. Therefore, $D \cap \Sigma = C_1, \dots, C_i, P_i, C_{i+n+1}, \dots, C_{2n}$, and $D$ is an analytic subvariety, and a suborbifold. Since $D = \bar{O}$ it is a toric variety. Since $X_t$ is real, $\bar{D} \subset X_t$. And $D \cup \bar{D} = X_t$ as both are degree two. □

Note that if the isotropy data of $\mathcal{M}$ is normalized to satisfy conditions (a) and (b) before Theorem 3.1, then we have the identification

$$\begin{aligned} \rho_1 = (m_1, n_1), \dots, \rho_{k+2} = (m_{k+2}, n_{k+2}), \rho_{k+3} = -(m_1, n_1), \\ \dots, \rho_{2k+4} = -(m_{k+2}, n_{k+2}) = (m_0, n_0). \end{aligned} \tag{62}$$

Here, as above, we identify $\rho_i$ with a lattice point in $N = \mathbb{Z} \times \mathbb{Z}$.

**3.3. Sasaki-embeddings.** Associated to each compact toric ASD Einstein orbifold $\mathcal{M}$ with $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$ is the twistor space $\mathcal{Z}$ and a family of embeddings $X_t \subset \mathcal{Z}$ where $t \in P \setminus \{t_1, t_2, \dots, t_{k+2}\}$ and $X = X_t$ is the symmetric toric Fano surface canonically associated to $\mathcal{M}$. We denote the family of embeddings by

$$\iota_t : X \to \mathcal{Z}. \tag{63}$$

Let $M$ be the total space of the $S^1$-Seifert bundle associated to $\mathbf{K}_X$ or $\mathbf{K}_X^{\frac{1}{2}}$, depending on whether $\mathrm{Ind}(X) = 1$ or 2.

**Theorem 3.16.** *Let $\mathcal{M}$ be a compact toric* ASD *Einstein orbifold with $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$. There exists a Sasakian structure $(\tilde{g}, \tilde{\eta}, \xi, \tilde{\Phi})$ on $M$. So that if $(X, \tilde{h})$ is the Kähler structure making $\pi : M \to X$ a Riemannian submersion, then we have the following diagram, where the horizontal maps are isometric embeddings and $(\tilde{g}, \tilde{\eta}, \xi, \tilde{\Phi})$ is the pull-back of the Sasaki structure*

*$(g, \eta_1, \xi_1, \Phi_1)$ under $\bar{\iota}_t$:*

(64)
$$\begin{array}{ccc} M & \xrightarrow{\bar{\iota}_t} & \mathcal{S} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota_t} & \mathcal{Z} \\ & & \downarrow \\ & & \mathcal{M}. \end{array}$$

*Furthermore, the image is $\bar{\iota}_t(M) = \{\eta^c(\bar{X}_t) = 0\} \subset \mathcal{S}$, where $\bar{X}_t \in \mathfrak{aut}(\mathcal{S}, g)$ denotes the induced vector field on $\mathcal{S}$, for $t \in P \setminus \{t_1, t_2, \ldots, t_{k+2}\}$.*

*If the 3-Sasakian space $\mathcal{S}$ is smooth, then so is $M$. If $M$ is smooth, then*

$$M \underset{diff}{\cong} \#k(S^2 \times S^3), \text{ where } k = 2b_2(\mathcal{S}) + 1.$$

**Proof.** The adjunction formula gives $\mathbf{K}_X \cong \mathbf{K}_{\mathcal{Z}} \otimes [X]|_X = \mathbf{K}_{\mathcal{Z}} \otimes \mathbf{K}_{\mathcal{Z}}^{-\frac{1}{2}}|_X = \mathbf{K}_{\mathcal{Z}}^{\frac{1}{2}}|_X$. Let $h$ be the Kähler–Einstein metric on $\mathcal{Z}$ related to the the 3-Sasakian metric $g$ on $\mathcal{S}$ by Riemannian submersion. So $\mathrm{Ric}_h = 8h$. Recall that $\mathcal{S}$ is the total space of the $S^1$-Seifert bundle associated to $\mathbf{L}^{-1}$, or $\mathbf{L}^{-\frac{1}{2}}$ iff $w_2(\mathcal{M}) = 0$. Also $M$ is the total space of the $S^1$-Seifert bundle associated to either $\mathbf{K}_X$ or $\mathbf{K}_X^{\frac{1}{2}}$. By the above adjunction isomorphism we lift $\iota_t$ to $\bar{\iota}_t$. Then we have

$$g = \eta \otimes \eta + \pi^* h,$$

and $\eta = \frac{d}{8}\theta$ with $\theta$ a connection on $\mathbf{L}^{-1}$ or $\mathbf{L}^{-\frac{1}{2}}$ and where $d = \mathrm{Ind}(\mathcal{Z}) = 2$ or 4 respectively. Then it is not difficult to see that by pulling the connection back by $\iota_t^* \mathbf{L}^{-1} \cong \mathbf{K}_X$ (or $\iota_t^* \mathbf{L}^{-\frac{1}{2}} \cong \mathbf{K}_X^{\frac{1}{2}}$) we can pull $\eta$ back to $\bar{\eta}$ on $M$. And define $\tilde{h} = \iota_t^*(h)$. Then

$$\tilde{g} = \tilde{\eta} \otimes \tilde{\eta} + \pi^* \tilde{h}$$

is a Sasakian metric on $M$.

If $\mathcal{S}$ is smooth, then locally the orbifold groups of $\mathcal{Z}$ act on $\mathbf{L}^{-1}$ (or $\mathbf{L}^{-\frac{1}{2}}$) without nontrivial stabilizers. Thus this holds for the bundle $\mathbf{K}_X$ (or $\mathbf{K}_X^{\frac{1}{2}}$) on $X$.

By a theorem in [30] $\pi_1^{\mathrm{orb}}(X) = e$ follows from $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$. Given a 4-dimensional orbifold $X$ with an effective 2-torus $T^2$ action, let Let $X_0$ be the open dense subset of 2-dimensional orbits. Then $W_0 = X_0/T^2$ is a 2-orbifold. The only other possible orbits are of dimensions 1 and 0, that is, with stabilizers of dimensions 1 and 2, respectively. Then $W = X/T^2$ is a compact connected oriented 2-orbifold with edges and corners with each edge labeled with a $\Lambda_i$, where $\Lambda_i$ is a rank 1 sublattice of $\Lambda$, the integral lattice such that $T^2 = \mathfrak{t}/\Lambda$, such that the two sublattices at a corner are linearly independent. Then we have the exact sequence

(65)
$$\pi_2^{\mathrm{orb}}(W_0) \to \Lambda / \textstyle\sum_i \Lambda_i \to \pi_1^{\mathrm{orb}}(X) \to \pi_1^{\mathrm{orb}}(W_0) \to e.$$

Since for both $\mathcal{M}$ and $X$, we have that $W$ is a polygon with no orbifold singularities, $\pi_1^{\mathrm{orb}}(X) = \pi_1^{\mathrm{orb}}(\mathcal{M}) = \Lambda/\sum_i \Lambda_i$.

Suppose $M$ is smooth. Since $\pi_1^{\mathrm{orb}}(X) = e$, the $\pi_i(M)$ must be finite. This is because on a Sasaki manifold only always has $H^1(M/\mathscr{F}_\xi) = H^1(M,\mathbb{R})$, where $H^1(M/\mathscr{F}_\xi) = H^1(X,\mathbb{R})$ is basic cohomology. Since $\pi_1^{\mathrm{orb}}(X) = e$, the universal cover $\tilde{M} \to M$ must be a root of the $S^1$-Seifert bundle over $X$. By Proposition 3.4 this is a trivial cover, so $\pi_1(M) = e$.

It is a result of H. Oh [39] that a simply connected 5-manifold with an effective $T^3$ action has $H_2(M,\mathbb{Z}) = \mathbb{Z}^{\ell-3}$, where $\ell$ is number of edges of $W$. Since $M$ is spin, the S. Smale classification of 5-manifolds give the diffeomorphism.

Recall the 1-form $\eta^c = \eta_2 - \sqrt{-1}\eta_3$ of Section 1.2 which is $(1,0)$ with respect to the CR structure $\Phi_1$. For $t \in \mathfrak{t}$ let $X_t$ denote the killing vector field on $\mathcal{Z}$ with lift $\bar{X}_t \in \mathfrak{aut}(\mathcal{S},g)$. Then $\theta(X_t) \in H^0(\mathcal{Z},\mathcal{O}(\mathbf{L}))$ which defines a holomorphic function on $\mathbf{L}^{-1}$. The $S^1$ subbundles of $\mathbf{L}^{-1}$ is identified with $\mathcal{S}$. In this way we get $\theta(X_t) = \eta(X_t)$ as holomorphic functions on $C(\mathcal{S})$. Complexifying gives the same equality for $t \in \mathfrak{t}_{\mathbb{C}}$. Thus for $t \in P \setminus \{t_1, t_2, \ldots, t_{k+2}\}$, we have $M_t := \bar{\iota}_t(M) = \{\eta^c(X_t) = 0\} \subset \mathcal{S}$.

Note that here we are setting $2/3$ s of the moment map to zero. □

## 4. Consequences

**4.1. Sasaki–Einstein metrics.** In this section we present the new infinite families of Sasakian–Einstein 5-manifolds.

**Theorem 4.1.** *Let $(\mathcal{S},g)$ be a toric 3-Sasakian 7-manifold with $\pi_1(\mathcal{S}) = e$. Canonically associated to $(\mathcal{S},g)$ are a special symmetric toric Fano surface $X$ and a toric Sasakian–Einstein 5-manifold $M$ which fit in the commutative diagram (64). We have $\pi_1^{\mathrm{orb}}(X) = e$ and $\pi_1(M) = e$. And*

$$M \underset{diff}{\cong} \#k(S^2 \times S^3), \text{ where } k = 2b_2(\mathcal{S}) + 1.$$

*Furthermore $(\mathcal{S},g)$ can be recovered from either $X$ or $M$ with their torus actions.*

**Proof.** The homotopy sequence

$$\cdots \to \pi_1(G) \to \pi_1(\mathcal{S}) \to \pi_1^{\mathrm{orb}}(\mathcal{M}) \to e,$$

where $G = \mathrm{SO}(3)$ or $\mathrm{Sp}(1)$, shows that $\pi_1^{\mathrm{orb}}(\mathcal{M}) = e$. The surface $X$ is uniquely determined by Theorem 3.15. It follows from the proof of Theorem 3.16 that $\pi_1^{\mathrm{orb}}(X) = e$ and we have the above diffeomorphism. An application of Theorem 2.21 and the remarks at the end of Section 1.1 give the Sasaki–Einstein structure on $M$. Given $X$ or $M$ with its Sasakian structure we can recover the orbifold $\mathcal{M}$, which has a unique toric ASD Einstein metric by Theorem 3.1. This uniquely determines the 3-Sasakian manifold by results of Section 1.2. □

**Theorem 4.2.** *For each odd $k \geq 3$ there is a countably infinite number of toric Sasaki–Einstein structures on $\#k(S^2 \times S^3)$.*

**Proof.** Recall from Corollary 1.14 there are infinitely homotopically distinct smooth simply connected 3-Sasakian manifolds $\mathcal{S}$ with $b_2(\mathcal{S}) = k$ for $k > 0$. From Theorem 4.1 associated to each $\mathcal{S}$ is a distinct Sasakian–Einstein manifold diffeomorphic to $\#m(S^2 \times S^3)$, where $m = 2k + 1$. □

The Sasaki–Einstein structures $(g, \eta, \xi, \Phi)$ of Theorem 4.2 have the property of being isomorphic to the conjugate structure $(g, -\eta, -\xi, -\Phi)$. This is because the Kähler–Einstein orbifold $(X, h)$ has an anti-holomorphic involution $\varsigma : X \to X$. Using a real embedding $X \subset \mathcal{Z}$ in Theorem 3.15 one gets a Kähler metric with Kähler form $\omega$ with $\omega \in 2\pi c_1^{\mathrm{orb}}(X)$ and $\varsigma^*\omega = -\omega$. Then in solving (36) one restricts to functions in $C^\infty(X)^G$ which are $\varsigma$-invariant.

The restriction of $k$ to be odd is merely a limitation on the techniques used. Subsequent to these examples appearing in the author's Ph.D. thesis, it was proved [19] that there are toric Sasaki–Einstein structures on $\#k(S^2 \times S^3)$ for all $k$.

If a simply connected 5-manifold has two Sasakian–Einstein structures with, nonproportional Reeb vector fields, for the same metric $g$, then it is $S^5$.

**Corollary 4.3.** *For each odd $k \geq 3$ there is a countably infinite number of cohomogeneity 2 Einstein metrics on $\#k(S^2 \times S^3)$. In particular, the identity component of the isometry group is $T^3$.*

These metrics have the following curious property.

**Proposition 4.4.** *For $M = \#k(S^2 \times S^3)$ with $k > 1$ odd, let $g_i$ be the sequence of Einstein metrics in the theorem normalized so that $\mathrm{Vol}_{g_i}(M) = 1$. Then we have $\mathrm{Ric}_{g_i} = \lambda_i g_i$ with the Einstein constants $\lambda_i \to 0$ as $i \to \infty$.*

**Proof.** We have

$$\mathrm{Vol}(M, g) = d\left(\frac{\pi}{3}\right)^3 \mathrm{Vol}(\Sigma_{-k}),$$

for the volume of a Sasakian–Einstein manifold with toric leaf space $X$ the anti-canonical polytope $\Sigma_{-k}$. This is because an argument in [28] shows that $\mathrm{Vol}(X_{\Sigma_{-k}}) = \mathrm{Vol}(\Sigma_{-k})$. We have $d = 1$ or 2. The above Sasakian–Einstein manifolds have leaf spaces $X_i$, where $X_i = X_{\Delta_i^*}$. Observe that the polygons $\Delta_i^*$ get arbitrarily large, and the anti-canonical polytopes $(\Sigma_{-k})_i$ satisfy

$$\mathrm{Vol}((\Sigma_{-k})_i) \to 0, \text{ as } i \to \infty.$$ □

This implies the following.

**Theorem 4.5.** *The moduli space of Einstein structures, with a $T^3$ isometry group, on each of the manifolds $\#k(S^2 \times S^3)$ for $k \geq 1$ odd has infinitely many connected components.*

The case $k = 1$ is covered by homogeneous examples by M. Wang and W. Ziller [48].

There are a couple of consequences of these examples following from some finiteness results. There is a result of M. Gromov [27] that says that a manifold which admits a metric of nonnegative sectional curvature satisfies a bound on the total Betti number depending only on the dimension. Further, he proved that if the diameter is bounded, then as the total Betti number goes to infinity the infimum of the sectional curvatures goes to $-\infty$. For any $\kappa \leq 0$ and a fixed diameter $D > 0$ there exists $k_0$ so that, for $k > k_0$, $\#k(S^2 \times S^3)$ does not admit a metric with sectional curvature $K \geq \kappa$ and diam $\leq D$. We have the following.

**Theorem 4.6.** *For any $\kappa \leq 0$ and fixed $\lambda > 0$ there are infinitely many simply connected Einstein 5-manifolds with Einstein constant $\lambda$ which do not admit Einstein metrics with Einstein constant $\lambda$ and sectional $K \geq \kappa$.*

One can also consider these examples in relation to a compactness result of M. Anderson [1]. He showed that the space of Riemannian n-manifolds $(M, g)$, $\mathscr{M}(\lambda, c, D)$ with $\operatorname{Ric}_g = \lambda g$, $\operatorname{inj}(g) \geq c > 0$, and diam $\leq D$ is compact in the $C^\infty$ topology. For fixed $k > 1$ odd in Theorem 4.2 the Sasakian–Einstein metrics $g_i$ on $M = \#k(S^2 \times S^3)$ have $\lambda = 4$. We have $\operatorname{Vol}_{g_i}(M) \to 0$ as $i \to \infty$, so no subsequence converges. We have the following.

**Theorem 4.7.** *For the sequence of Einstein manifolds $(M, g_i)$ we have $\operatorname{inj}(g_i) \to 0$ as $i \to \infty$. Also, take any sequence $k_i > 1$ of odd integers and examples from Theorem 4.2 $(\#k_i(S^2 \times S^3), g_i)$, then we have $\operatorname{inj}(g_i) \to 0$ as $i \to \infty$.*

Examples of Einstein 7-manifolds with properties as in Theorem 4.6 and in the second statement of Theorem 4.7 have been given in [14]. These are the toric 3-Sasakian 7-manifolds $\mathcal{S}_\Omega$ considered here.

**4.2. Space of toric ASD structures.** We will compute the dimension of $H^1(\mathcal{Z}, \Theta_{\mathcal{Z}})$ for a twistor space of a toric ASD Einstein orbifold $(\mathcal{M}, g)$. This will give both the dimension of the local deformation space of $\mathcal{Z}$ and the local deformation space of $[g]$ as an ASD conformal class. Recall that the infinitesimal deformations of $[g]$ as an ASD conformal class correspond to $\operatorname{Re} H^1(\mathcal{Z}, \Theta_{\mathcal{Z}})$. Although, $\mathcal{Z}$ is not smooth here, both the conformal class $[g]$ and the twistor space structure on $\mathcal{Z}$ are defined on local uniformizing charts of $\mathcal{M}$ and uniformizing charts these induce on $\mathcal{Z}$. Thus the twistor correspondence of [2] applies here.

In the following $X \subset Z$ will denote a relatively smooth divisor as in Theorem 3.15 with $\mathbf{K}_{\mathcal{Z}}^{-1} = [2X]$. All of the sheaves, divisors and bundles are in the orbifold sense, but the methods used below carry over to this case.

**Proposition 4.8.** $h^1(\mathcal{Z}, \Theta_{\mathcal{Z}}(-X)) = h^{1,1}(\mathcal{Z}) - 1$.

**Proof.** By Serre duality

$$H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}(-X)) = H^1\Big(\mathcal{Z}, \Theta_{\mathcal{Z}} \otimes \mathcal{O}\Big(\mathbf{K}_{\mathcal{Z}}^{\frac{1}{2}}\Big)\Big) = H^2\Big(\mathcal{Z}, \Omega^1 \otimes \mathcal{O}\Big(\mathbf{K}_{\mathcal{Z}}^{\frac{1}{2}}\Big)\Big) = H^2(\mathcal{Z}, \Omega^1(\mathbf{L}^{-1})).$$

From the restriction

$$0 \to \Omega^1_{\mathcal{Z}}(-X) \to \Omega^1_{\mathcal{Z}} \to \Omega^1_{\mathcal{Z}}|_X \to 0,$$

we obtain the exact sequence

$$0 \to H^{1,1}(\mathcal{Z}) \to H^1(X, \Omega^1_{\mathcal{Z}}) \to H^2(\mathcal{Z}, \Omega^1_{\mathcal{Z}}(-X)) \to 0, \tag{66}$$

because $H^1(\mathcal{Z}, \Omega^1(-X)) = 0$ by Kodaira–Nakano vanishing and $H^{1,2}(\mathcal{Z}) = 0$. It is proved in [44, 10] that all the cohomology of $\mathcal{Z}$ vanishes besides $H^{k,k}(\mathcal{Z})$.

Next, consider the conormal sequence

$$0 \to \mathcal{O}_{\mathcal{Z}}(-X)|_X \to \Omega^1_{\mathcal{Z}}|_X \to \Omega^1_X \to 0. \tag{67}$$

From Theorem 3.15 two distinct $X_{s_1},\ X_{s_2} \subset \mathcal{Z}$, $s_1, s_2 \in P \setminus \{t_1, t_2, \ldots, t_n\}$, have $X_{s_1} \cap X_{s_2} = \cup_{j=1}^{2k} C_k$, the anti-canonical divisor of $X_{s_1}$ and $X_{s_2}$. Then

$$\begin{aligned} H^1(X, \mathcal{O}_X(-X)) &= H^1(X, \mathcal{O}(\mathbf{K}_X)) = 0, \\ H^2(X, \mathcal{O}_X(-X)) &= H^2(X, \Omega^2_X) = \mathbb{C}. \end{aligned} \tag{68}$$

And note that

$$H^2(X, \Omega^1_{\mathcal{Z}}|_X) = H^0(X, \Theta_{\mathcal{Z}} \otimes \mathcal{O}_X(\mathbf{K}_X)) = 0. \tag{69}$$

This can be seen as follows. Consider

$$0 \to \Theta_{\mathcal{Z}}(-2X) \to \Theta_{\mathcal{Z}}(-X) \to \Theta_{\mathcal{Z}} \otimes \mathcal{O}_X(\mathbf{K}_X) \to 0. \tag{70}$$

Note that $H^0(\mathcal{Z}, \Theta_{\mathcal{Z}}(-X)) = 0$. If $\beta \in H^0(\mathcal{Z}, \Theta_{\mathcal{Z}}(-X))$, then restricting $\beta$ to the normal bundle of a generic twistor line $P_x$ gives a section of $N \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ vanishing to order 2, which therefore must vanish. Therefore, $\beta$ must be tangent to the twistor lines. But by the definition of the complex structure on $\mathcal{Z}$, that is impossible. And by Serre duality, $H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}(-2X)) = H^2(\mathcal{Z}, \Omega^1_{\mathcal{Z}}) = 0$. Then (69) follows from the cohomology sequence of (70).

The long exact sequence of (67) gives

$$0 \to H^1(X, \Omega^1_{\mathcal{Z}}) \to H^1(X, \Omega^1_X) \to \mathbb{C} \to 0. \tag{71}$$

Since $h^{1,1}(X) = 2h^{1,1}(\mathcal{Z})$ the proposition follows from (66) and (71). □

The arguments of [50] applied here prove the following.

**Lemma 4.9.** *Let $(\mathcal{M}, g)$ be a compact Einstein orbifold. If $g$ admits a conformal-Killing vector field which is not Killing, then $(\mathcal{M}, g)$ is isometric to $S^n/\Gamma$, where $S^n$ has the constant curvature metric and $\Gamma$ is a linear group of isometries fixing $(\pm 1, 0, \ldots, 0) \in S^n$.*

**Lemma 4.10.** *If $X$ is a toric Fano orbifold surface, then $H^1(X, \Theta_X) = 0$.*

**Proof.** Let $C = \sum_i C_i$ be the anti-canonical divisor. For each $\sigma \in \Delta^*$, let $\sigma'$ be the corresponding cone in the sublattice $N'$ as in Proposition 2.2.

Suppose, for the moment, that $X$ be a nonsingular toric variety, in particular $X = U_{\sigma'}$. If $\Omega^1_X$ denotes the algebraic sheaf of differential forms and $\Omega^1_X(\log C)$ the sheaf of differential forms with logarithmic poles along $C = \sum_i C_i$, then

$$0 \to \Omega^1_X \longrightarrow \Omega^1_X(\log C) \longrightarrow \bigoplus_{i=1}^{d} \mathcal{O}_{C_i} \to 0, \tag{72}$$

and

$$\Omega^1_X(\log C) \cong \mathcal{O}_X \oplus \mathcal{O}_X. \tag{73}$$

See [24] for a proof. Consider (72) on each uniformizing neighborhood $U_{\sigma'}$ of $X$. It is easy to see that (72) and the identification (73) are compatible with the identifications of the $U_{\sigma'}$. Extending to the structure sheaf of analytic functions we have

$$0 \to \Omega^1_X \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_X \longrightarrow \bigoplus_{i=1}^{d} \mathcal{O}_{C_i} \to 0, \tag{74}$$

where $\Omega^1_C$ denotes the coherent analytic sheaf associated to the orbifold bundle of 1-forms.

We have $H^1(X, \Theta_X) \cong H^1(X, \Omega^1(\mathbf{K}_X))$. Tensor (74) with $\mathcal{O}(\mathbf{K}_X)$ and take the long exact cohomology sequence

$$\cdots \to \bigoplus_i H^0(C_i, \mathcal{O}_{C_i}(\mathbf{K}_X)) \to H^1(X, \Omega^1(\mathbf{K}_X)) \to H^1(X, \mathcal{O}(\mathbf{K}_X))^{\oplus 2} \to \cdots .$$

Since $\mathbf{K}_X < 0$, Kodaira–Nakano vanishing shows that

$$H^0(C_i, \mathcal{O}_{C_i}(\mathbf{K}_X)) = H^1(X, \mathcal{O}(\mathbf{K}_X)) = 0,$$

thus $H^1(X, \Omega^1(\mathbf{K}_X)) = 0$. □

Let $\mathbf{N}_{X/\mathcal{Z}}$ be the normal orbifold bundle to $X$ in $\mathcal{Z}$ and $\mathcal{N}_{X/\mathcal{Z}}$ its sheaf of sections. Note that $\mathcal{N}_{X/\mathcal{Z}} = \mathcal{O}_X(X) = \mathcal{O}_X(\mathbf{L})$. Let $\Theta_{\mathcal{Z},X}$ be the sheaf of sections of $T\mathcal{Z}$ tangent to $X$. We will make use of the following exact sequences

$$0 \to \Theta_{\mathcal{Z},X} \longrightarrow \Theta_{\mathcal{Z}} \longrightarrow \mathcal{N}_{X/\mathcal{Z}} \to 0 \tag{75}$$

$$0 \to \Theta_{\mathcal{Z}}(-X) \longrightarrow \Theta_{\mathcal{Z},X} \longrightarrow \Theta_X \to 0. \tag{76}$$

Since $H^1(X, \mathcal{O}_X(\mathbf{L})) = H^1(X, \Omega^2_X(\mathbf{K}_X^{-2})) = 0$ by Kodaira–Nakano vanishing, (75) gives

$$h^1(\mathcal{Z}, \Theta_{\mathcal{Z}}) = h^1(\mathcal{Z}, \Theta_{\mathcal{Z},X}) - h^0(X, \mathcal{O}_X(X)) + h^0(\mathcal{Z}, \Theta_{\mathcal{Z}}) - h^0(\mathcal{Z}, \Theta_{\mathcal{Z},X}). \tag{77}$$

Since $H^0(\mathcal{Z}, \Theta_{\mathcal{Z}}(-X)) = 0$ and $H^1(X, \Theta_X) = 0$, from (76) we have

$$0 \to H^0(\mathcal{Z}, \Theta_{\mathcal{Z},X}) \to H^0(X, \Theta_X) \to H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}(-X)) \to H^1(\mathcal{Z}, \Theta_{\mathcal{Z},X}) \to 0. \tag{78}$$

Combining Proposition 4.8, (77) and (78) we get

$$h^1(\mathcal{Z},\Theta_{\mathcal{Z}}) = h^{1,1}(\mathcal{Z}) - 1 + h^0(\mathcal{Z},\Theta_{\mathcal{Z}}) + h^0(X,\Theta_X) - h^0(X,\mathcal{O}_X(X)). \tag{79}$$

If $\mathcal{M}$ is not conformally flat, i.e., $W_g^- \not\equiv 0$, then by Lemma 4.9 every conformal-Killing vector field of $(\mathcal{M}, g)$ is Killing and $H^0(\mathcal{Z},\Theta_{\mathcal{Z}}) \cong H^0(\mathcal{Z},\mathcal{O}(\mathbf{L}))$. Recall that $\operatorname{Re} H^0(\mathcal{Z},\Theta_{\mathcal{Z}})$ is isomorphic to the space of conformal-Killing vector fields of $(\mathcal{M}, g)$, and $\operatorname{Re} H^0(\mathcal{Z},\mathcal{O}(\mathbf{L}))$ the space of Killing vector fields.

One easily checks that $h^0(\mathcal{Z},\mathcal{O}(X)) = h^0(X,\mathcal{O}_X(X)) + 1$, and from (79) we have

$$h^1(\mathcal{Z},\Theta_{\mathcal{Z}}) = h^{1,1}(\mathcal{Z}) - h^0(X,\Theta_X). \tag{80}$$

Note that the compact torus $T^2$ acts on the above cohomology, and denote by $H^1(\mathcal{Z},\Theta_{\mathcal{Z}})^{T^2}$, etc., the fixed set. One can see that, as $X$ is a toric variety, $H^0(X,\Theta_X)^{T^2} = \mathfrak{t}^2_{\mathbb{C}}$ as follows. Suppose $\beta \in H^0(X,\Theta_X)^{T^2}$. Then $\beta$ is also invariant under $\mathbb{C}^* \times \mathbb{C}^*$. Let $\rho(t)$ be the one parameter group of transformations generated by $\beta$. Let $x \in U := X \setminus \cup_{i=1}^{2k} C_i$, the open orbit isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Fix $t_0 \in \mathbb{C}$ close to zero, and let $g \in \mathbb{C}^* \times \mathbb{C}^*$ be the element such that $g\rho(t_0)x = x$. Since $g\rho(t_0)$ commutes with $\mathbb{C}^* \times \mathbb{C}^*$, it fixes all of $U$ and thus $X$. Thus $\rho(t) \in \mathbb{C}^* \times \mathbb{C}^*$.

Since $T^2$ acts on all the sheafs and preserves the exact sequences above,

$$h^1(\mathcal{Z},\Theta_{\mathcal{Z}})^{T^2} = h^{1,1}(\mathcal{Z}) - h^0(X,\Theta_X)^{T^2} = h^{1,1}(\mathcal{Z}) - 2. \tag{81}$$

Therefore from (80) and (81) we must have

$$h^1(\mathcal{Z},\Theta_{\mathcal{Z}}) = h^1(\mathcal{Z},\Theta_{\mathcal{Z}})^{T^2} = h^{1,1}(\mathcal{Z}) - 2 = b_2(\mathcal{M}) - 1. \tag{82}$$

Note that this also proves that $h^0(X,\Theta_X) = 2$ when $W_g^- \not\equiv 0$, which is always the case when $b_2(\mathcal{M}) \geq 1$.

We have

$$H^3(\mathcal{Z},\Theta_{\mathcal{Z}}) = H^0(\mathcal{Z},\Omega^1(\mathbf{K}_{\mathcal{Z}})) = 0, \tag{83}$$

where the second equality holds for any twistor space. Also,

$$H^2(\mathcal{Z},\Theta_{\mathcal{Z}}) = H^1(\mathcal{Z},\Omega^1(\mathbf{K}_{\mathcal{Z}})) = 0, \tag{84}$$

by Kodaira–Nakano vanishing.

We have proved the following.

**Proposition 4.11.** *Let $\mathcal{Z}$ be the twistor space of a compact toric* ASD *Einstein orbifold* $(\mathcal{M}, g)$. *Then* $h^2(\mathcal{Z},\Theta_{\mathcal{Z}}) = h^3(\mathcal{Z},\Theta_{\mathcal{Z}}) = 0$. *If* $b_2(\mathcal{M}) \geq 1$, *then*

$$h^1(\mathcal{Z},\Theta_{\mathcal{Z}}) = h^1(\mathcal{Z},\Theta_{\mathcal{Z}})^{T^2} = b_2(\mathcal{M}) - 1.$$

*If* $b_2(\mathcal{M}) = 0$, *then* $\mathcal{M}$ *has an orbifold covering by* $S^4$, *which has the round metric.*

We show that the local space of ASD metrics coincides with the $T^2$-invariant ASD conformal metrics given by D. Joyce [33]. An explicit description of the toric ASD Einstein metrics $(\mathcal{M}, g)$ was given by D. Calderbank and M. Singer [18], which made use of the description in [16] of toric ASD Einstein metrics in terms of an eigen function potential on the hyperbolic plane. The conformal classes of these metrics are always given by the Joyce equation.

**Theorem 4.12.** *Let* $(\mathcal{M}, g)$ *be a compact toric* ASD *Einstein orbifold, then locally the space of* ASD *conformal classes near* $[g]$ *are those given by the Joyce ansatz. It is therefore a space of dimension* $b_2(\mathcal{M}) - 1$.

**Proof.** We first prove a lemma.

**Lemma 4.13.** *Suppose* $(\mathcal{M}, g)$ *is a compact toric* ASD *Einstein orbifold and* $W_g^- \not\equiv 0$. *If* $(\mathcal{M}, g)$ *is not homogeneous, then the connected component of the identity of the isometry group* $\mathrm{Isom}(g)_0$ *is either* $T^2$ *or* $U(2)$ *up to finite cover. In the second case* $b_2(\mathcal{M}) = 1$. *If* $(\mathcal{M}, g)$ *is homogeneous, then* $(\mathcal{M}, g)$ *is isometric to* $\overline{\mathbb{CP}}^2$ *with the Fubini–Study metric.*

**Proof.** We may assume that $b_2 > 0$, for otherwise the structure of toric 4-orbifolds [42, 30]) implies that $\mathcal{M}$ is diffeomorphic to $S^4/\Gamma$ where $\Gamma$ is a finite group acting as in Proposition 4.9.

Let $G = \mathrm{Isom}(g)_0$ be the connected component of the identity. Let $\alpha \in \mathcal{H}^2_{g\,-} \cong H^2(M, \mathbb{R})$ be nonzero. Note that standard Bochner techniques show that $\mathcal{H}^2_{g\,+} = 0$. Let $x \in \mathcal{M}$ be a point in the open dense subset of principle orbits where $\alpha_x \neq 0$. Since $G$ fixes $\alpha$, we have $H_x \subseteq U(2)$, where $H_x$ is the isotropy subgroup at $x$. We have $\dim G \leq 3 + \dim H_x \leq 7$. Note that $G$ cannot contain a 3-torus, because that 3-torus would give a cohomogeneity one action. And by the theory of such actions $\mathcal{M}$ would not be simply connected. Considering possible compact groups of rank 2 of these dimensions we see that $G$ is $T^2, \mathrm{U}(2)$ or $\mathrm{SO}(4)$ up to finite coverings. If $G = \mathrm{SO}(4)$ up to coverings, then $\dim H_x \geq 3$ and is disjoint from $T^2$ which is impossible. Suppose $G$ is $\mathrm{U}(2)$. Then the generic orbit $Gx$ is 3 dimensional and $H_x = S^1$. For if $\dim Gx = 2$, then $Gx \cong T^2$ and $H_x \cong T^2$ which is impossible, since $G/T^2 \cong S^2$ for any $T^2 \subset G$.

So $\mathcal{M}$ is of cohomogeneity 1. The orbifold isotropy group of a point $\Gamma_x$ is preserved by $G$. Standard arguments show that the set of smooth points of $\mathcal{M}$ is diffeomorphic to $(0,1) \times G/H$, where $H_0 = S^1$. Since otherwise the orbit space $\mathcal{M}/G \cong S^1$, would contradict $\pi_1(\mathcal{M}) = e$. Adding the two, possibly singular orbits at 0 and 1 gives a close dense subset of $\mathcal{M}$, so it must be all of $\mathcal{M}$. Thus the orbit space is $\pi : \mathcal{M} \to \mathcal{M}/G = [0,1]$. The each of the orbits $\pi^{-1}(0)$ and $\pi^{-1}(1)$ is either a point or $\mathbb{CP}^1$. Since $b_2 > 0$, at least one is $\mathbb{CP}^1$. It is easy to see that both orbits cannot be $\mathbb{CP}^1$, because in this case the isotropy subgroups of $T^2$ on these two $\mathbb{CP}^1$ must be equal, which is impossible by the results in[18]. Therefore $b_2(\mathcal{M}) = 1$.

If $\mathcal{M}$ is homogeneous then it must be smooth. Since $b_2 > 0$, the last statement of the proposition follows from [32]. □

We review the D. Joyce construction [33] of ASD conformal metrics with a surface orthogonal action of $T^2$ by conformal transformations. By *surface orthogonal* we mean that the orthogonal distribution to the $T^2$ orbits is integrable. Locally all toric surface orthogonal ASD metrics are of this form. The metrics are defined by linearly independent solutions to a linear equation on the spinor bundle $\mathcal{W} \to \mathcal{H}^2$ over the hyperbolic plane. See also [16, 17] for more details on the following.

Let $\mathbb{V}$ be a real 2-dimensional vector space with a symplectic form $\varepsilon(\cdot,\cdot)$. We consider bundle isomorphisms $\Phi : \mathcal{W} \to \mathcal{H} \times \mathbb{V}$, and define a $\mathbb{V}$-invariant metric on $\mathcal{H} \times \mathbb{V}$ in terms of $(\mathcal{H}, h)$ and $\Phi$. We define a family of metrics on $\mathbb{V}$ by

$$(u, v)_\Phi = h(\Phi^{-1}(u), \Phi^{-1}(v)),$$

and on $\mathcal{H} \times \mathbb{V}$ by

$$g_\Phi = \Omega^2(h + (\cdot,\cdot)_\Phi). \tag{85}$$

Fix the spinor bundle $\mathcal{W}$ by $\mathcal{W} \otimes_{\mathbb{C}} \mathcal{W} = TN$. We consider the half-space model of $\mathcal{H}$ with coordinates $(\eta, \rho)$, $\rho > 0$, with metric $h = (d\rho^2 + d\eta^2)/\rho^2$. Given a smooth section $\Phi \in C^\infty(\mathcal{W}^*)$ we have the *Joyce equation*

$$\bar{\partial}\Phi = \frac{1}{2}\bar{\Phi}. \tag{86}$$

We clarify the identifications made in (86). On the left-hand side

$$\bar{\partial}\Phi \in \Gamma(\overline{T^*\mathcal{H}} \otimes_{\mathbb{C}} \mathcal{W}^*),$$

while using the induce Hermitian metrics on the spinor bundles

$$\overline{T^*\mathcal{H}} \otimes_{\mathbb{C}} \mathcal{W}^* = \bar{\mathcal{W}}^* \otimes_{\mathbb{C}} \bar{\mathcal{W}}^* \otimes_{\mathbb{C}} \mathcal{W}^* = \bar{\mathcal{W}}^*.$$

Then $g_\Phi$ is an ASD metric if $\Phi \in C^\infty(\mathcal{H}, \mathcal{W}^* \otimes \mathbb{V})$ is a linearly independent solution to (86).

Considering $\mathcal{W}^*$ as a real bundle we can identify $S^2_0(\mathcal{W}^*)$ with $T^*\mathcal{H}$, in which there is an orthonormal frame $\lambda_0, \lambda_1$ of $\mathcal{W}^*$ and identifications $\lambda_0^2 - \lambda_1^2 = d\rho/\rho$ and $2\lambda_0\lambda_1 = d\eta/\rho$. Then a solution $\Phi \in C^\infty(\mathcal{H}, \mathcal{W}^* \otimes \mathbb{V})$ can be written

$$\Phi = \lambda_0 \otimes v_0 + \lambda_1 \otimes v_1,$$

with $v_0, v_1 \in C^\infty(\mathbb{V})$ satisfying the equations

$$\rho\partial_\rho v_0 + \rho\partial_\eta v_1 = v_0, \quad \rho\partial_\eta v_0 - \rho\partial_\rho v_1 = 0. \tag{87}$$

Then if $\mu_0, \mu_1$ is a dual frame to $\lambda_0, \lambda_1$,

$$\Phi^{-1} = \frac{\varepsilon(v_0,\cdot) \otimes \mu_1 - \varepsilon(v_1,\cdot) \oplus \mu_0}{\varepsilon(v_0, v_1)}.$$

D. Joyce made the observation that $-\lambda_1$ is obviously a solution to (87) and acting on it by $\mathrm{SL}(2,\mathbb{R})$ gives the family of *fundamental solutions* to (86)

$$\phi(\rho, \eta, x) = \frac{\rho\lambda_0 + (\eta - x)\lambda_1}{\sqrt{\rho^2 + (\eta - x)^2}}, \tag{88}$$

where $x \in \partial\mathcal{H}$.

Conditions were given in [16, 18] for the ASD structure $g_\Phi$ to be conformally Einstein. The condition is that the solution $\Phi$ to (86) comes from an eigenfunction of $\Delta_h$, the negative Laplacian on $(\mathcal{H}, h)$,

$$\Delta_h F = \frac{3}{4}F, \quad F \in C^\infty(\mathcal{H}).$$

We define $f(\rho,\eta) = \sqrt{\rho}F(\rho,\eta)$, then with

$$v_0 = (f_\rho, \eta f_\rho - \rho f_\eta), \quad v_1 = (f_\eta, \rho f_\rho + \eta f_\eta - f),$$

$\Phi = \lambda_0 \otimes v_0 + \lambda_1 \otimes v_1$ is a solution to (86) and

$$g_F = \frac{|F^2 - 4|dF|^2|}{4F^2}\Big(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\varepsilon(v_0,\cdot)^2 + \varepsilon(v_1,\cdot)^2}{\varepsilon(v_0,v_1)^2}\Big) \tag{89}$$

is an ASD Einstein equation with positive scalar curvature where $F^2 > 4|dF|^2$ and negative scalar curvature where $4|dF|^2 > F^2 > 0$.

By Theorem 3.1 the isotropy data $(m_1,n_1),(m_2,n_2),\dots,(m_{k+2},n_{k+2})$, with $(m_0,n_0) = (m_{k+2},n_{k+2})$, of the ASD Einstein space $\mathcal{M}$ can be arranged as follows, after possibly changing signs and acting by $\mathrm{SL}(2,\mathbb{Z})$. If we define $(a_i,b_i),\ i = 0,\dots,k+2$, by

$$2(a_i,b_i) = (m_i,n_i) - (m_{i-1},n_{i-1}),$$

then $a_i > 0$ and $x_i := b_i/a_i$ are increasing for $i = 0,\dots,k+2$, where we set $x_0 = -\infty$. It was proved in [18] that the potential

$$F = \sum_{i=1}^{k+2} \frac{\sqrt{a_i\rho^2 + (a_i\eta - b_i)^2}}{\sqrt{\rho}},$$

gives uniquely the ASD Einstein metric on $\mathcal{M}$ by (89). Then an easy computation gives

$$\Phi = \frac{1}{2}\sum_{i=1}^{k+2}\big(\phi(\rho,\eta,x_i) - \phi(\rho,\eta,x_{i-1})\big)\otimes(m_i,n_i). \tag{90}$$

Here the $x_1 < x_2 < \cdots < x_{k+2}$ are points on $\partial\mathcal{H}$ corresponding to the points on the boundary of $\mathcal{Q}$ fixed by $T^2$. Conversely, given a sequence of points $x_1 < x_2 < \cdots < x_{k+2}$ on $\partial\mathcal{H}$ the arguments in [33] show that (90) gives a solutions which compactifies on the toric orbifold $\mathcal{M}$ with the given isotropy data to a complete metric. In particular the same arguments there show that $\bigwedge^2\Phi$ is nonvanishing on $\mathcal{H}$. Clearly, $\mathrm{PSL}(2,\mathbb{R})$ acts on (90) giving isomorphic solutions and acts on the formula (90) by shifting the $x_1 < x_2 < \cdots < x_{k+2}$. Thus we have a $k-1$-parameter space of ASD structures. See Figure 4.

Suppose two metrics in this family are conformally isomorphic. So we have $\Phi_1$ defined by $x_1 < x_2 < \cdots < x_{k+2}$ and $\Phi_2$ defined by $z_1 < z_2 < \cdots < z_{k+2}$, metrics $g_1$ and $g_2$ defined in (85), and a diffeomorphism $\psi : \mathcal{M} \to \mathcal{M}$ so that $\psi^* g_2 = e^2 f g_1$. By changing the conformal factor $e^2 f$ we may assume that $\psi$

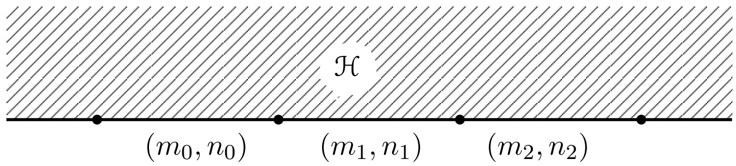


FIGURE 4. orbit space of $\mathcal{M}$

is an isomorphism. The arguments in Lemma 4.13 show that $\mathrm{Isom}(g_1)_0 = \mathrm{Isom}(g_2) = T^2$ unless $b_2(\mathcal{M}) = 1$, so we may assume that $\mathrm{Isom}(g_1)_0 = \mathrm{Isom}(g_2) = T^2$. So $\psi$ must map the Lie algebra $\mathfrak{t}$ of $T^2$ to itself, and $\psi$ is equivariant up to an automorphism $\psi_T \in \mathrm{GL}(2, \mathbb{Z})$. Since $\psi$ preserves the vector fields generated by $T^2$ it must preserve the orthogonal distributions to the torus orbits. Therefore $\phi$ descends to a conformal automorphism $\phi_{\mathcal{H}}$ of $\mathcal{H}$. Thus $\phi_{\mathcal{H}} \in \mathrm{PSL}^*(2, \mathbb{R})$, the isometry group of $(\mathcal{H}, h)$ generated by $\mathrm{PSL}(2, \mathbb{R})$ and the orientation reversing $\eta + \sqrt{-1}\rho \mapsto -\eta + \sqrt{-1}\rho$. Also, $\psi$ must map the fixed points $p_1, p_2, \ldots p_{k+2}$ of $T^2$ to themselves. If $\psi$ permutes the fixed points, then it must permute the edges of $\mathcal{Q}$ in such a way that the edge with stabilizer data $(m_i, n_i)$ goes to the edge with stabilizer data $\pm(\psi_T)_*(m_i, n_i)$. Thus $\psi_T$ is an automorphism of data of the toric orbifold. This is obviously a finite group.

Suppose that $\phi$ fixes the points $p_1, p_2, \ldots p_{k+2}$. Then $\phi_{\mathcal{H}} \in \mathrm{PSL}(2, \mathbb{R})$ and $(\psi_T)_*(m_i, n_i) = \pm(m_i, n_i)$. It is easy to see that $\psi_T = \pm\mathbf{1}$. And $\phi_{\mathcal{H}}$ maps $x_1 < x_2 < \cdots < x_{k+2}$ to $z_1 < z_2 < \cdots < z_{k+2}$. Therefore the map

$$\Big\{\{x_1 < x_2 < \ldots < x_{k+2}\} \subset \mathbb{RP}^1\Big\}/\,\mathrm{PSL}(2, \mathbb{R}) \to \Big\{\text{Conf. ASD str. on } \mathcal{M}\Big\}$$

is finite to one.

Let $g_0$ denote the ASD Einstein metric. Recall that ASD conformal classes $[g]$ on $\mathcal{M}$, with its fixed orbifold structure, are in correspondence with its twistor space $\mathcal{Z}$, with properties (a), (b) and (c) at the beginning of Section 3.2. Let $\mathcal{Z}_{[g]}$ denotes the twistor space of $(\mathcal{M}, [g])$ for $[g]$, ASD. Then by the orbifold Riemann-Roch theorem of [34]

$$\chi(\Theta_{\mathcal{Z}_{[g]}}) = h^0(\Theta_{\mathcal{Z}_{[g]}}) - h^1(\Theta_{\mathcal{Z}_{[g]}}) + h^2(\Theta_{\mathcal{Z}_{[g]}}), \tag{91}$$

is independent of $[g]$. By Lemma 4.13 we may assume that $\mathrm{Isom}(g_0)_0 = T^2$. Let $\mathcal{U}$ be a neighborhood of $[g_0]$ in the $C^\infty$ topology with $h^2(\Theta_{\mathcal{Z}_{[g]}}) = 0$. Let $\mathcal{J}$ be the space of toric ASD structures constructed above. Then for $[g] \in \mathcal{J} \cap \mathcal{U}$ we have $h^0(\Theta_{\mathcal{Z}_{[g]}}) = 2$ by the above assumption, and by (91) $h^1(\Theta_{\mathcal{Z}_{[g]}}) = b_2(\mathcal{M}) - 1$. Therefore $H^1(\Theta_{\mathcal{Z}_{[g]}}) = H^1(\Theta_{\mathcal{Z}_{[g]}})^{T^2}$ and the twistor spaces $\mathcal{Z}_{[g]}$ for $[g] \in \mathcal{J} \cap \mathcal{U}$ provide a real subspace of the deformation space of $\mathcal{Z}_{[g]}$ for $[g] \in \mathcal{J} \cap \mathcal{U}$. The total local deformation space of $\mathcal{Z}_{[g]}$ is a complex thickening of the real deformations. $\square$

**Remark 4.14.** This is in contrast to the case of ASD structures on $\#\ell\overline{\mathbb{CP}^2}$. There are many examples of toric ASD structures on $\#\ell\overline{\mathbb{CP}^2}$, $\ell \geq 3$, for which most deformations are not toric. It is a result of A. Fujiki [23] that the toric ASD conformal metrics on $\#\ell\overline{\mathbb{CP}^2}$ are the Joyce metrics, so each is in a $\ell - 1$ dimensional family. But $\chi(\Theta_{Z_{[g]}}) = \frac{1}{2}(15\chi + 29\tau) = 15 - 7\ell$ by a calculation originally due to N. Hitchin and I. Singer. See A. King and D. Kotschick [35] for more details on the moduli of ASD conformal metrics.

## 5. Examples

We consider some of the examples obtained starting with the simplest. In particular we can determine some of the spaces in diagram (1) associated to a smooth toric 3-Sasakian 7-manifold more explicitly in some cases.

**5.1. Smooth examples.** It is well-known that there exists only two complete examples of positive scalar curvature anti-self-dual Einstein manifolds [32, 22], $S^4$ and $\overline{\mathbb{CP}}^2$ with the round and Fubini–Study metrics respectively, where $\overline{\mathbb{CP}}^2$ denotes $\mathbb{CP}^2$ with the opposite of the usual orientation.

$\boldsymbol{\mathcal{M} = S^4}$**.** Considering the spaces in diagram (1) we have: $\mathcal{M} = S^4$ with the round metric; its twistor space $\mathcal{Z} = \mathbb{CP}^3$ with the Fubini–Study metric; the quadratic divisor $X \subset \mathcal{Z}$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$ with the homogeneous Kähler–Einstein metric; $M = S^2 \times S^3$ with the homogeneous Sasakian–Einstein structure; and $\mathcal{S} = S^7$ has the round metric. In this case diagram (1) becomes the following:

$$
\begin{array}{ccc}
S^2 \times S^3 & \longrightarrow & S^7 \\
\downarrow & & \downarrow \\
\mathbb{CP}^1 \times \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^3 \\
 & & \downarrow \\
 & & S^4.
\end{array}
\tag{92}
$$

This is the only example, I am aware of, for which the horizontal maps are isometric immersions when the toric surface and Sasakian space are equipped with the Einstein metrics.

$\boldsymbol{\mathcal{M} = \mathbb{CP}^2}$**.** In this case $\mathcal{M} = \mathbb{CP}^2$ with the Fubini–Study metric; its twistor space is $\mathcal{Z} = F_{1,2}$, the manifold of flags $V \subset W \subset \mathbb{C}^3$ with $\dim V = 1$ and $\dim W = 2$, with the homogeneous Kähler–Einstein metric. The projection $\pi : F_{1,2} \to \mathbb{CP}^2$ is as follows. If $(p, l) \in F_{1,2}$ so $l$ is a line in $\mathbb{CP}^2$ and $p \in l$, then $\pi(p, l) = p^\perp \cap l$, where $p^\perp$ is the orthogonal compliment with respect to the standard hermitian inner product. We can define $F_{1,2} \subset \mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ by

$$F_{1,2} = \left\{ ([p_0 : p_1 : p_2], [q^0 : q^1 : q^2]) \in \mathbb{CP}^2 \times (\mathbb{CP}^2)^* : \sum p_i q^i = 0 \right\}.$$

And the complex contact structure is given by $\theta = q^i dp_i - p_i dq^i$. Fix the action of $T^2$ on $\mathbb{CP}^2$ by

$$(e^{i\theta}, e^{i\phi})[z_0 : z_1 : z_2] = [z_0 : e^{i\theta} z_1 : e^{i\phi} z_2].$$

Then this induces the action on $F_{1,2}$

$$\begin{aligned}(e^{i\theta}, e^{i\phi})&([p_0 : p_1 : p_2], [q^0 : q^1 : q^2]) \\ &= ([p_0 : e^{i\theta} p_1 : e^{i\phi} p_2], [q^0 : e^{-i\theta} q^1 : e^{-i\phi} q^2]).\end{aligned}$$

Given $[a, b] \in \mathbb{CP}^1$ the one parameter group $(e^{ia\tau}, e^{ib\tau})$ induces the holomorphic vector field $W_\tau \in \Gamma(T^{1,0} F_{1,2})$ and the quadratic divisor $X_\tau = (\theta(W_\tau))$ given by

$$X_\tau = (ap_1 q^1 + bp_2 q^2 = 0, \quad p_i q^i = 0).$$

One can check directly that $X_\tau$ is smooth for $\tau \in \mathbb{CP}^1 \setminus \{[1, 0], [0, 1], [1, 1]\}$ and $X_\tau = \mathbb{CP}^2_{(3)}$, the equivariant blow-up of $\mathbb{CP}^2$ at 3 points. For $\tau \in \{[1, 0], [0, 1], [1, 1]\}$, $X_\tau = D_\tau + \bar{D}_\tau$ where both $D_\tau, \bar{D}_\tau$ are isomorphic to the Hirzebruch surface $F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1))$.

The Sasakian–Einstein space is $M = \#3(S^2 \times S^3)$. And we have $\mathcal{S} = \mathcal{S}(1, 1, 1) = SU(3)/U(1)$ with the homogeneous 3-Sasakian structure. This case has the following diagram:

$$\begin{array}{ccc} \#3(S^2 \times S^3) & \longrightarrow & SU(3)/U(1) \\ \downarrow & & \downarrow \\ \mathbb{CP}^2_{(3)} & \longrightarrow & F_{1,2} \\ & & \downarrow \\ & & \mathbb{CP}^2. \end{array} \tag{93}$$

**5.2. Galicki–Lawson quotients.** The simplest examples of quaternionic-Kähler quotients are the Galicki–Lawson examples first appearing in [26] and further considered in [14]. These are circle quotients of $\mathbb{HP}^2$. In this case the weight matrices are of the form $\Omega = \mathbf{p} = (p_1, p_2, p_3)$ with the admissible set

$$\mathcal{A}_{1,3}(Z) = \{\mathbf{p} \in \mathbb{Z}^3 : p_i \neq 0 \text{ for } i = 1, 2, 3 \text{ and } \gcd(p_i, p_j) = 1 \text{ for } i \neq j\}.$$

We may take $p_i > 0$ for $i = 1, 2, 3$. The zero locus of the 3-Sasakian moment map $N(\mathbf{p}) \subset S^{11}$ is diffeomorphic to the Stiefel manifold $V^{\mathbb{C}}_{2,3}$ of complex 2-frames in $\mathbb{C}^3$ which can be identified as $V^{\mathbb{C}}_{2,3} \cong U(3)/U(1) \cong SU(3)$. Let $f_{\mathbf{p}} : U(1) \to U(3)$ be

$$f_{\mathbf{p}}(\tau) = \begin{bmatrix} \tau^{p_1} & 0 & 0 \\ 0 & \tau^{p_2} & 0 \\ 0 & 0 & \tau^{p_3} \end{bmatrix}.$$

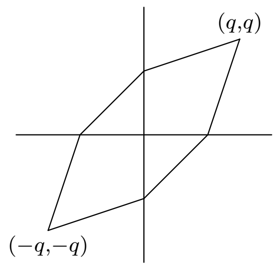


FIGURE 5. infinite Fano orbifold structures on $\mathbb{CP}^2_{(3)}$

Then the 3-Sasakian space $\mathcal{S}(\mathbf{p})$ is diffeomorphic to the quotient of $SU(3)$ by the action of $U(1)$

$$\tau \cdot W = f_{\mathbf{p}}(\tau) W f_{(0,0,-p_1-p_2-p_3)}(\tau) \quad \text{where } \tau \in U(1) \text{ and } W \in SU(3).$$

Thus $\mathcal{S}(\mathbf{p}) \cong SU(3)/U(1)$ is a biquotient similar to the examples considered by Eschenburg in [21].

The action of the group $SU(2)$ generated by $\{\xi^1, \xi^2, \xi^3\}$ on $N(\mathbf{p}) \cong SU(3)$ commutes with the action of $U(1)$. We have $N(\mathbf{p})/SU(2) \cong SU(3)/SU(2) \cong S^5$ with $U(1)$ acting by

$$\tau \cdot v = f_{(-p_2-p_3,-p_1-p_3,-p_1-p_2)} v \quad \text{for } v \in S^5 \subset \mathbb{C}^3.$$

We see that $\mathcal{M}_\Omega \cong \mathbb{CP}^2_{a_1,a_2,a_3}$ where $a_1 = p_2 + p_3, a_2 = p_1 + p_3, a_3 = p_1 + p_2$ and the quotient metric is anti-self-dual with the reverse of usual orientation. If $p_1, p_2, p_3$ are all odd then the generic leaf of the 3-Sasakian foliation $\mathcal{F}_3$ is SO(3). If exactly one is even, then the generic leaf is Sp(1). Denote by $X_{p_1,p_2,p_3}$ the toric Fano divisor, which can be considered as a generalization of $\mathbb{CP}^2_{(3)}$. We have the following spaces and embeddings:

$$\begin{array}{ccc} \#3(S^2 \times S^3) & \longrightarrow & \mathcal{S}(p_1,p_2,p_3) \\ \downarrow & & \downarrow \\ X_{p_1,p_2,p_3} & \longrightarrow & \mathcal{Z}(p_1,p_2,p_3) \\ & & \downarrow \\ & & \mathbb{CP}^2_{a_1,a_2,a_3}. \end{array} \tag{94}$$

A simple series of examples can be obtained by taking $\mathbf{p} = (2q-1,1,1)$ for any $q \geq 1$. Then the anti-self-dual Einstein space is $\mathcal{M} = \mathbb{CP}^2_{1,q,q}$ which is homeomorphic to $\mathbb{CP}^2$, but its metric is ramified along a $\mathbb{CP}^1$ to order $q$. For the toric divisor $X \subset \mathcal{Z}$ we have $X = \mathbb{CP}^2_{(3)}$ with the metric ramified along two $\mathbb{CP}^1$'s to order $q$. We get a sequence of distinct Sasakian–Einstein structures on $M \cong \#3(S^2 \times S^3)$.

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