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Jordan blocks and strong irreducibility

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Abstract. An operator is said to be strongly irreducible if its commutant has no nontrivial idempotent. This paper first shows that if an operator is not strongly irreducible then the set of idempotents in its commutant is either finite or uncountable. The second part of the paper focuses on the Jordan block which is a well-known class of irreducible operators, and determines when a Jordan block is strongly irreducible. This work is an interplay of operator theory and complex function theory.

Contents

1. Introduction

Let $\mathcal{H}$ denote a complex, separable, infinite dimensional Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For an operator T, $\operatorname{ran} T$ denotes its range and $\ker T$ denotes its kernel. T is said to be strongly irreducible (simply denoted by (SI)), if $\mathcal{A}'(T)$, the commutant of T, has no nontrivial idempotent. Clearly, every strongly irreducible operator is irreducible. In the case T is not strongly irreducible, a set of idempotent elements $\mathcal{P} = \{P_i\}_{i=1}^n,\ n < \infty$, is called a unit finite decomposition of T if the following conditions are satisfied:

(1) $P_i \in \mathcal{A}'(T)$.
(2) $P_iP_j = 0$ for $i \neq j$.
(3) $\sum_{i=1}^n P_i = I$.

If, in addition, for each $i = 1, 2, \ldots, n,\ T|_{\operatorname{ran} P_i}$ is stronly irreducible, then we call $\mathcal{P}$ a unit finite (SI) decomposition of T. Suppose T has finite (SI) decomposition. Further, if for any two unit finite (SI) decompositions of T,

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say $\mathcal{P} = \{P_i\}_{i=1}^n$ and $\mathcal{O} = \{O_i\}_{i=1}^m$, we have $m = n$ and $\mathcal{O}$ is a permutation of $\mathcal{P}$, then we say that T has unique (SI) decomposition.

A useful tool for the study of strong irreducibility is the Rosenblum operator $\tau_{T_iT_j}$ defined by

$$\tau_{T_iT_j}(X) = T_iX - XT_j, \quad T_i, T_j, X \in \mathcal{L}(\mathcal{H}).$$

For reference on this subject, we refer the readers to [3] and [4].

On the Hardy space over the unit disk $H^2(D)$, multiplication by coordinate function z is the unilateral shift, and its invariant subspace is of the form $\theta H^2(D)$, where θ is an inner function ([2]). The compression $S(\theta)$ of the unilateral shift to the quotient space $N := H^2(D) \ominus \theta H^2(D)$ is called a Jordan block. To be precise,

$$S(\theta)f = P_N zf, \quad f \in N,$$

where P_N is the projection from $H^2(D)$ onto N. Study of the unilateral shift and the Jordan block is a solid foundation for the development of nonselfadjoint operator theory ([1], [6]). A well-known fact is that every Jordan operator $S(\theta)$ is irreducible, in other words, the commutant $\mathcal{A}'(S(\theta))$ has no nontrivial projections.

In Section 2, we study the cardinality of the set of idempotents for non-strongly irreducible operators, and in Section 3, we study how the strong irreducibility of a Jordan block $S(\theta)$ is dependent on θ.

2. The cardinality of idempotents in $\mathcal{A}'(T)$

If Q is an idempotent, then its range $\operatorname{ran} Q$ is closed. In fact, it is not hard to check that $\operatorname{ran} Q = \ker(I - Q)$. For an idempotent $Q \in \mathcal{A}'(A)$, $\operatorname{ran} Q$ will be called a Banach reducing subspace for T. The following is the main theorem of this section.

Theorem 1. *The number of Banach reducing subspaces of any operator in $\mathcal{L}(\mathcal{H})$ is either finite or uncountably infinite. The former case occurs if and only if the operator is similar to the direct sum of finitely many strongly irreducible operators*

$$\sum_{i=1}^n \oplus T_i$$

with $\ker \tau_{T_iT_j} = \{0\}$ for any $i \neq j$. In this case, the number of Banach reducing subspaces is 2^n.

We will need the following lemmas to prove the theorem.

Lemma 2. *Assume that an operator T in $\mathcal{L}(\mathcal{H})$ is similar to the direct sum of finitely many strongly irreducible operators*

$$\sum_{i=1}^n \oplus T_i.$$

Then the following assertions are equivalent:

(a) *The (SI) decomposition of* T *is unique.*

(b) $\ker \tau_{T_iT_j} = \{0\}$ *for any* $i \neq j,\ i,j = 1,2,\dots,n$.

(c) *The number of Banach reducing subspace is* 2^n.

Proof. (b)⇒(a). Without loss of generality, we assume that

$$T = \sum_{i=1}^{n} \oplus T_i, \quad T_i \in \mathcal{L}(\mathcal{H}).$$

To verify uniqueness, we only need to show that every idempotent P in $\mathcal{A}'(T)$ has form:

$$P = \sum_{i=1}^{n} \oplus \delta_i I_i,$$

where $\delta_i = 0$ or 1, and I_i is the identity operator on $\mathcal{H}_i$.

Since $\ker \tau_{T_iT_j} = \{0\}$, every idempotent P in $\mathcal{A}'(T)$ can be written as

$$P = \sum_{i=1}^{n} \oplus P_i,$$

where $P_i \in \mathcal{A}'(T_i)$. To see this point, we write

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n-1} & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n-1} & P_{2n} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ P_{n-11} & P_{n-12} & \ddots & P_{n-1n-1} & P_{n-1n} \\ P_{n1} & P_{n2} & \cdots & P_{nn-1} & P_{nn} \end{bmatrix}.$$

Since $P \in \mathcal{A}'(T)$, $PT = TP$, and hence $P_{ij}T_i = T_iP_{ij}$. Since $\ker \tau_{T_iT_j} = 0$ for $i \neq j$, $P_{ij} = 0$ for $i \neq j$.

Now, by $P^2 = P$, we must have $P_i^2 = P_i$, $i = 1,2,\dots,n$. Since T_i is strongly irreducible, $P_i = I_i$ or $P_i = 0$.

(a)⇒(b). Let

$$T = \sum_{i=1}^{n} \oplus T_i \quad \text{on} \quad \mathcal{H} = \sum_{i=1}^{n} \oplus \mathcal{H}_i$$

be the unique (SI) decomposition of T. Next we prove $\ker \tau_{T_iT_j} = \{0\}$ for any $i \neq j$. For this, otherwise, assume that there is a nonzero operator Y such that $YT_i = T_jY$, where $1 \leq i < j < n$. For any scalar λ let

$$M_\lambda = \{0 \oplus \cdots \oplus \underset{i\text{th}}{x} \oplus 0 \oplus \cdots 0 \oplus \underset{j\text{th}}{\lambda Y x} \oplus \cdots \oplus 0 : x \in \mathcal{H}_i\}.$$

Since the (SI) decomposition of T is unique and finite, the number of reducing spaces of T is finitely many. But the M_λ's are distinct Banach reducing subspaces of T. This contradicts to our assumption. This completes our proof that (a)⇒(b).

(b)⇒(c) and (c)⇒(b) are obvious. □

Lemma 3. *Let P be a minimal idempotent of $\mathcal{A}'(T)$. Then $T|_{\operatorname{ran} P}$ is strongly irreducible.*

Proof. Otherwise, $T|_{\operatorname{ran} P}$ can be written as direct sum of two operators, i.e.,

$$T|_{\operatorname{ran} P} = T_1 \dot{+} T_2.$$

This shows that P is not minimal. □

Proof of Theorem 1. Assume that an operator T has a countably infinite number of Banach reducing subspaces.

Claim. *For every idempotent $P \in \mathcal{A}'(T)$, there exists a minimal idempotent $Q \in \mathcal{A}'(T)$ such that* $\operatorname{ran} Q \subset \operatorname{ran} P$.

Proof. Otherwise, we can find a sequence of idempotents $\{P_n\}_{n=1}^{\infty}$ in $\mathcal{A}'(T)$ satisfying $\operatorname{ran} P_n \underset{\neq}{\supset} \operatorname{ran} P_{n+1}$. Set $Q_i = P_i - P_{i+1}$, $i = 1, 2, 3, \ldots$. Then each Q_i is a nonzero idempotent in $\mathcal{A}'(T)$. Set

$$\mathcal{Q} = \{Q_i;\ i \geq 1;\ Q_i = P_i - P_{i+1} \in \mathcal{A}'(T)\}$$

and let Λ_1 and Λ_2 be subsets of the set of positive integers N satisfying $\Lambda_1 \cap \Lambda_2 = \emptyset$. Also let

$$Q_{\Lambda_1} = \bigvee_{\lambda \in \Lambda_1} \operatorname{ran} Q_\lambda, \quad Q_{\Lambda_2} = \bigvee_{\lambda \in \Lambda_2} \operatorname{ran} Q_\lambda.$$

Then Q_{Λ_1} and Q_{Λ_2} are different Banach reducing subspaces. Note that Λ is a infinite set. This implies that number of Banach reducing subspaces of T can not be countably infinite. This verifies our claim. □

By the claim and our assumption, we can find a sequence $\{Q_i\}_{i=1}^{l}$ of minimal idempotents of $\mathcal{A}'(T)$ such that

$$\sum_{i=1}^{l} Q_i = I \quad \text{and} \quad Q_i Q_j = 0$$

for any $i \neq j$. If $l = \infty$, we can induce that the number of Banach reducing subspaces of T can not be countably infinite by imitating the proof of the claim. This shows that T can only be written as the direct sum of finite many strongly irreducible operators, i.e., we can find finitely many minimal idempotents $\{Q_i\}_{i=1}^{n}$ in $\mathcal{A}'(T)$ such that

$$\sum_{i=1}^{n} Q_i = I \quad \text{and} \quad Q_i Q_j = 0$$

for any $i \neq j$. Without loss of generality, assume that

$$T = \sum_{i=1}^{n} \oplus T_i \quad \text{on} \quad \mathcal{H} = \sum_{i=1}^{n} \oplus \operatorname{ran} Q_i.$$

Then T_i's are strongly irreducible by Lemma 3. By our assumption and Lemma 2, there exist T_i and T_j such that $\ker \tau_{T_i T_j} \neq \{0\}$. Repeating the proof of Lemma 2, we can infer that the number of Banach reducing subspaces of T can not be countably infinite. This contradicts to our assumption on T.

This completes the proof of Theorem 1. □

3. Strong irreducibility of Jordan blocks

As we mentioned earlier, for every Jordan block $S(\theta)$, the commutant $\mathcal{A}'(S(\theta))$ has no nontrivial projections. So a natural question is whether $\mathcal{A}'(S(\theta))$ has nontrivial idempotents, i.e., whether $S(\theta)$ is strongly irreducible.

A key concept in this study is *corona decomposition*. An inner function θ is said to have a corona decomposition if θ can be decomposed as $\theta_1\theta_2$, where θ_1 and θ_2 are nonconstant inner functions such that

$$|\theta_1(z)| + |\theta_2(z)| \geq \epsilon, \quad \forall z \in D$$

for some positive constant ϵ.

For every $g \in H^\infty(D)$, we define an operator

$$S_g f = P_N g f, \quad f \in N.$$

Clearly, S_z is the Jordan operator $S(\theta)$. It is also not hard to see that $g \in \theta H^\infty(D)$ if and only if $S_g = 0$. Sarason's Theorem describes the commutant $\mathcal{A}'(S(\theta))$:

Sarason's Theorem ([5])**.** *A bounded linear operator A commutes with S_z on N if and only if $A = S_g$ for some $g \in H^\infty(D)$, and this g can be picked such that $\|g\|_\infty = \|A\|$.*

The following is the main result of this section.

Theorem 4. *$S(\theta)$ is not strongly irreducible if and only if θ has a corona decomposition.*

Proof. We first prove the sufficiency. If θ has a corona decomposition $\theta_1\theta_2$, then by the corona theorem, there exist $h_1,\ h_2 \in H^\infty(D)$ such that

$$\theta_1 h_1 + \theta_2 h_2 = 1.$$

Let $g_i = \theta_i h_i,\ i = 1,\ 2$. Clearly, at least one of g_1 and g_2 is not in $\theta H^2(D)$. Without loss of generality we assume g_1 is not in $\theta H^2(D)$. Then S_{g_1} is not equal to zero or the identity operator, and one checks that

$$\begin{aligned} S_{g_1}^2 - S_{g_1} &= S_{g_1^2 - g_1} \\ &= S_{g_1(g_1-1)} \\ &= -S_{\theta h_1 h_2} \\ &= 0. \end{aligned}$$

This means $S(\theta)$ is not strongly irreducible.

Next we prove the necessity. If there exists a nontrivial idempotent $A \in \mathcal{A}'(S(\theta))$, then by Sarason's theorem $A = S_g$ for some $g \in H^\infty(D)$ with $\|g\|_\infty = \|A\|$, and moreover $S_g^2 - S_g = 0$. It then follows that

$$g(g-1) = g^2 - g = \theta f$$

for some $f \in H^\infty(D)$. Let p_1h_1 be the inner-outer factorization of g and p_2h_2 be the inner-outer factorization of $1-g$, then

$$p_1h_1 + p_2h_2 = 1,$$

and hence

$$|p_1(z)| + |p_2(z)| \geq \epsilon, \quad \forall z \in D$$

for some positive constant ϵ. It is clear that θ is a factor of p_1p_2. Now let

$$\theta_1 = \gcd(p_1,\ \theta),$$

and $\theta_2 = \theta/\theta_1$. Then θ_2 is a factor of p_2, and it follows that

$$|\theta_1(z)| + |\theta_2(z)| \geq |p_1(z)| + |p_2(z)| \geq \epsilon, \quad \forall z \in D.$$

So $\theta_1\theta_2$ is a corona decomposition of θ. □

Intuitively speaking, an inner function θ has a corona decomposition if and only if the zeros of θ in the maximal ideal space are not connected.

Example 5. z^n and $e^{-\frac{1+z}{1-z}}$ have no corona decomposition and hence the corresponding Jordan blocks are (SI). If θ is Blaschke product with at least two different zeros, then one checks that it has a corona decomposition (though the decomposition may not be unique), and therefore the associated Jordan block is not (SI).

We point out that the proof of Theorem 4 in fact constructs a corresponding idempotent for each factor in the corona decomposition, for example $S_{\theta_1h_1}$ corresponds to θ_1 and $S_{\theta_2h_2}$ corresponds to θ_2. In simple cases, this correspondence enables one to count the number of idempotents in $\mathcal{A}'(S(\theta))$.

Example 6. Let

$$\theta(z) = \prod_{i=1}^{n} \frac{\lambda_i - z}{1 - \bar{\lambda}_i z}$$

be a finite Blaschke product with distinct zeros. Counting 1 and itself, θ has 2^n factors. So there are 2^n idempotents (including the trivial ones) in $\mathcal{A}'(S(\theta))$. In the case when θ is an infinite Blaschke product, it has 2^∞ many different corona decompositions, and hence has uncountably many idempotents. This observation somewhat illustrates the spirit of Theorem 1.

Moreover, the ranks of the idempotents can also be determined. Let $\theta_1\theta_2$ be a corona decomposition of θ and $h_1, h_2 \in H^\infty(D)$ be such that

$$\theta_1h_1 + \theta_2h_2 = 1.$$

For simplicity, we denote $\theta_1 h_1$ by g. Since S_g is an idempotent, $\operatorname{ran}(I-S_g) = \ker(S_g)$ as remarked at the beginning of Section 2. It is well-known in this case

$$H^2(D) \ominus \theta H^2(D) = \operatorname{span}\{(1-\overline{\lambda_i}z)^{-1} : \ i = 1, 2, \dots, n\}.$$

So for every $f \in H^2(D) \ominus \theta H^2(D)$,

$$\begin{aligned} \langle S_g f, \ (1-\overline{\lambda_i}z)^{-1}\rangle &= \langle gf, \ (1-\overline{\lambda_i}z)^{-1})\rangle \\ &= g(\lambda_i) f(\lambda_i) \\ &= \theta_1(\lambda_i) h_1(\lambda_i) f(\lambda_i). \end{aligned}$$

So if $\theta_1(\lambda_i) = 0$ then $(1-\overline{\lambda_i}z)^{-1}$ is orthogonal to $\operatorname{ran}(S_g)$, and it follows that

$$\dim \operatorname{ran}(S_g) \leq n - |Z(\theta_1)|,$$

where $Z(f)$ stands for the zero set of f in D and $|E|$ stands for the cardinality of E. Similarly, we have

$$\dim \ker(S_g) = \dim \operatorname{ran}(I - S_g) \leq n - |Z(\theta_2)|.$$

Adding the above two inequalities, we have

$$\begin{aligned} n &= \dim \operatorname{ran}(S_g) + \dim \ker(S_g) \\ &\leq 2n - |Z(\theta_1)| - |Z(\theta_2)| \\ &= n, \end{aligned}$$

and it follows that

$$\dim \operatorname{ran}(S_g) = n - |Z(\theta_1)|, \quad \dim \ker(S_g) = n - |Z(\theta_2)|.$$

So in conclusion, the idempotent corresponding to the factor θ_1 has rank $n - |Z(\theta_1)|$. When θ_1 has $n-1$ zeros, the corresponding idempotent is minimal.

References

[1] Bercovici Hari. Operator theory and arithmetic in H^∞. Mathematical Surveys and Monographs, 26. *American Mathematical Society, Providence, RI*, 1988. xii+275 pp. ISBN: 0-8218-1528-8. MR954383 (90e:47001), Zbl 0653.47004.

[2] Beurling Arne. On two problems concerning linear transformations in Hilbert space. *Acta Math.* **81** (1948), 239–255. MR0027954 (10,381e), Zbl 0033.37701.

[3] Jiang, Chunlan; Wang, Zongyao. Strongly irreducible operators on Hilbert space. Pitman Research Notes in Mathematics Series, 389. *Longman, Harlow*, 1998. x+243 pp. ISBN: 0-582-30594-2. MR1640067 (2000c:47033).

[4] Jiang, Chunlan; Wang, Zongyao. Structure of Hilbert space operators. *World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ*, 2006. x+248 pp. ISBN: 981-256-616-3. MR2221863 (2008j:47001), Zbl 1102.47001.

[5] Sarason Donald. Generalized interpolation in $H^\infty(D)$. *Trans. Amer. Math. Soc.* **127** (1967), 179–203. MR0208383 (34 # 8193), Zbl 0145.39303.

[6] Sz.-Nagy Béla.; Foiaş Ciprian. Harmonic analysis of operators on Hilbert space. Translated from the French and revised. *North-Holland, Amsterdam; American Elsevier, New York; Akad. Kiadó, Budapest*, 1970. xiii+389 pp. MR0275190 (43 # 947), Zbl 0201.45003.

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