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A new characterization for isometries by triangles

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Abstract. Let $\mathbb{R}^n$ be an n-dimensional Euclidean space and $\mathbb{D}^n$ be an n-dimensional hyperbolic space with the Poincaré metric for $n > 1$. In this paper, we shall prove the following results. (i) A bijection $f : \mathbb{D}^n \to \mathbb{D}^n$ is an isometry (Möbius transformation) if and only if f is triangle preserving. (ii) A bijection $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation if and only if f is triangle preserving.

Contents

1. Introduction

Let $\mathbb{R}^n$ be an n-dimension Euclidean space and

$$\mathbb{D}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$$

be an n-dimension hyperbolic space with the Poincaré metric for $n > 1$.

A Möbius transformation $f : \mathbb{D}^n \to \mathbb{D}^n$, which is equivalently an isometry under the Poincaré metric, has many beautiful properties. For example, f

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is bijective, holomorphic, circle-preserving, sphere-preserving, and geodesic-preserving . Moreover, f preserves angles and polygons.

An affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is line-preserving, super-plane-preserving, and keeps the parallel relation of two lines, and the rate of two segments in the same (parallel) line. The following results are known.

Theorem A ([3])**.** *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ *is surjective and line into line. Then f is an affine transformation.*

Theorem B ([8])**.** *Suppose that* $f : \mathbb{D}^n \to \mathbb{D}^n$ $(n > 1)$ *is a bijection that preserves geodesics. Then* f *is an isometry.*

Theorem C ([8])**.** *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ *is a bijection that preserves lines. Then* f *is an affine transformation.*

Here we say that f *preserves lines* if for any line l, $f(l)$ is a line.

Theorem D ([9])**.** *Suppose that* $f : \mathbb{D}^n \to \mathbb{D}^n$ *is geodesic-preserving. Then* f *is an isometry if and only if* f *is nondegenerate.*

Theorem E ([9])**.** *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is line-preserving. Then* f *is an affine transformation if and only if* f *is nondegenerate.*

Here, f is *nondegenerate* if the image of the whole space under f is more than a line or geodesic.

In the hyperbolic plane $\mathbb{D}$, Haruki and Rassias [6] gave a characterization of isometries by using Apollonius quadrilaterals and proved that if f is meromorphic and sends Apollonius quadrilaterals to Apollonius quadrilaterals, then f is Möbius. Here an Apollonius quadrilateral $ABCD$ satisfies $|AB| \cdot |CD| = |BC| \cdot |DA|$, where $|AB|$ denotes the length of the segment joining A and B. See [4] [5] [7] for other related results.

Yang and Fang ([13]) gave a characterization of isometries on $\mathbb{D}$ by using Lambert quadrilaterals.

Theorem F ([13])**.** *Suppose that* $f : \mathbb{D} \to \mathbb{D}$ *is a continuous bijection. Then* f *is Möbius if and only if* f *preserves Lambert quadrilaterals in* $\mathbb{D}$*.*

Recently, Yang ([12]) proved the following result on triangles in $\mathbb{D}$.

Theorem G ([12])**.** *Suppose that* $f : \mathbb{D} \to \mathbb{D}$ *is an injection. Then f is an isometry if and only if for some* $0 < \theta < \pi$, f *preserves triangles with an interior angle equal to* θ*.*

In this paper, we shall prove the following theorems.

Theorem 1. *Suppose that* $f : \mathbb{D}^n \to \mathbb{D}^n$ *is a bijection. Then* f *is an isometry if and only if* f *is triangle domain preserving.*

Here, *a triangle domain* is a closed domain bounded by a triangle.

Theorem 2. *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a bijection. Then* f *is an affine transformation if and only if* f *is triangle domain preserving.*

As an application of Theorems 1 and 2, we obtain our main results.

Theorem 3. *Suppose that $f : \mathbb{D}^n \to \mathbb{D}^n$ is a bijection. Then f is an isometry if and only if f is triangle preserving.*

Theorem 4. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection. Then f is an affine transformation, if and only if f is triangle preserving.*

2. Hyperbolic space

In this section, we shall first prove Theorem 1.

2.1. Triangle domain preserving maps for $n = 2$. We shall prove Theorem 1 for $n = 2$.

Denote points by $A, B, C, \dots$ in $\mathbb{D}$, the images under f by $A', B', C', \dots,$ the geodesic passing through A, B by L_{AB}, the segment ended by A, B by AB, a triangle domain by $\triangle$, the boundary of a triangle domain $\triangle$ by $\partial\triangle$, and the image triangle domain of a triangle domain $\triangle$ by $\triangle'$.

We have the following lemmas.

Lemma 2.1. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is a triangle domain preserving injection. Then for any triangle domain $\triangle$, $f(\partial\triangle) \subset \partial\triangle'$.*

In fact, for any $P \in \partial\triangle$, we can choose another triangle domain $\triangle_1$, such that $\triangle \cap \triangle_1 = \{P\}$. So $\triangle' \cap \triangle'_1 = \{P'\}$, and $P' \in \partial\triangle'$.

Lemma 2.2. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is a triangle domain preserving injection. Then the image of a segment under f is a segment.*

Proof. Given a segment L, one can find two triangle domains $\triangle_1$ and $\triangle_2$, such that $L = \triangle_1 \cap \triangle_2$. So the image $\triangle'_1 \cap \triangle'_2 = L'$ is a convex set by the convexity of triangle domains.

On the other hand, as $L \subset \partial\triangle_1 \cap \partial\triangle_2$, $L' \subset \partial\triangle'_1 \cap \partial\triangle'_2$ by Lemma 2.1. Therefore L' is a segment, and the proof is complete. □

Proof of Theorem 1 for $n = 2$. It follows from Lemma 2.2 that $f : \mathbb{D} \to \mathbb{D}$ is segment to segment, and geodesic to geodesic. Moreover, for any three collinear points P'_1, P'_2, P'_3, with $P'_2 \in P'_1P'_3$, we denote the inverse image points P_1, P_2, P_3. The image of the segment P_1P_3 is a segment containing P'_1 and P'_3, and so $P'_2 \in f(P_1P_3)$. That is, $P_2 \in P_1P_3$, and $f^{-1} : \mathbb{D} \to \mathbb{D}$ is geodesic to geodesic. Then f is geodesic-preserving. Therefore f is an isometry by Theorem B or D. □

2.2. Triangle domain preserving maps for $n > 2$. In this part, we shall prove Theorem 1 for $n > 2$. First we have the following lemmas.

Lemma 2.3. *Suppose that $f : \mathbb{D}^n \to \mathbb{D}^n$ is a triangle domain preserving bijection. Then the image of any 2-dimensional plane under f is in a 2-dimensional plane.*

Proof. For any 2-dimensional plane $\mathbb{D}$, we choose a triangle domain $\triangle \subset \mathbb{D}$, and denote the image triangle domain by $\triangle'$ in some 2-dimensional plane $\mathbb{D}'$.

For any point $P \in \mathbb{D}\backslash\triangle$, we can choose $A, B, A_1, B_1 \in \triangle$, such that $A_1 \in PA$, $B_1 \in PB$, and $L_{PA} \cap L_{PB} = P$.

Case 1. A', B', A_1', B_1' are noncollinear. Then the image of triangle domain $\triangle_{PAB}$ is a triangle domain passing through A', B', A_1', B_1', which is in $\mathbb{D}'$.

Case 2. A', B', A_1', B_1' is collinear. Choose a point $C' \in \triangle'$ noncollinear with them. Suppose that C is noncollinear with P, A. Then the image of triangle $\triangle_{PAC}$ is a triangle domain passing through A', A_1', C', which is in $\mathbb{D}'$.

So $P' \in \mathbb{D}'$ and the proof is complete. □

Proof of Theorem 1 for $n > 2$. By Lemma 2.3, the image of any 2-dimensional plane D is in a 2-dimensional plane. By composing some isometry, we may suppose that $f : \mathbb{D} \to \mathbb{D}$.

Therefore $f : \mathbb{D} \to \mathbb{D}$ is a triangle domain preserving injection. By Lemma 2.2, $f : \mathbb{D} \to \mathbb{D}$ is segment onto segment, and $f : \mathbb{D}^n \to \mathbb{D}^n$ is segment onto segment. As in the proof in the case of $n = 2$, we conclude that f and f^{-1} are geodesic to geodesic. So f is geodesic preserving. By Theorem B or D, f is an isometry, and the proof is complete. □

In the following we shall prove Theorem 3.

2.3. Triangle preserving maps for $n = 2$. In this part, we shall prove Theorem 3 for $n = 2$.

Denote the boundary of a triangle domain $\triangle$ by $\partial\triangle$, the image triangle of $\partial\triangle$ by $\partial'\triangle$. It is easy to see that $\partial\triangle$ divides $\mathbb{D}(\mathbb{R}^2)$ into two portions, the inner $\partial\triangle^\circ$ and the outer $\partial\triangle^c$.

It is obvious that $\partial\triangle^\circ$ is *connected* in $\mathbb{D}\backslash\partial\triangle$, which means that P, Q can be joined by some segments (triangle) in $\mathbb{D}\backslash\partial\triangle$ for any $P, Q \in \partial\triangle^\circ$. The image of any connected component in $\mathbb{D}\backslash\partial\triangle$ is connected in $\mathbb{D}\backslash\partial'\triangle$.

Let $NC(\partial\triangle)$ be the number of connected components of $\mathbb{D}\backslash\partial\triangle$. Obviously,

$$2 \leq NC(\partial'\triangle) \leq NC(\partial\triangle) \leq 4.$$

Then we have the following lemma.

Lemma 2.4. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is a triangle preserving bijection. Then f is triangle domain preserving.*

Proof. For any triangle domain $\triangle$, choose three noncollinear points A', B', C' in $\partial'\triangle$, such that $A, B, C \in \partial\triangle$ are noncollinear (except the vertices).

In fact, we can choose any three points A', B', C' from three different sides of $\partial'\triangle$ (not the vertices of $\partial'\triangle$ and the image of the vertices of $\partial\triangle$).

Suppose that A, B, C are in the same side of $\partial\triangle$. Choose a point A_1 in one of the other sides of $\partial\triangle$ (not the vertices or the inverse image of the vertices of $\partial'\triangle$), and A_1', B', C' are noncollinear.

It is obvious that $\partial\triangle_{ABC} \subset \triangle$, and $\partial\triangle_{ABC} \cap \partial\triangle^\circ \neq \emptyset$.

Note that the image triangle $\partial'\triangle_{ABC}$ passing through A', B', C' must have crossing points with $\partial'\triangle^\circ$(As in Figure 1). That is, $f(\partial\triangle^\circ) \cap \partial'\triangle^\circ \neq \emptyset$. So $f(\partial\triangle^\circ) \subset \partial'\triangle^\circ$.

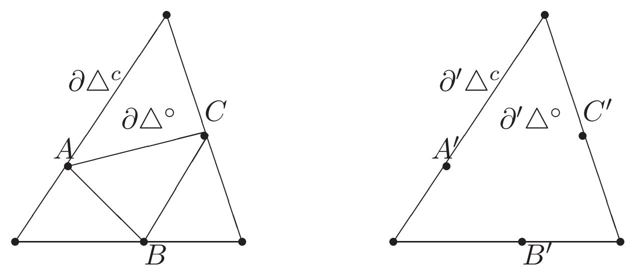

FIGURE 1

Case 1. $NC(\partial\triangle) = 2$. Then $f(\partial\triangle^\circ) = \partial'\triangle^\circ$, and $f(\triangle)$ is a triangle domain with boundary $\partial'\triangle$, denoted by $\triangle'$.

Case 2. $NC(\partial\triangle) = 3$ (or 4). The triangle domain $\triangle$ can be separated into two triangle domains by some segment S, say $\triangle_1, \triangle_2$, such that $NC(\partial\triangle_1) = NC(\partial\triangle_2) = 2$. Denote the image triangle domains $\triangle_1'$ and $\triangle_2'$.

Since the segment $S = \partial\triangle_1 \cap \partial\triangle_2$ and $S = \triangle_1 \cap \triangle_2$, the image $S' = \partial\triangle_1' \cap \partial\triangle_2'$ and $S' = \triangle_1' \cap \triangle_2'$ is a segment. So the image triangle $\partial'\triangle = \partial(\triangle_1' \cup \triangle_2')$. Note that $f(\triangle) = \triangle_1' \cup \triangle_2'$. Therefore $f(\triangle)$ is a triangle domain.

Therefore, for any triangle domain $\triangle$, $f(\triangle)$ is a triangle domain. This completes the proof. □

Proof of Theorem 3 for $n = 2$. In this case, Theorem 3 follows from Lemma 2.4 and Theorem 1. □

2.4. Triangle preserving maps for $n > 2$. In this part, we shall prove Theorem 3 for $n > 2$.

Lemma 2.5. *Suppose that $f : \mathbb{D}^n \to \mathbb{D}^n$ is a triangle preserving bijection. Then the image of any 2-dimensional plane is in a 2-dimensional plane.*

Proof. For any 2-dimensional plane $\mathbb{D}$, we choose a triangle $\partial\triangle \subset \mathbb{D}$, and denote the image triangle by $\partial'\triangle$ in some 2-dimensional plane $\mathbb{D}'$.

For any point $P \in \mathbb{D}$, we can choose $Q \in \mathbb{D}$, such that $L_{QP} \cap \partial\triangle = \{A, A_1\}$, and $P, A_1 \in QA$. Choose $B \in \partial\triangle$, such that $L_{QB} \cap \partial\triangle = \{B, B_1\}$, and $B_1 \in QB$.

Case 1. A', B', A_1', B_1' are noncollinear. Then the image of triangle $\partial\triangle_{QAB}$ is a triangle passing through A', B', A_1', B_1', which is in $\mathbb{D}'$.

Case 2. A', B', A'_1, B'_1 is collinear. Choose a point $C' \in \partial\triangle'$ noncollinear with them. Then the image of triangle $\partial\triangle_{QAC}$ is a triangle passing through A', A'_1, C', which is in $\mathbb{D}'$.

We have shown that $P' \in \mathbb{D}'$ for all cases. The proof is complete. □

Lemma 2.6. *Suppose that $f : \mathbb{D}^n \to \mathbb{D}^n$ is a triangle preserving bijection. Then for any 2-dimensional plane $\mathbb{D}$, $f(\mathbb{D})$ is a 2-dimensional plane.*

Proof. Choose a triangle $\partial\triangle \subset \mathbb{D}$ and the image triangle $\partial'\triangle \subset \mathbb{D}'$. By Lemma 2.3, $f(\mathbb{D}) \subset \mathbb{D}'$. Therefore we only need to prove that $f : \mathbb{D} \to \mathbb{D}'$ is surjective.

Suppose that there exists $P' \in \mathbb{D}' \backslash f(\mathbb{D})$. Then we can have its inverse image point $P \in \mathbb{D}^n \backslash \mathbb{D}$ since f is surjection. Without loss of generality, we may suppose that P' is in the outside of $\partial'\triangle$.

Denote 3-dimensional space containing $\mathbb{D}$ and P by $\mathbb{D}^3$.

Choose $A \in \partial\triangle$, such that $L_{A'P'} \cap \partial\triangle = \{A', A'_1\}$, and $A'_1 \in P'A'$. Given any $C \in \partial\triangle\backslash\{A, A_1\}$, the image triangle of $\partial\triangle_{PAC}$, passing through three noncollinear points P', A', C', is in $\mathbb{D}'$. So $f(PA) \subset \mathbb{D}'$.

Choose $E' \in \partial'\triangle\backslash\{A'\}$, such that any triangle in $\mathbb{D}'$, passing through P', E', has more than one crossing points with $\partial'\triangle$ (in fact, E' is in the interior of the farther side of P'). Choose $Q \in PA\backslash\{P, A\}$ and then

$$\partial\triangle_{PQE} \cap \partial\triangle = \{E\},$$

and their image triangles have more than one crossing point. This is a contradiction, and the proof is complete. □

Proof of Theorem 3 for $n > 2$. For any 2-dimensional plane $\mathbb{D} \in \mathbb{D}^n$, $f(\mathbb{D})$ is a 2-dimensional plane by Lemma 2.6. By composing some suitable isometry, we may suppose that $f : \mathbb{D} \to \mathbb{D}$ is a triangle preserving bijection. By the result of the case $n = 2$, $f : \mathbb{D} \to \mathbb{D}$ is an isometry and $f : \mathbb{D} \to \mathbb{D}$ is geodesic preserving. So $f : \mathbb{D}^n \to \mathbb{D}^n$ is geodesic preserving. By Theorem B or D, f is an isometry. This completes the proof. □

3. Euclidean space

By the same methods as in the case of hyperbolic space, we can prove Theorems 2 and 4 similarly. We omit the details.

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