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Asymptotic dimension of coarse spaces

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Abstract. We consider asymptotic dimension in the general setting of coarse spaces and prove some basic properties such as monotonicity, a formula for the asymptotic dimension of finite unions and estimates for the asymptotic dimension of the product of two coarse spaces.

Contents

1. Coarse geometry

Originally (cf. [Gro93]) coarse geometry was studied in the concrete setting of metric spaces. But it turned out that, similar to infinitesimal scale geometry (i.e., topology), there is an axiomatic description of large scale geometry.

Coarse concepts have applications, e.g., in geometric group theory. As a striking example, the strong Novikov conjecture for many groups can be proved using methods from coarse geometry. This started out using the large scale geometry of metric spaces, but one now observes that using general abstract coarse structures offers greater flexibility which is frequently very useful. Compare [Yu98], [Wri04] and [Mit01].

To make this paper self-contained I include a short discussion of the basic concepts of coarse geometry. More information can be found in Roe's textbook [Roe03] and in [Gra06].

Definition 1 (Coarse structure)**.** Let X be a set. A collection $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure*, and the elements of $\mathcal{E}$ will be called *entourages*, if the following axioms are fulfilled:

(a) A subset of an entourage is an entourage.
(b) A finite union of entourages is an entourage.
(c) The diagonal $\Delta_X := \{(x, x) \mid x \in X\}$ is an entourage.

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(d) The inverse E^{-1} of an entourage E is an entourage:

$$E^{-1} := \{(y,x) \in X \times X \mid (x,y) \in E\}.$$

(e) The composition E_1E_2 of entourages E_1 and E_2 is an entourage:

$$E_1E_2 := \{(x,z) \in X \times X \mid \exists_{y\in X}(x,y) \in E_1 \text{ and } (y,z) \in E_2\}.$$

The pair $(X, \mathcal{E})$ is called a *coarse space*.

A coarse space is called *connected* if every point of $X \times X$ is contained in an entourage.

Definition 2 (Bounded sets)**.** Let $(X, \mathcal{E})$ be a coarse space, $A \subseteq X$ and $E \in \mathcal{E}$. We define $E[A] := \{x \in X \mid (x,a) \in E \text{ for some } a \in A\}$. For a point $x \in X$ we will write $E(x)$ instead of $E[\{x\}]$. Sets of the form $E(x)$ with $x \in X$ and $E \in \mathcal{E}$ are called *bounded*.

Definition 3. Let X be a set and $\mathcal{M}$ a collection of subsets of $X \times X$. Since any intersection of coarse structures on X is itself a coarse structure, we can make the following definitions. We call the smallest coarse structure containing $\mathcal{M}$, i.e., the intersection of all coarse structures containing $\mathcal{M}$, the *coarse structure generated by* $\mathcal{M}$. In the same way, define the *connected coarse structure generated by* $\mathcal{M}$.

Definition 4. Let $(X, \mathcal{E}_X)$, $(Y, \mathcal{E}_Y)$ be coarse spaces and $f\colon X \to Y$ a map.

- We call f *coarsely proper* if the inverse image of each bounded set is bounded.
- We call f *coarsely uniform* if the image of each entourage under the map $f \times f\colon X \times X \to Y \times Y$ is an entourage.
- We call f a *coarse map* if it is coarsely proper and coarsely uniform.
- We call f a *coarse embedding* if f is coarsely uniform and the inverse image of an entourage under $f \times f$ is an entourage.

Definition 5 (Coarse equivalence)**.** We call a map $f\colon X \to Y$ a coarse equivalence if f is coarsely uniform and there exists a coarsely uniform map $g\colon Y \to X$ such that $g \circ f$ is close[1] to id_X and $f \circ g$ is close to id_Y.

2. Asymptotic dimension

In [Roe03], John Roe defined asymptotic dimension for coarse spaces in general. We are now going to give another definition of asymptotic dimension for coarse spaces which generalizes a different characterisation of asymptotic dimension for metric spaces.

Definition 6 (Asymptotic dimension of coarse spaces)**.** Let $(X, \mathcal{E})$ be a coarse space.

- Let $L \in \mathcal{E}$ be an entourage and $\mathcal{U}$ a cover of X. We say that $\mathcal{U}$ has *appetite* L if for all $x \in X$ there exists a set $U \in \mathcal{U}$ such that $L(x) \subseteq U$.
- We call a cover $\mathcal{U}$ *uniformly bounded* if $\Delta_{\mathcal{U}} := \bigcup_{U\in\mathcal{U}} U \times U$ is an entourage.
- Let $n \in \mathbb{N}$. We say $\operatorname{asdim}(X, \mathcal{E}) \leq n$ if for every[2] entourage $L \in \mathcal{E}$ there exists a cover $\mathcal{U}$ of X such that:

[1] Let $(X, \mathcal{E})$ be a coarse space and S a set. The maps $f\colon S \to X$ and $g\colon S \to X$ are called *close* if $\{(f(s), g(s)) \mid s \in S)\}$ is an entourage.

[2] An entourage $L \in \mathcal{E}$ is called symmetric if $L = L^{-1}$. We need to consider only symmetric entourages which contain the diagonal, because for any entourage $L \in \mathcal{E}$ we have $L \subseteq L \cup L^{-1} \cup \Delta_X \in \mathcal{E}$.

(1) The multiplicity $\mu(\mathcal{U})$ is at most $n+1$.
(2) $\mathcal{U}$ has appetite L.
(3) $\mathcal{U}$ is uniformly bounded.

The following is a redraft of Roe's definition of asymptotic dimension.

Definition 7. Let $(X, \mathcal{E})$ be a coarse space. We say $\operatorname{asdim}_{\mathrm{Roe}}(X, \mathcal{E}) \leq n$ if for every entourage $L \in \mathcal{E}$ there is a cover $\mathcal{U}$ of X such that:

(1) $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{n+1}$.
(2) Each of the families $\mathcal{U}_1, \ldots, \mathcal{U}_{n+1}$ is L-disjoint (i.e., whenever $A, B \in \mathcal{U}_i$ and $A \neq B$, then $A \times B \cap L = \emptyset$).
(3) $\mathcal{U}$ is uniformly bounded.

There is a small difference between Definition 7 and the definition given in [Roe03]. In Roe's original definition the cover $\mathcal{U}$ is supposed to be countable. We will not make any assumptions on the cardinality of $\mathcal{U}$.

A third version of asymptotic dimension will appear in Theorem 9.

Definition 8. $\operatorname{asdim}_{\mathrm{fam}}(X, \mathcal{E}) \leq n$ if for every entourage $L \in \mathcal{E}$ there is a cover $\mathcal{U}$ of X such that:

(1) $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{n+1}$, where each of the families $\mathcal{U}_i$ consists of disjoint sets.
(2) $\mathcal{U}$ has appetite L.
(3) $\mathcal{U}$ is uniformly bounded.

Theorem 9. *Let $(X, \mathcal{E})$ be a coarse space. Then*

$$\operatorname{asdim}(X, \mathcal{E}) = \operatorname{asdim}_{\mathrm{Roe}}(X, \mathcal{E}) = \operatorname{asdim}_{\mathrm{fam}}(X, \mathcal{E}).$$

Proof. We first prove $\operatorname{asdim}_{\mathrm{Roe}} \geq \operatorname{asdim}_{\mathrm{fam}}$. Assume $\operatorname{asdim}_{\mathrm{Roe}}(X, \mathcal{E}) = n \in \mathbb{N}$. Let L be a symmetric entourage which contains the diagonal. For $L^2 := LL \in \mathcal{E}$, there exists a cover $\mathcal{U}$ as in Definition 7. Since $A \times B \cap L^2 = \emptyset$ is equivalent to $L[A] \cap L[B] = \emptyset$, the cover $\mathcal{U}_L := \{L[U] \mid U \in \mathcal{U}\}$ meets all conditions required in Definition 8. Note that $\bigcup_{U \in \mathcal{U}} L[U] \times L[U] \subseteq L\left(\bigcup_{U \in \mathcal{U}} U \times U\right) L^{-1} \in \mathcal{E}$.

In a second step we have to prove $\operatorname{asdim}_{\mathrm{fam}} \geq \operatorname{asdim}$, but this is obvious, since condition (1) of Definition 8 implies condition (1) of Definition 6.

It remains to prove $\operatorname{asdim} \geq \operatorname{asdim}_{\mathrm{Roe}}$. For this purpose we need to construct a uniformly bounded cover $\mathcal{V}$ consisting of L-disjoint families from a uniformly bounded cover $\mathcal{U}$ with appetite L^{n+1}. The idea is to take all intersections of $n+1$ sets from $\mathcal{U}$ as one family, the intersections of exactly n sets from $\mathcal{U}$ as a second family, etc. However, we have to ensure these families to be L-disjoint.

Assume that $\operatorname{asdim}(X, \mathcal{E}) = n \in \mathbb{N}$. Let $L \in \mathcal{E}$ be a symmetric entourage that contains the diagonal. Let $\mathcal{U}$ be a uniformly bounded cover of X with appetite L^{n+1} and multiplicity at most $n+1$. For an entourage E and $U \subseteq X$ we define $\operatorname{Int}_E(U) := \{x \in X \mid E(x) \subseteq U\}$. Observe that $E_1 \subseteq E_2$ implies $\operatorname{Int}_{E_2}(U) \subseteq \operatorname{Int}_{E_1}(U)$. Some more definitions are needed to get $\mathcal{V}$:

$$\mathcal{U}_i := \{U_1 \cap \cdots \cap U_i \mid U_1, \ldots, U_i \in \mathcal{U} \text{ pairwise distinct}\}$$

$$S_i := \bigcup_{U \in \mathcal{U}_i} \operatorname{Int}_{L^{n+2-i}}(U) \quad \text{and} \quad S_{n+2} = \emptyset$$

$$\mathcal{V}_i := \{\operatorname{Int}_{L^{n+2-i}}(U) \backslash S_{i+1} \mid U \in \mathcal{U}_i\}$$

$$\mathcal{V} := \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{n+1}.$$

Since $\{\mathrm{Int}_{L^{n+1}}(U) \mid U \in \mathcal{U}\}$ is a cover of X, so is $\mathcal{V}$. Actually, $\mathcal{V}$ is a refinement of the cover $\mathcal{U}$. Therefore $\mathcal{V}$ is uniformly bounded.

It remains to prove that each of the families $\mathcal{V}_1, \ldots, \mathcal{V}_{n+1}$ is L-disjoint. Let $A, B \in \mathcal{V}_i$ such that $A \neq B$. There are $A_1, \ldots, A_i, B_1, \ldots, B_i \in \mathcal{U}$ such that $A = \mathrm{Int}_{L^{n+2-i}}(A_1 \cap \cdots \cap A_i) \backslash S_{i+1}$ and $B = \mathrm{Int}_{L^{n+2-i}}(B_1 \cap \cdots \cap B_i) \backslash S_{i+1}$. The sets $A_1, \ldots, A_i$ are supposed to be pairwise distinct as are the sets $B_1, \ldots, B_i$.

Let $(a, b) \in A \times B \;\cap\; L$ and observe the following facts:

$$a, b \notin S_{i+1} \tag{1}$$

$$a \;\in\; A \subseteq \mathrm{Int}_{L^{n+1-i}}(A_1 \cap \cdots \cap A_i) \tag{2}$$

$$b \;\in\; B \subseteq \mathrm{Int}_{L^{n+1-i}}(B_1 \cap \cdots \cap B_i) \tag{3}$$

$$a \;\in\; L[B] \subseteq L[\mathrm{Int}_{L^{n+2-i}}(B_1 \cap \cdots \cap B_i)] \tag{4}$$

$$b \;\in\; L[A] \subseteq L[\mathrm{Int}_{L^{n+2-i}}(A_1 \cap \cdots \cap A_i)]. \tag{5}$$

Since

$$\begin{aligned} L[\mathrm{Int}_{L^j}(U)] &= \left\{x \mid \exists_{y \in X} L^j(y) \subseteq U, x \in L(y)\right\} \;\subseteq\; \{x \mid L^{j-1}(x) \subseteq U\} \\ &= \mathrm{Int}_{L^{j-1}}(U), \end{aligned}$$

we get the following conclusions from (4) and (5):

$$a \in \mathrm{Int}_{L^{n+1-i}}(B_1 \cap \cdots \cap B_i)$$
$$b \in \mathrm{Int}_{L^{n+1-i}}(A_1 \cap \cdots \cap A_i).$$

Finally $a, b \in \mathrm{Int}_{L^{n+2-(i+1)}}(A_1 \cap \cdots \cap A_i \cap B_1 \cap \cdots \cap B_i)$. Since $A \neq B$, we know that the set $\{A_1, \ldots, A_i, B_1, \ldots, B_i\}$ contains at least $i+1$ elements. Thus $a, b \in S_{i+1}$, but this is a contradiction to (1). □

Theorem 10. *If $f\colon (X, \mathcal{E}) \to (Y, \mathcal{F})$ is a coarse embedding, then*

$$\mathrm{asdim}(X, \mathcal{E}) \leq \mathrm{asdim}(Y, \mathcal{F}).$$

Proof. Suppose that $n := \mathrm{asdim}(Y, \mathcal{F}) < \infty$. Let $E \in \mathcal{E}$ be an entourage and set $F := f \times f(E)$. Note that $E \subseteq (f \times f)^{-1}(F)$. There is a uniformly bounded cover $\mathcal{U}$ of Y with appetite F and multiplicity at most $n+1$. The inverse image of $\mathcal{U}$ is a uniformly bounded cover of X with appetite E and the same multiplicity as $\mathcal{U}$. □

Corollary 11 (Monotonicity of asymptotic dimension)**.** *Let $(X, \mathcal{E})$ be a coarse space and $A \subseteq X$. Note that the inclusion map is a coarse embedding. Hence*

$$\mathrm{asdim}(A, \mathcal{E}|_A) \leq \mathrm{asdim}(X, \mathcal{E}).$$

Corollary 12 (Coarse invariance of asymptotic dimension)**.** *Given a coarse equivalence $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$, we have*

$$\mathrm{asdim}(X, \mathcal{E}_X) = \mathrm{asdim}(Y, \mathcal{E}_Y).$$

Definition 13. Let $(X, \mathcal{E})$ be a coarse space and $A \subseteq X$. We call A a *substantial part* of X if $\mathrm{asdim}(A, \mathcal{E}|_A) = \mathrm{asdim}(X, \mathcal{E})$.

For a coarsely uniform map $(X, \mathcal{E}) \to (Y, \mathcal{F})$ which is also injective, there is no relation between the asymptotic dimensions of $(X, \mathcal{E})$ and $(Y, \mathcal{F})$.

Example 14. Let $(X, \mathcal{E})$ be a coarse space. Observe that the power set $\mathcal{P}(X \times X)$ is a coarse structure on X. The map $\mathrm{id}\colon (X, \mathcal{E}) \to (X, \mathcal{P}(X \times X))$ is coarsely uniform, but $\mathrm{asdim}(X, \mathcal{E}) \geq 0 = \mathrm{asdim}(X, \mathcal{P}(X \times X))$.

Example 15. Let $n \in \{2, 3, \dots\}$. By $\mathcal{E}_{\cdot}$ we denote the coarse structure coming from the one-point compactification[3] of $\mathbb{R}^n$ and by $\mathcal{E}_{\text{eucl}}$ the bounded coarse structure[4] corresponding to the euclidean metric of $\mathbb{R}^n$. It follows that the map $\mathrm{id}\colon (\mathbb{R}^n, \mathcal{E}_{\text{eucl}}) \to (\mathbb{R}^n, \mathcal{E}_{\cdot})$ is coarse, but $\operatorname{asdim}(\mathbb{R}^n, \mathcal{E}_{\text{eucl}}) = n > 1 = \operatorname{asdim}(\mathbb{R}^n, \mathcal{E}_{\cdot})$.

Proof. Since $\mathcal{E}_{\text{eucl}} \subseteq \mathcal{E}_{\cdot}$, the map id is coarsely uniform. A set B is bounded with respect to $\mathcal{E}_{\text{eucl}}$ if and only if B is precompact. The same is true for $\mathcal{E}_{\cdot}$. Thus id is coarsely proper.

It remains to prove $\operatorname{asdim}(\mathbb{R}^n, \mathcal{E}_{\cdot}) = 1$. For this we refer to Example 9.7 of [Roe03]. □

Proposition 16. *Let $(X, \mathcal{E})$ be a coarse space and $\mathcal{E}_{\text{cn}}$ the connected coarse structure generated by $\mathcal{E}$. Then* $\operatorname{asdim}(X, \mathcal{E}) = \operatorname{asdim}(X, \mathcal{E}_{\text{cn}})$.

Proof. Suppose $n := \operatorname{asdim}(X, \mathcal{E}_{\text{cn}}) < \infty$. Let $E \in \mathcal{E} \subseteq \mathcal{E}_{\text{cn}}$. There is a cover $\mathcal{U}$ of X with appetite E and multiplicity at most $n+1$ which is uniformly bounded with respect to $\mathcal{E}_{\text{cn}}$. Each $U \in \mathcal{U}$ can be written as the disjoint union of finitely many sets $U_1, \dots, U_k$ which are bounded with respect to $\mathcal{E}$ and such that the union of any two of the sets $U_1, \dots, U_k$ is not bounded with respect to $\mathcal{E}$. Define $\operatorname{comp}(U) := \{U_1, \dots, U_k\}$ and observe that $\mathcal{U}' := \bigcup_{U \in \mathcal{U}} \operatorname{comp}(U)$ is a cover of X with multiplicity at most $n+1$. Furthermore, $\mathcal{U}'$ has appetite E and is uniformly bounded with respect to $\mathcal{E}$. Hence $\operatorname{asdim}(X, \mathcal{E}) \le n$.

Set $n := \operatorname{asdim}(X, \mathcal{E})$. Let $E \in \mathcal{E}_{\text{cn}}$ be a symmetric entourage. This implies that $E = E' \cup \big(A_1 \times A_{\sigma(1)}\big) \cup \dots \cup \big(A_k \times A_{\sigma(k)}\big)$ with $E' \in \mathcal{E}$, $k \in \mathbb{N}$, $A_1, \dots, A_k$ bounded subsets of X (not necessarily pairwise distinct) and σ a permutation of the set $\{1, \dots, k\}$ with the additional property $\sigma \circ \sigma = \mathrm{id}$. Set $M := \Delta_X \cup A_1^2 \cup \dots \cup A_k^2$ and observe that $E'' := (E' \cup \Delta_X)M \in \mathcal{E}$. Let $\mathcal{U}''$ be a cover of X which is uniformly bounded with respect to $\mathcal{E}$ and which has appetite E'' and multiplicity at most $n+1$. Note that there are sets $U_1, \dots, U_k \in \mathcal{U}''$ such that $E'[A_i] \cup A_i \subseteq U_i$. We define the cover $\mathcal{U} := \mathcal{U}'' \cup \{U_1 \cup \dots \cup U_k\} \setminus \{U_1, \dots, U_k\}$ of X. Observe that $\mathcal{U}$ is uniformly bounded with respect to $\mathcal{E}_{\text{cn}}$ and has multiplicity at most $n+1$.

Moreover, $\mathcal{U}$ has appetite E. To see this, let $x \in X$. If $x \in A_i$, then

$$E(x) = E'(x) \cup A_{\sigma(i)} \subseteq U_i \cup U_{\sigma(i)} \subseteq U_1 \cup \dots \cup U_k.$$

If $x \notin \{A_1 \cup \dots \cup A_k\}$, then $E(x) = E'(x) \subseteq E''(x)$. □

Proposition 17 (Asymptotic dimension of finite unions)**.** *Let $(X, \mathcal{E})$ be a coarse space and $A, B \subseteq X$ with $A \cup B = X$. Then*

$$\operatorname{asdim}(X, \mathcal{E}) = \max\{\operatorname{asdim}(A, \mathcal{E}|_A), \operatorname{asdim}(B, \mathcal{E}|_B)\}.$$

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[3] Let X be a Hausdorff space and $\overline{X}$ a compactification of X, i.e., X is a dense and open subset of the compact set $\overline{X}$. The collection

$$\mathcal{E}_{\overline{X}} := \{E \subseteq X \times X \mid \overline{E} \subseteq X \times X \cup \Delta_{\overline{X}}\}$$

of all subsets $E \subseteq X \times X$, whose closure meets the boundary $(\overline{X} \times \overline{X}) \setminus (X \times X)$ only in the diagonal, is a connected coarse structure on X.

[4] Let (X, d) be a metric space. Set $\Delta_r := \{(x, y) \in X \times X \mid d(x, y) < r\}$ and define

$$\mathcal{E}_d := \{E \subseteq X \times X \mid E \subseteq \Delta_r \text{ for some } r > 0\}.$$

It is easy to verify that $\mathcal{E}_d$ is the (connected) coarse structure generated by $\{\Delta_r \mid r > 0\}$. It is called the *bounded coarse structure* corresponding to the metric space (X, d).

Proof. The proof of $\geq$ follows from monotonicity. To see $\leq$, we generalize an argument of Bell and Dranishnikov (see [BD01]).

Let n be the maximum of $\operatorname{asdim}(A, \mathcal{E}|_A)$ and $\operatorname{asdim}(B, \mathcal{E}|_B)$ and take a symmetric entourage $L \in \mathcal{E}$ which contains Δ_X. For $\mathcal{U} \subseteq \mathcal{P}(X)$ and $V \subseteq X$ we define

$$N_L(V,\mathcal{U}) \;:=\; V \;\cup \bigcup_{\substack{U \in \mathcal{U} \\ L \cap U \times V \neq \emptyset}} U.$$

There is a uniformly bounded cover $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{n+1}$ of A consisting of L-disjoint families $\mathcal{U}_i$. Moreover, there is a uniformly bounded cover $\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{n+1}$ of B consisting of $(L\Delta_{\mathcal{U}}L\Delta_{\mathcal{U}}L)$-disjoint families $\mathcal{V}_i$. For $i \in \{1, \ldots, n+1\}$ set

$$\mathcal{W}_i := \big\{N_L(V,\mathcal{U}_i) \mid V \in \mathcal{V}_i\big\} \cup \big\{U \in \mathcal{U}_i \mid L \cap U \times V = \emptyset \text{ for all } V \in \mathcal{V}_i\big\}.$$

Observe that $N_L(V,\mathcal{U}_i) \subseteq \Delta_{\mathcal{U}}L[V]$. Hence, we get a uniformly bounded cover $\mathcal{W} = \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_{n+1}$ of X where $\mathcal{W}_i$ is L-disjoint for $1 \leq i \leq n+1$. This proves $\operatorname{asdim}(X, \mathcal{E}) \leq n$. ☐

Proposition 18 (Asymptotic dimension of coproducts)**.** *Let* Λ *be any set and let* $(X_\lambda, \mathcal{E}_\lambda)$ *be a coarse space for every* $\lambda \in \Lambda$. *Define* $X := \coprod_{\lambda \in \Lambda} X_\lambda$. *If* $(X, \mathcal{E})$ *is the coproduct in the category of coarse spaces and uniformly bounded maps, then* $\operatorname{asdim}(X, \mathcal{E}) = \sup_{\lambda \in \Lambda} \operatorname{asdim}(X_\lambda, \mathcal{E}_\lambda)$.

Proof. Set $n := \sup_{\lambda \in \Lambda} \operatorname{asdim}(X_\lambda, \mathcal{E}_\lambda)$. Monotonicity of asymptotic dimension implies $\operatorname{asdim}(X, \mathcal{E}) \geq n$.

We will now prove $\operatorname{asdim}(X, \mathcal{E}) \leq n$. Take an entourage $L \in \mathcal{E}$ which contains Δ_X. Then there are $\lambda_1, \ldots, \lambda_k \in \Lambda$ and $L_{\lambda_i} \in \mathcal{E}_{\lambda_i}$ such that $L = L_{\lambda_1} \cup \cdots \cup L_{\lambda_k} \cup \Delta_X$. For $i \in \{1, \ldots, k\}$ choose a uniformly bounded cover $\mathcal{U}_{\lambda_i}$ of X_{λ_i} with appetite L_{λ_i} and multiplicity at most $\operatorname{asdim}(X_{\lambda_i}, \mathcal{E}_{\lambda_i}) + 1$. For $\lambda \in \Lambda \backslash \{\lambda_1, \ldots, \lambda_k\}$ set $\mathcal{U}_\lambda := \{\{x\} \mid x \in X_\lambda\}$. The union $\mathcal{U} := \bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda$ is a uniformly bounded cover of X with appetite L whose multiplicity does not exceed $n+1$. ☐

Definition 19. For $i \in \{1, \ldots, k\}$ let $(X_i, \mathcal{E}_i)$ be a coarse space. By $p_i \colon X_1 \times \cdots \times X_k \to X_i$ we denote the projection to the i-th factor. The *product coarse structure* is defined as follows:

$$\mathcal{E}_1 * \cdots * \mathcal{E}_k := \big\{E \subseteq (X_1 \times \cdots \times X_k)^2 \mid (p_i \times p_i)(E) \in \mathcal{E}_i \text{ for } i \in \{1, \ldots, k\}\big\}.$$

If $(X, \mathcal{E})$ is a coarse space, we will sometimes write $\mathcal{E}^{*k}$ for the product coarse structure on X^k.

It is easy to prove that $\mathcal{E}_1 * \cdots * \mathcal{E}_k$ actually is a coarse structure. The product coarse structure $\mathcal{E}_1 * \cdots * \mathcal{E}_k$ is connected if and only if the coarse structures $\mathcal{E}_i$ are connected. Moreover, we have the following formulas:

$$\Delta_{X_1 \times \cdots \times X_k} = \Delta_{X_1} \times \cdots \times \Delta_{X_k}$$
$$(E_1 \times \cdots \times E_k)(E_1' \times \cdots \times E_k') = E_1E_1' \times \cdots \times E_kE_k'.$$

One remark on our notation: If $E \subseteq X \times X$, we should not confuse the composition $E^k = E \cdots E$ and the product $E^{\times k} = E \times \cdots \times E$.

Proposition 20 (Asymptotic dimension of products)**.** *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces. Then*

$$\operatorname{asdim}(X \times Y, \mathcal{E}_X * \mathcal{E}_Y) \leq \operatorname{asdim}(X, \mathcal{E}_X) + \operatorname{asdim}(Y, \mathcal{E}_Y),$$
$$\operatorname{asdim}(X, \mathcal{E}_X) \leq \operatorname{asdim}(X \times Y, \mathcal{E}_X * \mathcal{E}_Y) \quad \text{if } Y \neq \emptyset.$$

Proof. Compare [Roe03] for the special case of bounded coarse structures.

Set $n := \operatorname{asdim}(X)$ and $m := \operatorname{asdim}(Y)$. Let $E \in \mathcal{E}_X * \mathcal{E}_Y$. There are symmetric entourages $E_X \in \mathcal{E}_X$ and $E_Y \in \mathcal{E}_Y$ containing the diagonals Δ_X and Δ_Y respectively such that $E \subseteq E_X \times E_Y$.

There is a uniformly bounded cover $\mathcal{U}$ of X with appetite E_X^{n+m+1} and multiplicity $\mu(\mathcal{U}) \leq n+1$. There is also a uniformly bounded cover $\mathcal{V}$ of Y with appetite E_Y^{n+m+1} and multiplicity $\mu(\mathcal{V}) \leq m+1$. We get a uniformly bounded cover $\mathcal{U} \times \mathcal{V} := \{U \times V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$ of $X \times Y$ with appetite E^{n+m+1} and multiplicity $\leq (n+1)\cdot(m+1) = n\cdot m + n + m + 1$. Thus, we need to improve the multiplicity.

We proceed similar as in the proof of Theorem 9 and begin with some definitions. Let $k \in \{2, \ldots, n+m+2\}$. Set

$$\begin{aligned}
\mathcal{A}_k &:= \{U_1 \cap \cdots \cap U_p \times V_1 \cap \cdots \cap V_q \mid \\
&\qquad\qquad p + q = k, U_i \in \mathcal{U}, V_i \in \mathcal{V} \text{ pairwise distinct } \}\\
B_k &:= \bigcup_{A \in \mathcal{A}_k} \operatorname{Int}_{E^{n+m+3-k}}(A) \quad \text{ and } \quad B_{n+m+3} := \emptyset\\
\mathcal{W}_k &:= \{\operatorname{Int}_{E^{n+m+3-k}}(U) \setminus B_{k+1} \mid U \in \mathcal{A}_k\}\\
\mathcal{W} &:= \mathcal{W}_2 \cup \cdots \cup \mathcal{W}_{n+m+2}.
\end{aligned}$$

Notice that $\mathcal{W}$ is a uniformly bounded cover of $X \times Y$ consisting of the $n+m+1$ disjoint families $\mathcal{W}_2, \ldots, \mathcal{W}_{n+m+2}$. It remains to prove that $\mathcal{W}_k$ is E-disjoint for $k \in \{2, \ldots, n+m+2\}$.

For this purpose let $M, N \in \mathcal{W}_k$ with $M \neq N$ and suppose $M \times N \cap E \neq \emptyset$. Choose $((x_M, y_M), (x_N, y_N)) \in M \times N \cap E$. There are $p_M, q_M \in \mathbb{N}$ with $p_M + q_M = k$ and $M_1, \ldots, M_{p_M} \in \mathcal{U}$, $M_1', \ldots, M_{q_M}' \in \mathcal{V}$ such that

$$M = \operatorname{Int}_{E^{n+m+3-k}}(\underbrace{M_1 \cap \cdots \cap M_{p_M} \times M_1' \cap \cdots \cap M_{q_M}'}_{=:MM'}) \setminus B_{k+1}.$$

Similarly, there are $p_N, q_N \in \mathbb{N}$ with $p_N + q_N = k$ and sets $N_1, \ldots, N_{p_N} \in \mathcal{U}$ and $N_1', \ldots, N_{q_N}' \in \mathcal{V}$ such that

$$N = \operatorname{Int}_{E^{n+m+3-k}}(\underbrace{N_1 \cap \cdots \cap N_{p_N} \times N_1' \cap \cdots \cap N_{q_N}'}_{=:NN'}) \setminus B_{k+1}.$$

Observe that the following relations hold:

$$\begin{aligned}
&(x_M, y_M) \notin B_{k+1}\\
&(x_M, y_M) \in M \ \subseteq \ \operatorname{Int}_{E^{n+m+2-k}}(MM')\\
&(x_M, y_M) \in E[N] \ \subseteq \ E[\operatorname{Int}_{E^{n+m+3-k}}(NN')] \ \subseteq \ \operatorname{Int}_{E^{n+m+2-k}}(NN').
\end{aligned}$$

It follows that $(x_M, y_M) \in \operatorname{Int}_{E^{n+m+3-(k+1)}}(MM' \cap NN')$. Since $M \neq N$, the set $\{M_1, \ldots, M_{p_M}, M_1', \ldots, M_{q_M}', N_1, \ldots, N_{p_N}, N_1', \ldots, N_{q_N}'\}$ contains at least $k+1$

different elements. Hence $(x_M, y_M) \in B_{k+1}$. But this is a contradiction to what we found before. □

The equality $\operatorname{asdim}(X \times Y) = \operatorname{asdim}(X) + \operatorname{asdim}(Y)$ is not true in general. Compare [BL] and Corollary 5.9 of [Gra06].

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