New York Journal of Mathematics

New York J. Math. **10** (2004) 209–229.

A classification result for simple real approximate interval algebras

P. J. Stacey

Abstract. A classification in terms of K-theory and tracial states is obtained for those real structures which are compatible with the inductive limit structure of a simple C^*-inductive limit of direct sums of algebras of continous matrix valued functions on an interval.

Contents

1. Introduction 209
2. A uniqueness theorem 212
3. Injective connecting maps and approximate divisibility 217
4. An existence result 221
5. The classification theorem 226
References 228

1. Introduction

There has been remarkable progress in recent years in the classification of simple amenable C^*-algebras, following the program set down by George Elliott. See, for example, the surveys [7], [13], [16].

By contrast there has been little attention paid to real C^*-algebras other than the real AF-algebras considered in [9], [12], [19]. The purpose of the present paper is to show, by concentrating on the very basic example of real AI-algebras, that it can be expected that there will be appropriate real counterparts to all the complex results.

Many of the classification results for simple C^*-algebras have exploited an assumed inductive limit structure in the algebra. It is not clear that, if the complexification of a real C^*-algebra possesses such an inductive limit structure, then so does the algebra itself: this is open even for the CAR (or 2^∞ UHF) algebra. Therefore the present paper will restrict attention to the situation where the real algebra

Received October 14, 2003.

Mathematics Subject Classification. 46L35, 46L05.

Key words and phrases. Real C^*-algebras, classification, AI-algebras.

ISSN 1076-9803/04

does have such an inductive limit structure, giving an AI-structure in its complexification. More precisely the real algebras will be assumed to be inductive limits, under real $*$-homomorphisms, of algebras A_n, where the complexification $A_n \otimes_{\mathbb{R}} \mathbb{C}$ of A_n is a direct sum of algebras $C([0,1], M_q(\mathbb{C}))$ of continuous $q \times q$ matrix valued functions on $[0,1]$, for varying $q \geq 1$. Equivalently, $A_n = \{a \in B : \Phi(a) = a^*\}$ where Φ is an involutory $*$-antiautomorphism of a direct sum B of algebras of the form $C([0,1], M_q(\mathbb{C}))$. If e is a minimal central projection in B then either $\Phi(e) = e$, in which case Φ restricts to an antiautomorphism of $eB \cong C([0,1], M_q(\mathbb{C}))$ for some q, or $\Phi(e) \neq e$, in which case Φ interchanges the two summands of $(e + \Phi(e))B \cong C([0,1], M_q(\mathbb{C})) \oplus C([0,1], M_q(\mathbb{C}))$. In the latter case, the associated real algebra $\{(eb, \Phi(eb)^*) : b \in B\}$ is (real linearly) isomorphic to $C([0,1], M_q(\mathbb{C}))$. In the former, the restriction of Φ to the centre $C([0,1], \mathbb{C})$ gives rise to a period 2 homeomorphism of $[0,1]$, which is conjugate either to the identity map id or the reflection $1 - \mathrm{id}$. It follows that Φ is conjugate to an antiautomorphism for which the restriction to the centre is either the identity or satisfies $(\Phi f)(t) = f(1-t)$ for each $f \in C([0,1], \mathbb{C})$ and each $t \in [0,1]$.

When Φ restricts to the identity on $C([0,1], \mathbb{C})$, the proof of Theorem 3.3 of [17], together with the remarks before that theorem, show that the real algebra associated with Φ is the cross-section algebra of a locally trivial bundle over $[0,1]$ with fibres either all isomorphic to $M_q(\mathbb{R})$ or all isomorphic to $M_{q/2}(\mathbb{H})$. All such bundles over $[0,1]$ are trivial and hence the associated real algebra is isomorphic either to $C([0,1], M_q(\mathbb{R})$ or $C([0,1], M_{q/2}(\mathbb{H}))$. Here $\mathbb{H}$ denotes the algebra of quaternions, which can be identified with the real subalgebra of $M_2(\mathbb{C})$ generated by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

When the restriction of Φ to $C([0,1])$ satisfies $(\Phi f)(t) = f(1-t)$ then for each $t \in [0,1]$ there exists an antiautomorphism Φ_t of $M_q(\mathbb{C})$ such that $(\Phi f)(t) = \Phi_t(f(1-t))$ for each $f \in C([0,1], M_q(\mathbb{C}))$ and $\Phi_t \Phi_{1-t} = \mathrm{id}$ for each $t \in [0,1]$. (One way of seeing this is to note that if $(\Psi f)(t) = f(1-t)^{\mathrm{tr}}$, where tr denotes the transpose, then $\Phi\Psi$ restricts to the identity on $C([0,1], \mathbb{C})$ and hence is inner, by 1.6 of [15].) It follows that the restriction map onto $[0, \frac{1}{2}]$ is an isomorphism on eBe, with image $\{f \in C([0, \frac{1}{2}], M_q(\mathbb{C})) : \Phi_{\frac{1}{2}}(f(\frac{1}{2})) = f(\frac{1}{2})^*\}$. Furthermore, there exists an automorphism $\mathrm{Ad}u$ of $M_q(\mathbb{C})$ such that $\mathrm{Ad}u(\{A : \Phi_{\frac{1}{2}}(A) = A^*\})$ is either $M_q(\mathbb{R})$ or $M_{q/2}(\mathbb{H})$. Regarding u as a constant function on $[0, \frac{1}{2}], \mathrm{Ad}(u)$ then gives an isomorphism from $\{f \in C([0, \frac{1}{2}], M_q(\mathbb{C})) : \Phi_{\frac{1}{2}}(f(\frac{1}{2})) = f(\frac{1}{2})^*\}$ onto either $\{f \in C([0, \frac{1}{2}], M_q(\mathbb{C})) : f(\frac{1}{2}) \in M_q(\mathbb{R})\}$ or $\{f \in C([0, \frac{1}{2}], M_q(\mathbb{C})) : f(\frac{1}{2}) \in M_{q/2}(\mathbb{H})\}$.

So the basic building blocks to consider are $C([0,1], M_q(\mathbb{C}))$, $C([0,1], M_q(\mathbb{R}))$, $C([0,1], M_{q/2}(\mathbb{H}))$ for q even,

$$\begin{aligned} A(q, \mathbb{R}) &= \{f \in C([0,1], M_q(\mathbb{C})) : f(t) = \overline{f(1-t)} \text{ for } 0 \leq t \leq 1\} \\ &\cong \left\{ f \in C\left(\left[0, \frac{1}{2}\right], M_q(\mathbb{C})\right) : f\left(\frac{1}{2}\right) \in M_q(\mathbb{R}) \right\} \end{aligned}$$

and

$$\begin{aligned} A(q/2, \mathbb{H}) &= \{f \in C([0,1], M_q(\mathbb{C})) : f(t) = \Phi_{\mathbb{H}}(f(1-t))^* \text{ for } 0 \leq t \leq 1\} \\ &\cong \left\{ f \in C\left(\left[0, \frac{1}{2}\right], M_q(\mathbb{C})\right) : f\left(\frac{1}{2}\right) \in M_{q/2}(\mathbb{H}) \right\} \end{aligned}$$

for q even, where $\Phi_{\mathbb{H}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Note that, arising from the usual identification $C([0,1], M_q(\mathbb{R})) = C([0,1],\mathbb{R}) \otimes_{\mathbb{R}} M_q(\mathbb{R})$, $A(q,\mathbb{R}) = A(1,\mathbb{R}) \otimes_{\mathbb{R}} M_q(\mathbb{R})$ and $A(q/2,\mathbb{H}) = A(1,\mathbb{R}) \otimes_{\mathbb{R}} M_{q/2}(\mathbb{H})$ and that $A(1,\mathbb{R})$ is generated as a real C^*-algebra by the constant 1 and the skew-adjoint map $g : t \mapsto i(\frac{1}{2} - t)$. To see the latter claim, note that 1 and g generate $C([0,1],\mathbb{C})$ as a complex algebra and the real algebra they generate is contained in (and hence is equal to)

$$\{f \in C([0,1],\mathbb{C}) : f(t) = \overline{f(1-t)} \text{ for all } t\}.$$

As with the complex case, considered in [6], the classification of simple real AI algebras uses tracial states and the pairing of traces with K_0. It is thus appropriate to recall that, as described in Chapter 14 of [11], a state k on a unital real C^*-algebra A is a positive linear map $k : A \to \mathbb{R}$ which, by definition, satisfies $k(1) = 1$ and $k(a) = k(a^*)$ for each $a \in A$. Each such positive map extends uniquely to a complex linear state $k : A^{\mathbb{C}} \to \mathbb{C}$, where $A^{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of A. Furthermore $\Phi_A^* k = k$, where $\Phi_A^* k = k \circ \Phi_A$ and where $\Phi_A(a + ib) = a^* + ib^*$ for $a, b \in A$, so $A = \{a \in A^{\mathbb{C}} : \Phi_A(a) = a^*\}$. Conversely, each complex state k of $A^{\mathbb{C}}$ satisfying $\Phi_A^* k = k$ restricts to a real state of A (but unless $\Phi_A^* k = k$ the restriction may not satisfy $k(a) = k(a^*)$). This correspondence produces a bijection between the real tracial states of A and the tracial states τ of $A^{\mathbb{C}}$ satisfying $\Phi_A^* \tau = \tau$. The unique extension map from the real tracial state space $T(A)$ of A to the tracial state space $T(A^{\mathbb{C}})$ of $A^{\mathbb{C}}$ produces a map $\mathrm{Aff}(T(A^{\mathbb{C}})) \to \mathrm{Aff}(T(A))$ between the associated spaces of continuous real affine functions and the affine automorphism Φ_A^* of $T(A^{\mathbb{C}})$ produces an involution $\hat{\Phi}_A$ on $\mathrm{Aff}(T(A^{\mathbb{C}}))$ by $\hat{\Phi}_A a = a \circ \Phi_A^*$. Furthermore the natural map $\theta : K_0(A^{\mathbb{C}}) \to \mathrm{Aff}(T(A^{\mathbb{C}}))$ automatically gives rise to $\theta : K_0(A) \to \mathrm{Aff}(T(A))$ by means of the following diagram:

$$\begin{array}{ccc} K_0(A) & \longrightarrow & \mathrm{Aff}(T(A)) \\ \big\downarrow & & \big\uparrow \\ K_0(A^{\mathbb{C}}) & \longrightarrow & \mathrm{Aff}(T(A^{\mathbb{C}})). \end{array}$$

If a positive unital map M from $\mathrm{Aff}(T(A^{\mathbb{C}}))$ to $\mathrm{Aff}(T(B^{\mathbb{C}}))$ obeys $\hat{\Phi}_B M = M \hat{\Phi}_A$ then it gives rise to a map from $\mathrm{Aff}(T(A))$ to $\mathrm{Aff}(T(B))$ and if a map ϕ from $T(A^{\mathbb{C}})$ to $T(B^{\mathbb{C}})$ satisfies $\phi \Phi_A^* = \Phi_B^* \phi$ then it gives rise to a map from $T(A)$ to $T(B)$. However an isomorphism from $T(A)$ to $T(B)$ does not necessarily extend to an isomorphism from $T(A^{\mathbb{C}})$ to $T(B^{\mathbb{C}})$, for example if $A = \mathbb{R}$ and $B = \mathbb{C}$, and in the classification result 5.3 the tracial state space of the complexification is part of the invariant.

When $A = C([0,1],\mathbb{R})$, Φ^* is the identity on the set $M_1^+[0,1]$ of Borel probability measures on $[0,1]$, so that $T(A)$ is identified with $T(A^{\mathbb{C}})$. $\mathrm{Aff}(T(A^{\mathbb{C}}))$ can be identified with $C([0,1],\mathbb{R})$ and $\hat{\Phi}_A = \mathrm{id}$. When $A = A(1,\mathbb{R}) = \{f \in C([0,1],\mathbb{C}) : f(t) = \overline{f(1-t)}\}$ then $(\Phi^* \mu)(E) = \mu(1 - E)$ for each $\mu \in M_1^+[0,1]$ and for each Borel set E in $[0,1]$, so that $T(A)$ is identified with $\{\mu : \mu(E) = \mu(1-E) \text{ for all } E\} \cong M_1^+[0,\frac{1}{2}]$. $\mathrm{Aff}(T(A^{\mathbb{C}}))$ can be identified with $C([0,1],\mathbb{R})$ and $(\hat{\Phi}_A f)(t) = f(1-t)$ for each $f \in T(A^{\mathbb{C}})$ and each $0 \le t \le 1$. When $A = C([0,1],\mathbb{C})$ then $\Phi^*(\mu,\nu) = (\nu,\mu)$ for $(\mu,\nu) \in M_1^+[0,1] \oplus M_1^+[0,1]$, so that $T(A)$ can be identified with $\{(\mu,\mu) : \mu \in M_1^+[0,1]\} \cong M_1^+[0,1]$. $\mathrm{Aff}(T(A^{\mathbb{C}}))$ can be identified with $C([0,1],\mathbb{R}) \oplus C([0,1],\mathbb{R})$

and $\hat{\Phi}_A(f,g) = (g,f)$ for each $f,g \in C([0,1],\mathbb{R})$. In each case, taking the tensor product with $M_q(\mathbb{R})$ or $M_{q/2}(\mathbb{H})$ does not change the tracial state space.

2. A uniqueness theorem

Let A, B be finite direct sums of basic building blocks and let ϕ,ψ be unital homomorphisms from A to B with complexifications $\phi^{\mathbb{C}},\psi^{\mathbb{C}}$ from $A^{\mathbb{C}}$ to $B^{\mathbb{C}}$. Theorem 6 of [6] gives sufficient conditions for there to exist a unitary $u \in B^{\mathbb{C}}$ such that $\phi^{\mathbb{C}}$ and $(\mathrm{Ad}u)\psi^{\mathbb{C}}$ agree to within $\frac{3}{n}$ on the canonical generators of $A^{\mathbb{C}}$. In the present section a minor variation of this result is obtained, with slightly strengthened conditions, which enable the unitary u to be chosen to belong to B. The first three lemmas enable reduction to the cases where $A = C([0,1],\mathbb{R})$ or $A = A(1,\mathbb{R}) = \{f \in C([0,1],\mathbb{C}) : f(t) = \overline{f(1-t)}\}$. The first lemma reduces to the case of a single block.

Lemma 2.1. *Let A and B be finite direct sums of basic building blocks and let ϕ,ψ be unital homomorphisms from A to B giving rise to the same map from $K_0(A)$ to $K_0(B)$. Then there exists a unitary $u \in B$ such that $\phi(e) = u\psi(e)u^*$ for each minimal central projection $e \in A$.*

Proof. From the K_0 equalities $[\phi(e)] = [\psi(e)]$ and $[1-\phi(e)] = [1-\psi(e)]$ it follows by Propositions 4.2.5 and 4.6.5 of [1], which also apply to real algebras, that there exists $u_e \in B$ with $\phi(e) = u_e\psi(e)u_e^*$. Then $u = \sum_e \phi(e)u_e\psi(e)$ is a unitary with $\phi(e) = u\psi(e)u^*$ for each minimal central projection $e \in A$. $\square$

The next lemma reduces to the case $A = C([0,1],\mathbb{R})$ or $A = A(1,\mathbb{R})$, except when the centre of A is isomorphic to $C([0,1],\mathbb{C})$.

Lemma 2.2. *Let A be a basic building block with a unital subalgebra C isomorphic to $M_q(\mathbb{R})$ or $M_{q/2}(\mathbb{H})$ for some q. If ϕ,ψ are homomorphisms from A to a finite direct sum B of basic building blocks with $\phi(1) = \psi(1) = e$ then there exists a unitary $v \in eBe$ with $\phi(c) = v\psi(c)v^*$ for each $c \in C$.*

Proof. It suffices to consider the case where eBe has a single summand, which will be of the form $Z \otimes_{\mathbb{R}} M_q(\mathbb{R})$ or $Z \otimes_{\mathbb{R}} M_{q/2}(\mathbb{H})$ where Z, the centre of eBe, is either isomorphic to $C([0,1],\mathbb{R})$, $C([0,1],\mathbb{C})$ or $A(1,\mathbb{R})$. In each case $\psi(C)$ and $\phi(C)$ induce tensor product decompositions of eBe of the form $\psi(C) \otimes_{\mathbb{R}} C_\psi \otimes_{\mathbb{R}} Z$ and $\phi(C) \otimes_{\mathbb{R}} C_\phi \otimes_{\mathbb{R}} Z$ where C_ψ and C_ϕ are subalgebras of eBe isomorphic to the same full real or quaternionic matrix algebra. Thus there is an automorphism γ of eBe, equal to the identity on Z, with $\gamma\psi(c) = \phi(c)$ for each $c \in C$. By Lemma 1.6 of [15] the complexification of γ on $eB^{\mathbb{C}}e$, which is isomorphic to $C([0,1],M_q(\mathbb{C}))$ or $C([0,1],M_q(\mathbb{C})^2)$, is inner. If $\gamma = \mathrm{Ad}u$ and Φ is the involutory antiautomorphism of $eB^{\mathbb{C}}e$ corresponding to eBe, then $\gamma\Phi = \Phi\gamma$ so $w = u^*\Phi(u^*) \in Z^{\mathbb{C}}$ and $\Phi(w) = w$. The centre $Z^{\mathbb{C}}$ of $eB^{\mathbb{C}}e$ is isomorphic either to $C([0,1],\mathbb{C})$ or $C([0,1],\mathbb{C}^2)$. When $Z^{\mathbb{C}}$ is isomorphic to $C([0,1],\mathbb{C})$ then Φ either satisfies $\Phi f = f$ or $(\Phi f)(t) = f(1-t)$ for all $f \in C([0,1],\mathbb{C})$, so there exists a unitary square root $w^{1/2}$ of w in $Z^{\mathbb{C}}$ with $\Phi(w^{1/2}) = w^{1/2}$. When $Z^{\mathbb{C}}$ is isomorphic to $C([0,1],\mathbb{C}^2)$ then $\Phi(f,g) = (g,f)$ for each $f,g \in C([0,1],\mathbb{C})$. Therefore, in this case as well, there exists a unitary square root $w^{1/2}$ of w in $Z^{\mathbb{C}}$ with $\Phi(w^{1/2}) = w^{1/2}$. Then $\Phi(w^{1/2}u) = \Phi(u)w^{1/2} = u^*w^*w^{1/2} = u^*w^{1/2*} = (w^{1/2}u)^*$ and $\gamma = \mathrm{Ad}(w^{1/2}u)$, as required. $\square$

The remaining case is when the centre of A is isomorphic to $C([0,1],\mathbb{C})$.

Lemma 2.3. *Let A be a basic building block $C([0,1],M_q(\mathbb{C}))$ and let ϕ,ψ be real-linear homomorphisms from A to a finite direct sum B of basic building blocks, with $\phi(1)=\psi(1)=e$, giving rise to the same map from $K_0(A^{\mathbb{C}})$ to $K_0(B^{\mathbb{C}})$. Then there exists a unitary $v\in eBe$ with $\phi(c)=v\psi(c)v^*$ for each $c\in C$, the algebra of constant functions in A.*

Proof. It suffices to consider the case where eBe has a single summand. If D is a subalgebra of C isomorphic to $M_q(\mathbb{R})$ then, by Lemma 2.2, there exists $u\in eBe$ with $\phi(d)=u\psi(d)u^*$ for $d\in D$. Replacing ψ by $\mathrm{Ad}(u)\circ\psi$ and eBe by the commutant of $\phi(D)$ in eBe, it therefore further suffices to consider the case where $A=C([0,1],\mathbb{C})$ so $C=\mathbb{C}1$. Then $C^{\mathbb{C}}$ will be isomorphic to $\mathbb{C}^2$, with C embedded as $\{(z,\bar{z}):z\in\mathbb{C}\}$. From the K_0 equalities $[\phi^{\mathbb{C}}(1,0)]=[\psi^{\mathbb{C}}(1,0)]$ and $[\phi^{\mathbb{C}}(0,1)]=[\psi^{\mathbb{C}}(0,1)]$ it follows that there is a unitary u in $eB^{\mathbb{C}}e$ with

$$\begin{aligned}u\phi(i)u^* &= u\phi^{\mathbb{C}}(i,-i)u^* = iu\phi^{\mathbb{C}}(1,0)u^* - iu\phi^{\mathbb{C}}(0,1)u^* = i\psi^{\mathbb{C}}(1,0) - i\psi^{\mathbb{C}}(0,1)\\ &= \psi^{\mathbb{C}}(i,-i) = \psi(i).\end{aligned}$$

Let $P=\phi^{\mathbb{C}}(1,0)$, so $\phi(i)=iP-i(e-P)$, and let Φ be the involutory antiautomorphism of $eB^{\mathbb{C}}e$ corresponding to eBe.

From $\Phi(\phi(i))=\phi(i)^*=-\phi(i)$ it follows that $\Phi(P)=e-P$; from $\Phi(u\phi(i)u^*)=\Phi(\psi(i))=-\psi(i)=-u\phi(i)u^*$ it follows that $\Phi(u^*)\phi(i)\Phi(u)=u\phi(i)u^*$ and hence $\Phi(u^*)P\Phi(u)=uPu^*$. Let $v=uP+\Phi(u^*)(e-P)$. Then $\Phi(v)=(e-P)\Phi(u)+Pu^*=v^*$, $vv^*=uPu^*+\Phi(u^*)(e-P)\Phi(u)=uPu^*+u(e-P)u^*=e$ and

$$\begin{aligned}v\phi(i)v^* &= [uP+\Phi(u^*)(e-P)][iP-i(e-P)][Pu^*+(e-P)\Phi(u)]\\ &= iuPu^* - i\Phi(u^*)(e-P)\Phi(u)\\ &= iuPu^* - iu(e-P)u^*\\ &= u\phi(i)u^* = \psi(i).\end{aligned}$$

Since B is finite, $v^*v=e$, so v is the required unitary. □

The proof of the appropriate version of Theorem 6 of [6] is thus reduced to the cases $A=C([0,1],\mathbb{R})$ or $A=A(1,\mathbb{R})$, both of which have $A^{\mathbb{C}}=C([0,1],\mathbb{C})$, with B a single building block. It is then required to find $u\in B$ such that $\phi^{\mathbb{C}}$ and $(\mathrm{Ad}u)\psi^{\mathbb{C}}$ agree to within $\frac{3}{n}$ on the generator $h(t)=t$ of $C([0,1],\mathbb{C})$. This will be achieved by obtaining a diagonal (or other canonical) form for the images of $\phi(h)$ and $\psi(h)$ in the case $A=C([0,1],\mathbb{R})$ and for the images of $\phi(g)$ and $\psi(g)$ in the case $A=A(1,\mathbb{R})$, where $g(t)=i(\frac{1}{2}-t)$ is a skew-adjoint generator for $A(1,\mathbb{R})$.

Lemma 2.4. *Let $\epsilon>0$, let B be a basic building block with $B^{\mathbb{C}}=C([0,1],M_q(\mathbb{C}))$ or $B=C([0,1],M_q(\mathbb{C}))$ and let $f\in B$ satisfy $f=kf^*$ where $k=\pm1$.*

(a) *Unless $k=1$ and either $B=C([0,1],M_{q/2}(\mathbb{H}))$ or $B=A(q/2,\mathbb{H})$ then there exists $g\in B$ with $g=kg^*$ and $\|g-f\|<\epsilon$ such that, for each $0\le t\le 1$, $g(t)$ has q distinct complex eigenvalues.*

(b) *When $f=f^*$ and $B=A(q/2,\mathbb{H})$ there exists $g\in B$ with $g=g^*$ and $\|g-f\|<\epsilon$ such that, for each $t\neq\frac{1}{2}$, $g(t)$ has q distinct complex eigenvalues and $g(\frac{1}{2})$ has $q/2$ distinct eigenvalues each of multiplicity 2. Furthermore, g can be chosen to have continuous eigenprojections.*

(c) *When $f = f^*$ and $B = C([0,1], M_{q/2}(\mathbb{H}))$ there exists $g \in B$ with $g = g^*$ and $\|g - f\| < \epsilon$ such that, for each $0 \le t \le 1$, $g(t) = \sum_{j=1}^{q/2} \lambda_j(t) P_j(t)$ where $t \mapsto P_j(t)$ is a continuous family of two-dimensional projections and $t \mapsto \lambda_j(t)$ is a continuous real-valued function for each $1 \le j \le q/2$.*

Proof. The proof is identical to the relevant part of the proof of Theorem 4 of [3] except for the choices needed to ensure that g belongs to B. Firstly note that any skew-adjoint element of $M_q(\mathbb{R}), M_q(\mathbb{C})$ or $M_{q/2}(\mathbb{H})$ or any self-adjoint element of $M_q(\mathbb{R})$ or $M_q(\mathbb{C})$ can be given an arbitrarily small perturbation to produce a skew-adjoint or self-adjoint element with q distinct complex eigenvalues. Any self-adjoint element of $M_{q/2}(\mathbb{H})$ (regarded as an element of $M_q(\mathbb{C})$) necessarily has each eigenvalue of even multiplicity, but it can be given an arbitrarily small perturbation to produce a self-adjoint element with $q/2$ distinct eigenvalues, each of multiplicity 2.

Thus when f is approximated arbitrarily closely by a piecewise linear element of B then, except in case (c), the approximation can be taken to have q distinct complex eigenvalues at one point and hence at all but finitely many points. In case (c) it can be arranged that there are $q/2$ distinct eigenvalues, each of multiplicity two, except at finitely many points. As in [3] by passing to subintervals there can be assumed to be only one such point. In the self-adjoint case, for which the eigenvalues are real, small constant perturbations give a reduction to the case where just two eigenvalues coincide at each of the degenerate points. In the skew adjoint case, for which the eigenvalues are purely imaginary, at a point t for which $\Phi(f(t)) = f(t)^*$ for an antiautomorphism Φ of $M_q(\mathbb{C})$ the eigenvalues occur in complex conjugate pairs with orthogonal eigenprojections $P(t)$ and $\Phi(P(t))$. When $B = C([0,1], M_q(\mathbb{R}))$ or $B = C([0,1], M_{q/2}(\mathbb{H}))$ this holds for all t and suitable perturbations are obtained by adding small imaginary constants $i\epsilon_j, -i\epsilon_j$ to each pair $\lambda_j(t), \overline{\lambda_j(t)}$ of corresponding eigenvalues. The perturbation $\epsilon_j(t) = i\epsilon_j P_j(t) - i\epsilon_j \Phi(P_j(t))$ of $f(t)$ satisfies $\Phi(\epsilon_j(t)) = \epsilon_j(t)^*$ for each t, so belongs to B. When $B = A(q, \mathbb{R})$ or $A(q/2, \mathbb{H})$ the small imaginary constants $i\epsilon_j, -i\epsilon_j$ are added to pairs of eigenvalues $\lambda_j(t), \lambda_j'(t)$ for which $\lambda_j(\frac{1}{2}) = \overline{\lambda_j'(\frac{1}{2})}$.

If at the remaining single point t_0 of pairwise degeneracy the corresponding eigenvalue functions $\lambda_j(t), \lambda_k(t)$ touch but do not cross at t_0, then in the skew adjoint case the corresponding complex conjugate functions also touch and the degeneracy (other than the forced double degeneracy when $f = f^*$ and $B = C([0,1], M_{q/2}(\mathbb{H}))$ or $B = A(q/2, \mathbb{H})$), can be entirely removed by either a small real perturbation to $\lambda_j(t)$ in the self-adjoint case or a pair of conjugate purely imaginary perturbations to $\lambda_j(t), \overline{\lambda_j(t)}$ in the skew-adjoint case.

If the eigenvalue functions λ_j and λ_k cross at t_0 and have eigenprojections P_j and P_k then, in the self-adjoint case, consider $\lambda_j P_j + \lambda_k P_k$ which belongs to B. Firstly pick an interval $[a, b]$ containing t_0 on which $\lambda_j P_j + \lambda_k P_k$ is sufficiently close to $\lambda_j(t_0) P_j(t_0) + \lambda_k(t_0) P_k(t_0)$, with $\lambda_j(a) < \lambda_k(a)$ and $\lambda_j(b) > \lambda_k(b)$. Then let $\{Q(t) : a \le t \le b\}$ be a path of projections with $Q(t) \le P_j(t) + P_k(t)$, $Q(a) = P_j(a)$ and $Q(b) = P_k(b)$. The combination $\min(\lambda_j, \lambda_k) Q + \max(\lambda_j, \lambda_k)(P_j + P_k - Q)$ agrees with $\lambda_j P_j + \lambda_k P_k$ at a and b, is close to $\lambda_j P_j + \lambda_k P_k$ on $[a, b]$ and has touching rather than crossing eigenvalue functions at t_0, which can be removed as before. In the skew adjoint case a slight modification of this approach is needed when $B = C([0,1], M_q(\mathbb{R}))$ or $B = C([0,1], M_{q/2}(\mathbb{H}))$. If Φ is the corresponding antiautomorphism of $M_q(\mathbb{C})$ then consider $\lambda_j(P_j - \Phi P_j) + \lambda_k(P_k - \Phi P_k)$. The simultaneous crossings of λ_j with λ_k and $\bar{\lambda}_j = -\lambda_j$ with $\bar{\lambda}_k = -\lambda_k$ can be removed simultaneously using a path $Q + \Phi(Q)$ of projections with $Q(t) \leq P_j(t) + P_k(t)$ and an appropriate combination of $Q + \Phi(Q)$ and $P_j + P_k + \Phi(P_j) + \Phi(P_k) - Q - \Phi(Q)$.

The resulting perturbation has q distinct eigenvalues at each point except when $f = f^*$ and $B = C([0,1], M_{q/2}(\mathbb{H}))$ or $B = A(q/2, \mathbb{H})$, when it has only the forced double degeneracies. The construction produces continuous eigenvalues and continuous eigenprojections, which are of rank 2 when $B = C([0,1], M_{q/2}(\mathbb{H}))$. □

Lemma 2.5. (a) *Let B be a basic building block with $B^{\mathbb{C}} = C([0,1], M_q(\mathbb{C}))$ or $B = C([0,1], M_q(\mathbb{C}))$, let $f = f^* \in B$ and let $f(t)$ have q distinct eigenvalues for $t \neq \frac{1}{2}$. Then there exists $u \in B$ such that $(ufu^*)(t)$ is real and diagonal for each $0 \leq t \leq 1$.*

(b) *Let $B = C([0,1], M_{q/2}(\mathbb{H}))$ and let $f = f^* = \sum \lambda_j P_j \in A$ where for each $1 \leq j \leq q/2$, $\lambda_j \in C([0,1], \mathbb{R}), P_j \in B$ and, for each $0 \leq t \leq 1, P_j(t)$ is a two-dimensional projection. Then there exists $u \in B$ such that $(ufu^*)(t)$ is real and diagonal for each $0 \leq t \leq 1$.*

(c) *Let $B = C([0,1], M_q(\mathbb{C})), B = C([0,1], M_{q/2}(\mathbb{H}))$ or $B = A(q/2, \mathbb{H})$, let $f = -f^* \in B$ and let $f(t)$ have q distinct eigenvalues for $0 \leq t \leq 1$. Then there exists $u \in B$ such that $(ufu^*)(t)$ is purely imaginary and diagonal for each $0 \leq t \leq 1$.*

(d) *Let $B = C([0,1], M_q(\mathbb{R}))$ or $B = A(q, \mathbb{R})$, let $f = -f^* \in B$ and let $f(t)$ have q distinct eigenvalues for $0 \leq t \leq 1$. Then there exists $u \in B$ such that, for each $0 \leq t \leq 1$, $(wufu^*w^*)(t)$ is purely imaginary and diagonal, where w consists of 2×2 diagonal blocks $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$, together with a 1×1 block if $f(t)$ has a zero eigenvalue for all t.*

Proof. Case (a) is standard linear algebra. In case (b) let K be the antilinear unitary map on $\mathbb{C}^q$ with $K(x_1, x_2, x_3, x_4, \dots) = (-\overline{x}_2, \overline{x}_1, -\overline{x}_4, \overline{x}_3, \dots)$ and let $\Phi(a) = Ka^*K^*$ for each $a \in M_q(\mathbb{C})$. For each $1 \leq j \leq q/2$ let $t \mapsto e_j(t)$ be a continuous choice of elements from $t \mapsto P_j(t)\mathbb{C}^q$. Then the transition map from the standard basis to $\{e_j, Ke_j : 1 \leq j \leq q/2\}$ belongs to $C([0,1], M_{q/2}(\mathbb{H}))$, giving the required result.

In case (c) the result is immediate when $B = C([0,1], M_q(\mathbb{C}))$. When $B = C([0,1], M_{q/2}(\mathbb{H}))$, first pick a continuous choice of eigenvectors $t \mapsto e_j(t)$ associated with $\lambda_j(t)$, then choose $t \mapsto Ke_j(t)$ for the eigenvectors associated with $-\lambda_j(t)$. When $B = A(q/2, \mathbb{H})$, first pick a choice of eigenvectors $t \mapsto e_j(t)$ associated with $\lambda_j(t)$ and then, if $\lambda_j(\frac{1}{2}) = -\lambda_i(\frac{1}{2})$, let $e_i(t) = Ke_j(1-t)$, so the corresponding eigenvalues and eigenprojections satisfy $\lambda_i(t) = -\lambda_j(1-t)$ and $P_i(t) = \Phi P_j(1-t)$. The result then follows as in case (b).

In case (d) when $B = C([0,1], M_q(\mathbb{R}))$, a continuous choice of eigenvectors $t \mapsto e_j(t)$ is first made for $\lambda_j(t)$ and then the choice $t \mapsto \overline{e_j(t)}$ is made for the eigenvalue associated with $\overline{\lambda_j(t)}$. When $B = A(q, \mathbb{R})$ the choice $t \mapsto e_k(t)$, where $e_k(t) = \overline{e_j(1-t)}$, is made for the eigenvector associated with λ_k where $\lambda_k(\frac{1}{2}) = \overline{\lambda_j(\frac{1}{2})}$. After reordering so that $k = j + 1$, the transition matrix from the standard basis to the basis of eigenvectors has adjacent columns of the form $(x_1(t), \dots, x_q(t))$ and

$(\overline{x_1(t)},\ldots,\overline{x_q(t)})$ or $(x_1(t),\ldots,x_q(t))$ and $(\overline{x_1(1-t)},\ldots,\overline{x_q(1-t)})$. Multiplying on the right by w then produces a matrix $u^* \in B$. $\square$

Following Theorem 6 of [6] let the n real functions $h_1,\ldots,h_n$ in $C([0,1],\mathbb{R})$ be defined by

$$h_r(t) = \begin{cases} 0 & 0 \le t \le \frac{r-1}{n} \\ n(t-\frac{r-1}{n}) & \frac{r-1}{n} \le t \le \frac{r}{n} \\ 1 & \frac{r}{n} \le t \le 1 \end{cases}$$

and let k_r be the characteristic function of the interval $[\frac{r}{n},1]$ for $1 \le r \le n-1$, so that $h_r k_r = k_r$ and $k_r h_{r+1} = h_{r+1}$ for each $1 \le r \le n-1$. The following minor variation of Theorem 6 of [6] can now be proved.

Proposition 2.6. *Let A,B be direct sums of basic building blocks and let ϕ and ψ be unital homomorphisms from A to B giving rise to the same map from the pair $K_0(A) \to K_0(A \otimes_{\mathbb{R}} \mathbb{C})$ to the pair $K_0(B) \to K_0(B \otimes_{\mathbb{R}} \mathbb{C})$. Let $n > 0$ be an integer and suppose that for some $\delta > 0$ each primitve quotient in $B^{\mathbb{C}}$ of the image under each of $\phi^{\mathbb{C}}$ and $\psi^{\mathbb{C}}$ of the canonical self adjoint generator of the centre of each minimal direct summand of $A^{\mathbb{C}}$ has at least the fraction δ of its eigenvalues in each of the n consecutive subintervals of $(0,1]$ of length $\frac{1}{n}$. Suppose that the maps from $TB^{\mathbb{C}}$ to $TA^{\mathbb{C}}$ arising from $\phi^{\mathbb{C}}$ and $\psi^{\mathbb{C}}$ agree to strictly within δ on the n central functions $h_1,\ldots,h_n$ of each minimal direct summand of $A^{\mathbb{C}}$.*

It follows that there exists a unitary $u \in B$ such that $\phi^{\mathbb{C}}$ and $(\mathrm{Ad}u)\psi^{\mathbb{C}}$ agree to within $\frac{3}{n}$ on the canonical generators of $A^{\mathbb{C}}$.

Proof. By Lemmas 2.1, 2.2 and 2.3 the proof is reduced to the case where A is either $C([0,1],\mathbb{R})$ or $A(1,\mathbb{R}) = \{f \in C([0,1],\mathbb{C}) : f(t) = \overline{f(1-t)}\}$ and B is a single building block. Let $h(t) = t$ be the self-adjoint generator of $C([0,1],\mathbb{R})$ and $g(t) = i(\frac{1}{2}-t)$ be the skew-adjoint generator of $A(1,\mathbb{R})$. In the latter case, the canonical self-adjoint generator of $C([0,1],\mathbb{C}) = A^{\mathbb{C}}$ is given by $h(t) = \frac{1}{2} + ig(t)$.

By Lemmas 2.4 and 2.5, when $A = C([0,1],\mathbb{R})$, $\phi^{\mathbb{C}}(h)$ and $\psi^{\mathbb{C}}(h)$ can be given arbitrarily small perturbations so that there exist $u_\phi, u_\psi \in B$ with $(\mathrm{Ad}u_\phi)\phi^{\mathbb{C}}(h)$ and $(\mathrm{Ad}u_\psi)\psi^{\mathbb{C}}(h)$ diagonal with elements in increasing order. The proof of Theorem 6 of [6] then applies directly to give the required result.

When $A = A(1,\mathbb{R})$ then, by Lemma 2.4, $\phi(g)$ and $\psi(g)$ can be given an arbitrary small perturbation to have q distinct eigenvalues. When $B = C([0,1],M_q(\mathbb{C}))$, $B = C([0,1],M_{q/2}(\mathbb{H}))$ or $B = A(q/2,\mathbb{H})$ there therefore exist $u_\phi, u_\psi \in B$ such that $(\mathrm{Ad}u_\phi)\phi(g)$ and $(\mathrm{Ad}u_\psi)\psi(g)$ are diagonal, with purely imaginary eigenvalues. In the last two cases $(\mathrm{Ad}u_\phi)\phi^{\mathbb{C}}(h)$ and $(\mathrm{Ad}u_\psi)\psi^{\mathbb{C}}(h)$ are also diagonal, with real values which can be taken to be in increasing order. In the first case $\mathrm{Ad}(u_\phi,\bar{u}_\phi)\phi^{\mathbb{C}}(h)$ and $\mathrm{Ad}(u_\psi,\bar{u}_\psi)\psi^{\mathbb{C}}(h)$ are of the form (α,α) where α is real and diagonal, where the elements can again be taken to be in increasing order. In all three cases the proof of Theorem 6 of [6] can therefore be applied directly to give the required result.

In the remaining case, when $B = C([0,1],M_q(\mathbb{R}))$ or $B = A(q,\mathbb{R})$ then, after perturbation,there exist $u_\phi, u_\psi \in B$ such that $(\mathrm{Ad}wu_\phi)\phi(g)$ and $(\mathrm{Ad}wu_\phi)\psi(g)$ are diagonal with purely imaginary eigenvalues, where w consists of 2×2 diagonal blocks $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$, so $(\mathrm{Ad}wu_\psi)\psi^{\mathbb{C}}(h)$ and $(\mathrm{Ad}wu_\phi)\phi^{\mathbb{C}}(h)$ consist of real diagonal blocks $\begin{pmatrix} \frac{1}{2}+\alpha & 0 \\ 0 & \frac{1}{2}-\alpha \end{pmatrix}$ where the elements α can be taken to be in increasing order.

Theorem 6 of [6] then shows that $\mathrm{Ad}(wu_\phi)\phi^{\mathbb{C}}(h)$ and $\mathrm{Ad}(wu_\psi)\psi^{\mathbb{C}}(h)$ agree to within $\frac{3}{n}$ as therefore do $(\mathrm{Ad}u_\phi)\phi^{\mathbb{C}}(h)$ and $(\mathrm{Ad}u_\psi)\psi^{\mathbb{C}}(h)$. □

3. Injective connecting maps and approximate divisibility

As in [14], an inductive limit of basic building blocks can be written as an inductive limit of these blocks with injective connecting maps. The proof follows [14] but is easier.

Lemma 3.1. *If A is a basic building block, B is a unital real C^*-algebra, $\phi : A \to B$ is a unital $*$-homomorphism, F is a finite subset of $\phi(A)$ and $\epsilon > 0$, there exists a subalgebra B_1 of $\phi(A)$, isomorphic to a direct sum of basic building blocks and finite dimensional real C^*-algebras, such that F is approximately contained in B_1 to within ϵ.*

Proof. If A is either $C([0,1], M_q(\mathbb{C})), C([0,1], M_q(\mathbb{R}))$ or $C([0,1], M_{q/2}(\mathbb{H}))$ then $\phi(A)$ is isomorphic to either $C(X, M_q(\mathbb{C})), C(X, M_q(\mathbb{R}))$ or $C(X, M_{q/2}(\mathbb{H}))$ for X a closed subset of $[0,1]$. In either of the other two cases $\phi(A)$ is isomorphic to $C(X, M_q(\mathbb{C}))$ or $\{f \in C(X, M_q(\mathbb{C}) : f(\frac{1}{2}) \in R\}$ where R is isomorphic to $M_q(\mathbb{R})$ or $M_{q/2}(\mathbb{H})$ and $X \subseteq [0, \frac{1}{2}]$.

Let $F = \{f_1, \dots, f_r\}$ and, regarding these as continuous matrix valued functions on X, pick δ such that $\|f_i(s) - f_i(t)\| < \epsilon/2$ for each i whenever $|s - t| < \delta$. By Lemma 1.3 of [14], there exists a finite union Y of points and closed intervals with $Y \subseteq X$ and a retraction α from X onto Y such that $\sup_t |\alpha(t) - t| < \delta$ for each $t \in X$. Y can be taken to include the connected component of X containing $\frac{1}{2}$ and α to be the identity on this connected component. Let $\theta : D \to C(X, M)$ be defined by $\theta(f) = f \circ \alpha$ for $M \in \{M_q(\mathbb{C}), M_q(\mathbb{R}), M_{q/2}(\mathbb{H})\}$, where $D = C(Y, M)$ unless $A = \{f \in C(X, M_q(\mathbb{C})) : f(\frac{1}{2}) \in R\}$ and $\frac{1}{2} \in X$, in which case $D = \{f \in C(Y, M_q(\mathbb{C})) : f(\frac{1}{2}) \in R\}$.

Using the identification of $\phi(A)$ with either $C(X, M)$ or $\{f \in C(X, M) : f(\frac{1}{2}) \in R\}$, θ is an injective unital $*$-homomorphism from D to $\phi(A)$. D is a sum of basic building blocks and finite-dimensional algebras. Furthermore F is approximately contained in $B_1 = \theta(D)$ to within ϵ: given an element of $F \subseteq \phi(A)$ let f_i be the associated element of $C(X, M)$ and note that

$$\|f_i - \theta(f_i|_Y)\| = \sup_t \|f_i(t) - f_i(\alpha(t))\| < \epsilon.$$ □

Lemma 3.2. *Let B be a simple unital real infinite-dimensional AF algebra. Then B contains a self-adjoint element with spectrum $[0,1]$.*

Proof. $K_0(B)$ is a simple dimension group other than $\mathbb{Z}$ and so, by Lemma A4.1 in [8], there are positive elements $1 > a_{n,1} > \cdots > a_{n,2^n-1} > 0$ in $K_0(B)$ with $a_{n,i} = a_{n+1,2i}$ for each $1 \le i \le 2^n - 1$. There exist orthogonal projections $p_{n,1}, \dots, p_{n,2^n}$ in B corresponding to $1 - a_{n,1}, a_{n,1} - a_{n,2}, \dots, a_{n,2^n-2} - a_{n,2^n-1}, a_{n,2^n-1}$ with $p_{n,i} = p_{n+1,2i-1} + p_{n+1,2i}$ for each i. Let $a_n = \sum_{r=1}^{2^n} \frac{r}{2^n} p_{n,r}$ so $a_n - a_{n+1} = \sum_{r=1}^{2^n} \frac{2r}{2^{n+1}}(p_{n+1,2r-1} + p_{n+1,2r}) - \sum_{r=1}^{2^{n+1}} \frac{r}{2^{n+1}} p_{n+1,r} = \sum_{r=1}^{2^n} \frac{1}{2^{n+1}} p_{n+1,2r-1}$ and therefore $\|a_{n+1} - a_n\| = \frac{1}{2^{n+1}}$. Then a_n converges in B to a self-adjoint element a, which has spectrum $[0,1]$. □

Lemma 3.3. *Let B be a separable real C^*-algebra such that, for every finite subset $F \subseteq B$ and every $\epsilon > 0$ there exists a direct sum of basic building blocks $C \subseteq B$ which contains F to within ϵ. Then B is isomorphic to an inductive limit of a sequence of basic building blocks with injective unital connecting $*$-homomorphisms.*

Proof. The proof follows the usual complex argument, outlined in Lemma 1.4 of [14], using the methods of Theorem 4.3 of [5], Theorem 2.2 of [2] and the earlier work in [10]. The most difficult extra ingredient in the real case involves the quaternionic cases, which are handled using the following lemma. □

Lemma 3.4. *Let A, B be real C^*-algebras with $A \subseteq B$ and let $E,\ I,\ J \in B$ with $E^2 = E = E^*$, $I^* = -I$, $I^2 = -E$, $J^* = -J$, $J^2 = -E$ and $IJ = -JI$ (so that $I,\ J$ generate a copy of $\mathbb{H}$). If $\epsilon > 0$ there exists $\beta > 0$ such that whenever there exist $E', I', J' \in A$ with $\|E - E'\| < \beta$, $\|I - I'\| < \beta$, $\|J - J'\| < \beta$ then there exist $E'', I'', J'' \in A$ with*

$$E''^2 = E'' = E''^*, \quad I'' = -I''^*, \quad J'' = -J''^*,$$
$$I''^2 = -E'', \quad J''^2 = -E'', \quad J''I'' = -J''I'',$$
$$\|E - E''\| < \epsilon, \quad \|I - I''\| < \epsilon \quad \textit{and} \quad \|J - J''\| < \epsilon.$$

Proof. In the complexification $B^{\mathbb{C}}$ of B let $E_{12} = \frac{1}{2}(J - iIJ)$, $E_{11} = E_{12}E_{12}^*$, $E_{22} = E_{12}^*E_{12}$ and $E_{21} = E_{12}^*$. Then (corresponding to $M_2(\mathbb{C})$ being the complexification of $\mathbb{H}$) E_{ij} form a set of 2×2 matrix units in $B^{\mathbb{C}}$ with $\Phi(E_{12}) = -E_{12}$, and hence $\Phi(E_{11}) = E_{22}$, where Φ is the antiautomorphism of $B^{\mathbb{C}}$ associated with B. I and J are given by $I = iE_{11} - iE_{22}$ and $J = E_{12} - E_{21}$.

For $\alpha > 0$ let $\gamma(\alpha)$ and $\delta(\alpha)$ be the values defined in the statements of Lemmas 1.6 and 1.9 of [10]. Let $\delta_1 = \min(\frac{1}{36}, \frac{\epsilon}{62})$, $\delta_2 = \min(\delta(\delta_1), 1)$ and $\beta = \min(\frac{1}{32}, \frac{1}{16}\gamma(\frac{1}{40}\delta_2), \frac{1}{640}\delta_2)$. Let $x = \frac{1}{2}(J' - iI'J')$, where I', J', E' are as defined in the lemma. Then

$$\begin{aligned}\|x - E_{12}\| &\leq \frac{1}{2}\|J' - J\| + \frac{1}{2}\|I'J' - I'J\| + \frac{1}{2}\|I'J - IJ\| \\ &< \frac{1}{2}\beta + \frac{1}{2}\|I'\|\beta + \frac{1}{2}\beta \\ &< 2\beta < \delta_2.\end{aligned}$$

Also

$$\begin{aligned}\|xx^* - E_{11}\| &\leq \|xx^* - E_{12}x^*\| + \|E_{12}x^* - E_{12}E_{12}^*\| \\ &< 2\|x\|\beta + 2\beta \\ &< (1 + 2\beta)2\beta + 2\beta < 8\beta\end{aligned}$$

and similarly $\|x^*x - E_{22}\| \leq 8\beta$ so, putting $r = \frac{1}{2}(xx^* + x^*x + \Phi(xx^*) + \Phi(x^*x))$, $r = r^* = \Phi(r)$ and $\|r - (E_{11} + E_{22})\| < 16\beta < \gamma(\frac{1}{40}\delta_2)$. By Lemma 1.6 of [10] and its proof there exists a projection e in $A^{\mathbb{C}}$ with $\|e - (E_{11} + E_{22})\| < \frac{1}{40}\delta_2$ and $\Phi(e) = e$. Let $t = \frac{1}{2}(xx^* - x^*x - \Phi(xx^*) + \Phi(x^*x))$ and $s = ete$, so that $\Phi(s) = -s = -s^*$

and

$$\begin{aligned}
&\|s-(E_{11}-E_{22})\| \\
&\le \|ete-(E_{11}+E_{22})te\| + \|(E_{11}+E_{22})te-(E_{11}+E_{22})t(E_{11}+E_{22})\| \\
&\quad + \|(E_{11}+E_{22})t(E_{11}+E_{22})-(E_{11}+E_{22})(E_{11}-E_{22})(E_{11}+E_{22})\| \\
&< \frac{1}{40}\|t\|\delta_2 + \frac{1}{40}\|t\|\delta_2 + 16\beta \\
&< \frac{1}{20}(1+16\beta)\delta_2 + 16\beta \le \frac{3}{40}\delta_2 + \frac{1}{40}\delta_2 = \frac{1}{10}\delta_2.
\end{aligned}$$

Then

$$\begin{aligned}
\|s^2-e\| &\le \|s^2-s(E_{11}-E_{22})\| + \|s(E_{11}-E_{22})-(E_{11}-E_{22})^2\| \\
&\quad + \|E_{11}+E_{22}-e\| \\
&< \frac{1}{10}\|s\|\delta_2 + \frac{1}{10}\delta_2 + \frac{1}{40}\delta_2 \le \frac{1}{10}\left(1+\frac{1}{10}\delta_2\right)\delta_2 + \frac{5}{40}\delta_2 \\
&< \frac{3}{10}\delta_2.
\end{aligned}$$

Considering the commutative C^*-algebra generated by s and e (for which e is the identity), the spectrum of s is contained in $[-1-\frac{3}{5}\delta_2, -1+\frac{3}{5}\delta_2] \cup [1-\frac{3}{5}\delta_2, 1+\frac{3}{5}\delta_2]$. Let f be the odd continuous function on $[-1-\frac{3}{5}\delta_2, 1+\frac{3}{5}\delta_2]$ which is linear on $[0, 1-\frac{3}{5}\delta_2]$ and equal to 1 on $[1-\frac{3}{5}\delta_2, 1+\frac{3}{5}\delta_2]$ and let $s' = f(s)$. Then ${s'}^2 = e$, $\Phi(s') = -s' = -s'^*$ and $\|s-s'\| \le \frac{3}{5}\delta_2$. Let $e_{11} = \frac{1}{2}(e+s')$ so

$$e_{11}^2 = e_{11} = e_{11}^*, \Phi(e_{11})e_{11} = 0$$

and $e_{11} + \Phi(e_{11}) = e$. Then

$$\begin{aligned}
\|e_{11}-E_{11}\| &\le \frac{1}{2}\|s'-s\| + \frac{1}{2}\|s-(E_{11}-E_{22})\| + \frac{1}{2}\|e-(E_{11}+E_{22})\| \\
&< \frac{3}{10}\delta_2 + \frac{1}{20}\delta_2 + \frac{1}{80}\delta_2 < \delta_2
\end{aligned}$$

and so $\|\Phi(e_{11}) - E_{22}\| < \delta_2$. Thus, by Lemma 1.9 of [10], there exists a partial isometry w in $A^{\mathbb{C}}$ with $ww^* = e_{11}, w^*w = \Phi(e_{11})$ and $\|w - E_{12}\| < \delta_1$.

Next let $v = \frac{1}{2}e_{11}(w-\Phi(w))e_{22}$ and note that

$$\|e_{11}-E_{11}\| = \|ww^*-E_{12}E_{12}^*\| \le \|ww^*-wE_{12}^*\| + \|wE_{12}^*-E_{12}E_{12}^*\| < 2\delta_1,$$

$\|e_{22}-E_{22}\| < 2\delta_1$, $\|\frac{1}{2}(w-\Phi(w)) - E_{12}\| < \delta_1$ and thus that

$$\begin{aligned}
\|v-E_{12}\| &\le \|v-e_{11}E_{12}e_{22}\| + \|e_{11}E_{12}e_{22}-E_{11}E_{12}e_{22}\| + \|E_{12}e_{22}-E_{12}\| \\
&< \delta_1 + 2\delta_1 + 2\delta_1 = 5\delta_1
\end{aligned}$$

and

$$\begin{aligned}
\|v^*v-e_{22}\| &\le \|v^*v-v^*E_{12}\| + \|v^*E_{12}+E_{22}\| + \|E_{22}-e_{22}\| \\
&< 5\delta_1 + 5\delta_1 + 2\delta_1 = 12\delta_1.
\end{aligned}$$

Thus v^*v is an invertible element of $e_{22}B^{\mathbb{C}}e_{22}$ and

$$\|(v^*v)^{-1/2} - e_{22}\| < \left(\frac{1}{1-12\delta_1}\right) - 1 < 24\delta_1.$$

Let $u = v(v^*v)^{-1/2}$ so that

$$\begin{aligned}\|u - E_{12}\| &\le \|v(v^*v)^{-1/2} - ve_{22}\| + \|ve_{22} - E_{12}e_{22}\| + \|E_{12}e_{22} - E_{12}E_{22}\| \\ &< 24\delta_1 + 5\delta_1 + 2\delta_1 = 31\delta_1\end{aligned}$$

and

$$u^*u = (v^*v)^{-1/2}v^*v(v^*v)^{-1/2} = e_{22},$$

from which u is a partial isometry. The initial projection $uu^* = v(v^*v)^{-1}v^*$ satisfies $e_{11}uu^* = uu^*e_{11} = uu^*$ and

$$\begin{aligned}\|uu^* - e_{11}\| &\le \|v(v^*v)^{-1}v^* - ve_{22}v^*\| + \|ve_{22}v^* - e_{11}\| \\ &< 24\delta_1 + \|vv^* - e_{11}\| < 36\delta_1 < 1,\end{aligned}$$

so $uu^* = e_{11}$. $\Phi(u)$ is also a partial isometry with $\Phi(u)^*\Phi(u) = \Phi(uu^*) = \Phi(e_{11}) = e_{22}$ and $\Phi(u)\Phi(u^*) = \Phi(u^*u) = \Phi(e_{22}) = e_{11}$. From $\Phi(v) = -v$, $\Phi(v^*v) = vv^*$ and $\Phi((v^*v)^{-1/2}) = (vv^*)^{-1/2}$, where the latter inverse is taken in $e_{11}B^{\mathbb{C}}e_{11}$. Also $(u(v^*v)^{1/2}u^*)^2 = u(v^*v)^{1/2}u^*u(v^*v)^{1/2}u^* = uv^*vu^* = vv^*$, so $(vv^*)^{1/2} = u(v^*v)^{1/2}u^*$ and therefore

$$\begin{aligned}\Phi(u) &= \Phi((v^*v)^{-1/2})\Phi(v) = -(vv^*)^{-1/2}v \\ &= -u(v^*v)^{-1/2}u^*v = -u(v^*v)^{-1/2}(v^*v)^{-1/2}v^*v \\ &= -u.\end{aligned}$$

Let $E'' = uu^* + u^*u$, $I'' = iuu^* - iu^*u$ and $J'' = u - u^*$. Then $\Phi(E'') = E'' = E''^*$, $\Phi(I'') = -I'' = I''^*$, $\Phi(J'') = -J'' = J''^*$, $I''^2 = -E''$, $J''^2 = -E''$, $J''I'' = -I''J''$,

$$\begin{aligned}\|E'' - E\| &= \|e_{11} + e_{22} - E_{11} - E_{22}\| < 4\delta_1 < \epsilon, \\ \|I'' - I\| &= \|ie_{11} - ie_{22} - iE_{11} + iE_{22}\| < 4\delta_1 < \epsilon \quad \text{and} \\ \|J'' - J\| &= \|u - u^* - E_{12} + E_{21}\| \le 2\|u - E_{12}\| < 62\delta_1 < \epsilon.\end{aligned}$$

□

Theorem 3.5. *Let A be a simple unital infinite-dimensional real C^*-algebraic direct limit of direct sums of basic building blocks $A_1 \underset{\phi_1}{\rightarrow} A_2 \underset{\phi_2}{\rightarrow} \cdots$. Then A is also the direct limit of a system of direct sums of basic building blocks with unital injective maps.*

Proof. If $\phi_{n,\infty} : A_n \to A$ is the canonical map then there exists N such that, for $n \ge N$, $\phi_{n,\infty}(1) = 1$. Omitting $A_1, \dots, A_{N-1}$ and substituting $\phi_n(1)A_{n+1}\phi_n(1)$ for A_{n+1} for $n \ge N$ it can be assumed that each ϕ_n is unital.

Let F be a finite subset of A. There exists N such that F is contained within $\phi_{N,\infty}(A_N)$ up to $\epsilon/2$. Let F' be a subset of $\phi_{N,\infty}(A_N)$ with each element within $\epsilon/2$ of an element of F. By Lemma 3.1 there exists $B = B_1 \oplus B_2 \subseteq \phi_{N,\infty}(A_N)$ containing F' to within $\epsilon/2$, where B_1 is finite-dimensional with identity p and B_2 is a direct sum of basic building blocks. The relative commutant of B_1 in pAp is a direct sum of simple inductive limit algebras, each of which contains a simple real AF algebra (with the same K_0 group) and therefore, applying Lemma 3.2, contains a self-adjoint element h with spectrum $[0,1]$. Then the real C^*-algebra generated by B and h is a finite direct sum of basic building blocks containing F to within ϵ. The result follows by Lemma 3.3. □

The existence of a sequence with injective connecting maps is used on page 377 of [6] to establish approximate divisibility. It will now be shown that the finite dimensional unital subalgebra produced in this construction can be chosen to be invariant under the appropriate involutory antiautomorphism.

Proposition 3.6. *Let A be a simple separable unital real C^*-algebra, which is the direct limit of a unital system $A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$, where each A_i is a direct sum of basic building blocks and each ϕ_i is injective. Then, for each i, each N, each finite set $F \subseteq A_i^{\mathbb{C}}$ and each $\epsilon > 0$ there exists $j \geq i$ and a homomorphism $\psi : A_i \to A_j$, such that $\psi^{\mathbb{C}}$ agrees with $\phi_{j-1}^{\mathbb{C}} \circ \cdots \circ \phi_i^{\mathbb{C}}$ on F to within ϵ, and a unital finite dimensional subalgebra H of A_j in the commutant of $\psi(A_i)$ such that each summand of H has order at least N.*

Proof. It suffices to consider A_i to be a single basic building block which, by the results of Section 1, is of the form $C([0,1],\mathbb{R}) \otimes_{\mathbb{R}} R$ or $A(1,\mathbb{R}) \otimes_{\mathbb{R}} R$ for some algebra R isomorphic to $M_q(\mathbb{R})$ or $M_{q/2}(\mathbb{H})$ or $M_q(\mathbb{C})$. It therefore suffices to consider $A_i = C([0,1],\mathbb{R})$ or $A_i = A(1,\mathbb{R})$, which are generated by $h(t) = t$ and $g(t) = i(\frac{1}{2} - t)$ respectively. As in [6], given δ, j can be chosen so that the eigenvalues of each primitive quotient of $k = \phi_{j-1} \circ \cdots \circ \phi_i(h)$ or $\ell = \phi_{j-1} \circ \cdots \circ \phi_i(g)$ are $\frac{\delta}{4N}$ dense in [0,1] or $i[-\frac{1}{2}, \frac{1}{2}]$. It was shown in Lemma 2.4 that there is an arbitrarily small perturbation of each summand of k of the form $\sum \lambda_r P_r \in A_j$, where each P_r is a 1 or 2 dimensional projection valued function in A_j and each $\lambda_r(t) \in \mathbb{R}$. It was also shown in Lemma 2.4 that there is an arbitrarily small perturbation of each summand of ℓ of the form $\sum \lambda_r P_r \in A_j$, where each $\lambda_r(t)$ is purely imaginary and each $P_r(t)$ is 1-dimensional. In this case either $P_r \in A_j$ or there is a partner P_s with $i(P_r - P_s) \in A_j$ and $P_r + P_s \in A_j$.

In each case, by coalescing eigenvalues as on page 377 of [6], a perturbation $\sum \lambda_r Q_r$ of each summand of k or ℓ can be found in A_j such that, for each eigenprojection Q_r, the dimension of each $Q_r(t)$ is at least $2N$. For the perturbation of each summand of k, each Q_r belongs to A_j and there exists a unital finite-dimensional subalgebra H_r of $Q_r A_j Q_r$. For the perturbation of each summand of ℓ, either the same applies or for given r there exists s such that both $i(Q_r - Q_s)$ and $Q_r + Q_s$ belong to A_j and the element $i(Q_r - Q_s)$ of $(Q_r + Q_s)A_j(Q_r + Q_s)$ commutes with a unital finite-dimensional subalgebra H_r of order at least N. (The complexification of H_r has order at least $2N$.) The direct sum of the subalgebras H_r, for varying r, gives the required algebra H. □

4. An existence result

In this section we obtain an appropriate version of Theorem 5 of [6]. To produce homomorphisms of real C^*-algebras consistent with prescribed K-theoretic maps the method previously employed for real AF-algebras, using standard homomorphisms, can be used. There is an apparent problem obtaining the other required condition, approximate consistency with a given Markov map between affine function spaces on tracial state spaces. For example, the only nonzero homomorphism from $A(1,\mathbb{R}) = \{f \in C([0,1],\mathbb{C}) : f(t) = \overline{f(1-t)}\}$ to $B = C([0,1],\mathbb{R})$ maps f to the constant function $f(\frac{1}{2})$. Thus some Markov maps from A to B, such as that mapping f to the constant $\frac{1}{2}f(0) + \frac{1}{2}f(1)$, cannot be approximated by convex combinations of homomorphisms. The algebra $A(1,\mathbb{R})$ is similar to the dimension drop algebras considered in [20], (noting that $A(1,\mathbb{R})$ is isomorphic to $\{f \in C([0,\frac{1}{2}],\mathbb{C}) : f(\frac{1}{2}) \in \mathbb{R}\}$) and, as in [20], the solution will be to seek approximating convex combinations of homomorphisms from A into a matrix algebra over B, in this case $M_2(B)$. The first lemma establishes the required version of Theorem 2.1 of [21].

Lemma 4.1. *Let $A = C([0,1],\mathbb{R})$, let $\theta_1, \theta_2 \in \{\mathrm{id}, 1-\mathrm{id}\}$ be homeomorphisms of $[0,1]$, let $\hat{\Phi}_1, \hat{\Phi}_2$ be the associated involutions of A (with $\hat{\Phi}_i f = f \circ \theta_i$) and let $M : A \to A$ be a unital positive operator with $M\hat{\Phi}_1 = \hat{\Phi}_2 M$. Given $\delta > 0$ and a finite subset F of $C([0,1],\mathbb{R})$ there exist $N > 0$ and continuous functions $\mu_1, \ldots, \mu_{2N}$ from $[0,1]$ to $[0,1]$ with $\mu_i\theta_2 = \theta_1\mu_{2N+1-i}$ for each i such that*

$$\left\| M(f) - \frac{1}{2N}\sum_{i=1}^{2N} f \circ \mu_i \right\| < \delta$$

for all $f \in F$.

Proof. When $\theta_1 = \theta_2 = \mathrm{id}$ use Theorem 2.1 of [21] and its proof to approximate $M(f)$ by $\frac{1}{N}\sum_{i=1}^{N} f \circ \mu_i$ and then define $\mu_{2N+1-i} = \mu_i$ for $1 \le i \le N$. When $\theta_1 = \mathrm{id}$ and $\theta_2 = 1-\mathrm{id}$, $(Mf)(t) = (Mf)(1-t)$ for each $0 \le t \le 1$ so Mf can be regarded as an element of $C([0,\frac{1}{2}])$. Use Theorem 2.1 of [21] and its proof to approximate $M(f)$ by $\frac{1}{N}\sum_{i=1}^{N} f \circ \mu_i$ where $\mu_i : [0,\frac{1}{2}] \to [0,1]$. Extend each μ_i to $[0,1]$ by $\mu_i(t) = \mu_i(1-t)$ and, as before, define $\mu_{2N+1-i} = \mu_i$ for $1 \le i \le N$.

When $\theta_1 = 1-\mathrm{id}$ and $\theta_2 = \mathrm{id}$ then $Mf = M\hat{\Phi}_1 f = M(\frac{1}{2}f + \frac{1}{2}\hat{\Phi}_1 f)$ and $(\frac{1}{2}f + \frac{1}{2}\hat{\Phi}_1 f)(t) = (\frac{1}{2}f + \frac{1}{2}\hat{\Phi}_1 f)(1-t)$ for each $0 \le t \le 1$. Thus M can be regarded as a map from $C([0,\frac{1}{2}])$ to $C([0,1])$ and therefore $M(f)$ can be approximated by $\frac{1}{N}\sum_{i=1}^{N}(\frac{1}{2}f + \frac{1}{2}\hat{\Phi}_1 f) \circ \mu_i$ where $\mu_i : [0,1] \to [0,\frac{1}{2}]$. For $1 \le i \le N$ define $\mu_{2N+1-i} = \theta_1\mu_i$, to obtain the required approximation.

When $\theta_1 = \theta_2 = \theta = 1-\mathrm{id}$, the required result is obtained by a minor refinement of the proof of Theorem 2.1 of [21]. Firstly note that if $a \in C([0,1],\mathbb{R})$ and $y \in [0,\frac{1}{2}]$ then both $(Ma)(y)$ is approximated by a convex combination $\sum \lambda_i a(x_i)$ and $(Ma)(1-y)$ is approximated by $\sum \lambda_i a(1-x_i)$; in particular if $y = \frac{1}{2}$ then $(Ma)(\frac{1}{2})$ is approximated both by $\sum \lambda_i a(x_i)$ and $\sum \lambda_i a(1-x_i)$ and thus by

$$\frac{1}{2}\sum \lambda_i(a(x_i) + a(1-x_i)).$$

As in Theorem 2.1 of [21] choose an open covering $\{U_j : j = 1,\ldots,N\}$ of $[0,\frac{1}{2}]$ on which the approximation by convex combinations persists, but choose $\frac{1}{2} \in U_N$ and $\frac{1}{2} \notin U_j$ for $1 \le j < N$. Then define an open cover $\{V_1,\ldots,V_{2N-1}\}$ of $[0,1]$ by $V_i = U_i$ for $1 \le i \le N-1$, $V_i = \theta U_{2N-i}$ for $N+1 \le i \le 2N-1$ and $V_N = U_N \cup \theta U_N$. A partition of unity $\{h_1,\ldots,h_{2N-1}\}$ subordinate to $\{V_1,\ldots,V_{2N-1}\}$ can then be found with $h_i = \theta h_{2N-i}$ for $1 \le i \le N-1$ and $h_N = \theta h_N$ (with $h_N(\frac{1}{2}) = 1$). The corresponding Markov operator $Va = \sum_{i=1}^{2n} g_i a(x_i)$ obtained in [21] can be chosen to have $g_{2n+1-i} = \theta g_i$ and $x_i = 1 - x_{2n+1-i}$. Then let $G_j = \sum_{i=1}^{j} g_i$ so that $G_{j-1}(y) < t < G_j(y)$ if and only if $G_{2n-j}(1-y) < 1-t < G_{2n-j+1}(1-y)$. Thus, using the notation $U_j = \{(y,t) \in [0,1]\times[0,1] : G_{j-1}(y) < t < G_j(y)\}$ introduced in the proof of Theorem 2.1 of [21], (y,t) belongs to U_j if and only if $(1-y, 1-t)$ belongs to U_{2n+1-j}. The maps $\phi_1,\ldots,\phi_{2n}$ supported on $U_1,\ldots,U_n$

can then be chosen to satisfy $\phi_j(y,t) = \phi_{2n+1-j}(1-y,1-t)$ and, with $x_0 = \frac{1}{2}$, the continuous maps ψ_j can be chosen to have $\psi_j(t) = 1 - \psi_{2n+1-j}(t)$. The continuous map $h : [0,1] \times [0,1] \to [0,1]$ defined by $h(y,t) = \psi_j(\phi_j(y,t))$ satisfies

$$h(y,t) = 1 - h(1-y, 1-t)$$

for each (y,t) and thus, with $\mu_i(y) = h(y, \frac{2i-1}{4N})$ for $1 \le i \le 2N$,

$$\mu_i(1-y) = 1 - h\left(y, \frac{2(2N+1-i)-1}{4N}\right) = 1 - \mu_{2N+1-i}(y).$$

The proof of Theorem 2.1 of [21] shows that $\mu_1, \ldots, \mu_{2N}$ have the required properties. □

The continuous maps μ_i found in the previous lemma do not give rise to homomorphisms between the real algebras associated with θ_1, θ_2. The next lemma shows however that they can be combined in pairs to give appropriate homomorphisms into matrix algebras.

Lemma 4.2. *Let $\mu_1, \mu_2 : [0,1] \to [0.1]$ be continuous, let $\theta_1, \theta_2 \in \{\mathrm{id}, 1 - \mathrm{id}\}$ and let $\mu_1\theta_2 = \theta_1\mu_2$. For $f \in C([0,1],\mathbb{C})$ and $i \in \{1,2\}$ let $\Phi_i f = f \circ \theta_i$ and $M(f) = \frac{1}{2} f \circ \mu_1 + \frac{1}{2} f \circ \mu_2$. Then there exists a homomorphism*

$$\psi : C([0,1],\mathbb{C}) \to C([0,1],\mathbb{C}) \otimes M_2(\mathbb{C}) \quad \textit{with} \quad (\Phi_2 \otimes \mathrm{Tr})\psi = \psi\Phi_1.$$

where Tr *is the transpose map on $M_2(\mathbb{C})$. Furthermore, when the tracial state spaces of $C([0,1],\mathbb{C})$ and $C([0,1],M_2(\mathbb{C}))$ are identified, $M(f) = \psi(f)$ as affine functions on the tracial state space.*

Proof. Let $W = 1 \otimes \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} i & -i \\ 1 & 1 \end{smallmatrix}\right)$ and let

$$\begin{aligned} \psi(f) &= W \mathrm{diag}(f \circ \mu_1, f \circ \mu_2) W^* \\ &= \tfrac{1}{2} \begin{pmatrix} f \circ \mu_1 + f \circ \mu_2 & i(f \circ \mu_1 - f \circ \mu_2) \\ i(f \circ \mu_2 - f \circ \mu_1) & f \circ \mu_1 + f \circ \mu_2 \end{pmatrix}. \end{aligned}$$

From $\mu_1\theta_2 = \theta_1\mu_2$ it also follows that $\theta_1\mu_1 = \mu_2\theta_2$ and so

$$\begin{aligned} (\Phi_2 \otimes \mathrm{Tr})(\psi(f)) &= \tfrac{1}{2} \begin{pmatrix} f \circ \mu_1 \circ \theta_2 + f \circ \mu_2 \circ \theta_2 & i(f \circ \mu_1 \circ \theta_2 - f \circ \mu_2 \circ \theta_2) \\ i(f \circ \mu_2 \circ \theta_2 - f \circ \mu_1 \circ \theta_2) & f \circ \mu_1 \circ \theta_2 + f \circ \mu_2 \circ \theta_2 \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} f \circ \theta_1 \circ \mu_1 + f \circ \theta_1 \circ \mu_2 & i(f \circ \theta_1 \circ \mu_1 - f \circ \theta_1 \circ \mu_2) \\ i(f \circ \theta_1 \circ \mu_2 - f \circ \theta_1 \circ \mu_1) & f \circ \theta_1 \circ \mu_1 + f \circ \theta_1 \circ \mu_2 \end{pmatrix} \\ &= \psi(\Phi_1(f)). \end{aligned}$$

Furthermore, if $\widetilde{\tau}$ is a trace on $C([0,1],M_2(\mathbb{C}))$ corresponding to τ on $C([0,1],\mathbb{C})$ then

$$\widetilde{\tau}(\psi(f)) = \widetilde{\tau}\mathrm{diag}(f \circ \mu_1, f \circ \mu_2) = \widetilde{\tau}\mathrm{diag}(M(f), M(f)) = \tau(M(f)).$$ □

The next lemma is just the appropriate version of Lemma 4.2 of [14]. It uses certain standard homomorphisms betweeen finite-dimensional real C^*-algebras which were defined in [18] and [9].

Lemma 4.3. *Let $A = \oplus_{i=1}^r A_i$ and $B = \oplus_{j=1}^s B_j$ where each A_i and B_j is a basic building block. Let $T(A^{\mathbb{C}}), T(B^{\mathbb{C}})$ be the tracial state spaces of $A^{\mathbb{C}}, B^{\mathbb{C}}$, and let Φ_A^*, Φ_B^* be the affine homeomorphisms of $T(A^{\mathbb{C}}), T(B^{\mathbb{C}})$ arising from the involutory antiautomorphisms Φ_A, Φ_B of $A^{\mathbb{C}}, B^{\mathbb{C}}$ associated with the real algebras A, B. Let F*

be a finite subset of Aff$(T(A^{\mathbb{C}}))$*, the continuous affine functions on* $T(A^{\mathbb{C}})$*, let* $\delta > 0$*, let* $M : \mathrm{Aff}(T(A^{\mathbb{C}})) \to \mathrm{Aff}(T(B^{\mathbb{C}}))$ *be unital and positive, with* $M\hat{\Phi}_A = \hat{\Phi}_B M$*, and let* $h : K_0(A) \to K_0(B)$*,* $h^{\mathbb{C}} : K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(B \otimes_{\mathbb{R}} \mathbb{C})$*,* $h^{\mathbb{H}} : K_0(A \otimes_{\mathbb{R}} \mathbb{H}) \to K_0(B \otimes_{\mathbb{R}} \mathbb{H})$ *be such that the following two diagrams commute, where* ρ_A *is the natural map from* $K_0(A^{\mathbb{C}})$ *into* $\mathrm{Aff}(T(A^{\mathbb{C}}))$*:*

$$\begin{array}{ccc} K_0(A^{\mathbb{C}}) & \xrightarrow{\rho_A} & \mathrm{Aff}(T(A^{\mathbb{C}})) \\ {\scriptstyle h^{\mathbb{C}}}\downarrow & & {\scriptstyle M}\downarrow \\ K_0(B^{\mathbb{C}}) & \xrightarrow[\rho_B]{} & \mathrm{Aff}(T(B^{\mathbb{C}})) \end{array}$$

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A \otimes_{\mathbb{R}} \mathbb{C}) & \longrightarrow & K_0(A \otimes_{\mathbb{R}} \mathbb{H}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^{\mathbb{C}}} & & \downarrow{\scriptstyle h^{\mathbb{H}}} \\ K_0(B) & \longrightarrow & K_0(B \otimes_{\mathbb{R}} \mathbb{C}) & \longrightarrow & K_0(B \otimes_{\mathbb{R}} \mathbb{H}). \end{array}$$

Then there exist $k \in \mathbb{N}$ *and* $*$*-homomorphisms* $\lambda_i : A \to B \otimes_{\mathbb{R}} M_2(\mathbb{R}), i = 1, \dots, k$ *such that* $\lambda_{i*} = d_* \circ h$ *on* $K_0(A), \lambda_{i*}^{\mathbb{C}} = d_* \circ h^{\mathbb{C}}$ *on* $K_0(A \otimes_{\mathbb{R}} \mathbb{C}), \lambda_{i*}^{\mathbb{H}} = d_* \circ h^{\mathbb{H}}$ *on* $K_0(A \otimes_{\mathbb{R}} \mathbb{H})$ *and* $\|\frac{1}{k}\sum \hat{\lambda}_i^{\mathbb{C}}(f) - M(f)\| < \delta$ *for all* $f \in F$ *where, for* $\tau \in T(A^{\mathbb{C}}) = T(A^{\mathbb{C}} \otimes M_2(\mathbb{C}))$*,* $\hat{\lambda}_i^{\mathbb{C}}(f)(\tau) = f(\tau \circ \lambda_i^{\mathbb{C}})$ *and* d_* *arises from the diagonal embedding* $B \to B \otimes_{\mathbb{R}} M_2(\mathbb{R})$*.*

Proof. By considering each summand separately it suffices to consider B to be a single building block. Let p_d be the identity of the summand A_d of A and let $\{q_d : 1 \le d \le r\}$ be a set of orthogonal projections in B such that $h([p_d]) = [q_d]$ which exist because $K_0(B) \cong \mathbb{Z}$, with generator given by a minimal projection in B, and $h([1]) = [1]$.

It will suffice to replace A by A_d, B by $q_d B q_d$ and, if $q_d \neq 0$, M by $\hat{q}_d^{-1} M \circ \mathrm{id}_d$ where id_d is the dth coordinate embedding and $\hat{q}_d$ is the ratio $[q_d]/[1]$ when both are regarded as elements of $\mathbb{Z} \cong K_0(B)$. (If $q_d = 0$, the compatibility between $h^{\mathbb{C}}$ and M forces M to be zero on the dth summand: in this case let $k = 1$ and $\lambda_1 = 0$).

Let Z be the centre of A and Z' the centre of B. Then $A = M_q(\mathbb{R}) \otimes_{\mathbb{R}} Z$ or $A = M_q(\mathbb{H}) \otimes_{\mathbb{R}} Z$, for some q, with a similar result for B, so that M can be regarded as a map from $\mathrm{Aff}(T(Z^{\mathbb{C}}))$ to $\mathrm{Aff}(T({Z'}^{\mathbb{C}}))$. Exactly as in Lemma 4.2 of [14] both $\mathrm{Aff}(T(Z^{\mathbb{C}}))$ and $\mathrm{Aff}(T({Z'}^{\mathbb{C}}))$ can be identified with either $C([0,1],\mathbb{R})$ or $C([0,1],\mathbb{R}^2)$.

The first step, except when $Z = C([0,1],\mathbb{C})$ and $Z' = C([0,1],\mathbb{C})$, is to find unital homomorphisms $\psi_i : Z^{\mathbb{C}} \to {Z'}^{\mathbb{C}} \otimes M_2(\mathbb{C})$, mapping Z to $Z' \otimes_{\mathbb{R}} M_2(\mathbb{R})$, such that $\frac{1}{k}\sum \psi_i$ approximates M on a given finite set. When Z and Z' are both equal to either $C([0,1],\mathbb{R})$ or $A(1,\mathbb{R})$ such homomorphisms exist by Lemmas 4.1 and 4.2.

When Z is either $C([0,1],\mathbb{R})$ or $A(1,\mathbb{R})$ and $Z' = C([0,1],\mathbb{C})$ then

$$M : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}^2)$$

and, from the compatibility condition $M\hat{\Phi}_A = \hat{\Phi}_B M$, M is of the form $M(f) = (m(f), m(f \circ \theta))$, where θ is the homeomorphism of $[0,1]$ associated with Z and m is unital and positive. By Lemma 4.2 of [14] there exist continuous maps $\mu_1, \dots, \mu_k$ such that $m(f)$ is approximated by $\frac{1}{k}\sum f \circ \mu_i$ for f in $F \cup \hat{\Phi}_A(F)$. Then $\psi_i : Z^{\mathbb{C}} \to {Z'}^{\mathbb{C}} \otimes M_2(\mathbb{C})$ defined by $\psi_i(f) = (f \circ \mu_i, f \circ \theta \circ \mu_i) \otimes I_2$ have the required approximation property and map elements of Z, for which $f^* = f \circ \theta$, to elements of $Z' \otimes_{\mathbb{R}} M_2(\mathbb{R})$.

When $Z = C([0,1],\mathbb{C})$ and Z' is either $C([0,1],\mathbb{R})$ or $A(1,\mathbb{R})$ then

$$M : C([0,1],\mathbb{R}^2) \to C([0,1],\mathbb{R})$$

and, from the compatibility of M with $h^{\mathbb{C}} : \mathbb{Z}^2 \to \mathbb{Z}$, $M(1,0)$ is a constant function. The compatibility condition $M\hat{\Phi}_A = \hat{\Phi}_B M$ then gives $M(0,1) = \hat{\Phi}_B M(1,0) = M(1,0) \circ \theta = M(1,0)$ and, from $M(1,1) = 1$, it then follows that $M(1,0) = M(0,1) = \frac{1}{2}$. Thus Lemma 4.2 of [14] can be applied to $m : f \mapsto 2M(f,0)$ to produce continuous maps $\mu_1, \ldots, \mu_k$ for which $\frac{1}{k}\sum f \circ \mu_i$ approximates $m(f)$ and $\frac{1}{k}\sum g \circ \mu_i$ approximates $m(g)$ whenever $(f,g) \in F$ and therefore for which $\frac{1}{2k}\sum(f \circ \mu_i + g \circ \mu_i \circ \theta)$ approximates $M(f,g)$. Let p be the projection $\frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ i & 1 \end{smallmatrix}\right) \in M_2(\mathbb{C})$ and define ψ_i to be the homomorphism $\psi_i(f,g) = (f \circ \mu_i)p + (g \circ \mu_i \circ \theta)(1-p)$. Then $\psi_i(f,f^*)^{\mathrm{Tr}} = (f \circ \mu_i)(1-p) + (f^* \circ \mu_i \circ \theta)p = \psi_i(f,f^*)^* \circ \theta$, so that ψ_i maps Z to $Z' \otimes M_2(\mathbb{R})$, and $\frac{1}{k}\sum \psi_i$ approximates M on F.

Except when $Z = Z' = C([0,1],\mathbb{C})$, the homomorhisms λ_i can now be defined by $\lambda_i = \alpha_i \otimes \psi_i : M_q(\mathbb{F}) \otimes_{\mathbb{R}} Z \to M_m(\mathbb{F}') \otimes_{\mathbb{R}} Z' \otimes_{\mathbb{R}} M_2(\mathbb{R})$ where $\mathbb{F}, \mathbb{F}'$ are either $\mathbb{R}$ or $\mathbb{H}$ and α_i is the appropriate standard homomorphism, as used in the real AF situation in Lemma 2.2 of [19] and either Theorem 3.3 of [18] or Proposition 3.6 of [9]. The effect on K-theory is correct because, as in Lemma 6.6 of [20], the evaluation of an element of $M_q(\mathbb{F}) \otimes_{\mathbb{R}} Z$ at $\frac{1}{2}$ is a split homomorphism and therefore gives an isomorphism between the K-theory sequences for $M_q(\mathbb{F}) \otimes_{\mathbb{R}} Z$ and $M_q(\mathbb{F})$.

When $Z = Z' = C([0,1],\mathbb{C})$ the K-theory and affine approximation must be addressed simultaneously. In this case the K-theory data is

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{(\mathrm{id}_1,\mathrm{id}_2)} & \mathbb{Z}^2 & \xrightarrow{\mathrm{id}_1+\mathrm{id}_2} & \mathbb{Z} \\ h\downarrow & & \downarrow h^{\mathbb{C}} & & \downarrow h^{\mathbb{H}} \\ \mathbb{Z} & \xrightarrow{(\mathrm{id}_1,\mathrm{id}_2)} & \mathbb{Z}^2 & \xrightarrow{\mathrm{id}_1+\mathrm{id}_2} & \mathbb{Z}\,. \end{array}$$

If $h^{\mathbb{C}}(1,0) = (k,\ell)$ let $h^{\mathbb{C}}(0,1) = (k',\ell')$. Then $k' + \ell' = h^{\mathbb{H}}(1) = k + \ell$ and $(k+k', \ell+\ell') = (h(1),h(1))$. So $k + k' = \ell + \ell'$ and therefore $k' = \ell$ and $\ell' = k$. The positive map $M : C([0,1],\mathbb{R}^2) \to C([0,1],\mathbb{R}^2)$ is of the form $M(f,g) = (m(f,g), m(g,f))$ for some positive unital map m and the compatibility with $h^{\mathbb{C}}$ implies that $m(1,0) = \frac{k}{k+\ell}$ and $m(0,1) = \frac{\ell}{k+\ell}$. If $k \neq 0$ and $(f,g) \in F$, the map $f \mapsto \frac{k+\ell}{k}m(f,0)$ can be approximated by a sum $\frac{1}{N}\sum f \circ \mu_i$ and if $\ell \neq 0$, the map $g \mapsto \frac{k+\ell}{\ell}m(0,g)$ can be approximated by a sum $\frac{1}{M}\sum g \circ \nu_i$. Repeating elements as necessary, we can assume that $M = N$, so that $m(f,g)$ is approximated by $\frac{1}{N(k+\ell)}\sum(kf \circ \mu_i + \ell g \circ \nu_i)$. This also holds when $k = 0$ or $\ell = 0$. Then let

$$\lambda_i = \left(\begin{pmatrix} (f \circ \mu_i) \otimes I_k & 0 \\ 0 & (g \circ \nu_i) \otimes I_\ell \end{pmatrix}, \begin{pmatrix} (g \circ \mu_i) \otimes I_k & 0 \\ 0 & (f \circ \nu_i) \otimes I_\ell \end{pmatrix}\right) \otimes I_2.$$

The combination $\frac{1}{N}\sum \hat{\lambda}_i$ approximates M on F and each λ_i has the required K-theoretic properties. □

The existence theorem now follows as in Corollary 4.3 of [14].

Proposition 4.4. *Let $A = \bigoplus_{i=1}^r A_i$ and $B = \bigoplus_{j=1}^s B_j$ where each A_i and B_j is a basic building block, let $T(A^{\mathbb{C}}), T(B^{\mathbb{C}})$ be the trace state spaces of $A^{\mathbb{C}}, B^{\mathbb{C}}$, let*

$F \subseteq \mathrm{Aff}(T(A^{\mathbb{C}}))$ be a finite subset and let $\delta > 0$. Further let $M : \mathrm{Aff}(T(A^{\mathbb{C}})) \to \mathrm{Aff}(T(B^{\mathbb{C}}))$ be unital and positive, with $M\hat{\Phi}_A = \hat{\Phi}_B M$, where $\hat{\Phi}_A, \hat{\Phi}_B$ arise from the involutory antiautomorphisms of $A^{\mathbb{C}}, B^{\mathbb{C}}$ associated with A, B, and let $h : K_0(A) \to K_0(B)$, $h^{\mathbb{C}} : K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(B \otimes_{\mathbb{R}} \mathbb{C})$ and $h^{\mathbb{H}} : K_0(A \otimes_{\mathbb{R}} \mathbb{H}) \to K_0(B \otimes_{\mathbb{R}} \mathbb{H})$ be such that the following two diagrams commute:

$$\begin{array}{ccc} K_0(A^{\mathbb{C}}) & \xrightarrow{\rho_A} & \mathrm{Aff}(T(A^{\mathbb{C}})) \\ \downarrow h^{\mathbb{C}} & & \downarrow M \\ K_0(B^{\mathbb{C}}) & \xrightarrow[\rho_B]{} & \mathrm{Aff}(T(B^{\mathbb{C}})) \end{array}$$

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A \otimes_{\mathbb{R}} \mathbb{C}) & \longrightarrow & K_0(A \otimes_{\mathbb{R}} \mathbb{H}) \\ h \downarrow & & \downarrow h^{\mathbb{C}} & & \downarrow h^{\mathbb{H}} \\ K_0(B) & \longrightarrow & K_0(B \otimes_{\mathbb{R}} \mathbb{C}) & \longrightarrow & K_0(B \otimes_{\mathbb{R}} \mathbb{H}). \end{array}$$

Then there exists $T \in \mathbb{N}$ so that for each set $\{\ell_1, \ldots, \ell_R\}$ of integers with $\ell_j \geq T$ for each j, there is a unital $*$-homomorphism $\psi : A \to B \otimes_{\mathbb{R}} H$, where $H = M_{\ell_1}(\mathbb{R}) \oplus \cdots \oplus M_{\ell_R}(\mathbb{R})$, such that $\psi_* = d_* \circ h$ on $K_0(A), \psi_*^{\mathbb{C}} = d_*^{\mathbb{C}} \circ h^{\mathbb{C}}$ on $K_0(A \otimes_{\mathbb{R}} \mathbb{C}), \psi_*^{\mathbb{H}} = d_*^{\mathbb{H}} \circ h^{\mathbb{H}}$ on $K_0(A \otimes_{\mathbb{R}} \mathbb{H})$ and $\|\hat{\psi}^{\mathbb{C}}(f) - (\hat{d}^{\mathbb{C}} \circ M)(f)\| < \delta$ for all $f \in F$, where $d : B \to B \otimes_{\mathbb{R}} H$ is the $*$-homomorphism $d(b) = b \otimes 1_H$.

Proof. The method of Corollary 4.3 of [14] applies when modified, as in Lemma 6.6 of [20], to combine homomorphisms $\psi_i : A \to B \otimes_{\mathbb{R}} M_2(\mathbb{R})$ rather than homomorphisms $\psi_i : A \to B$. □

5. The classification theorem

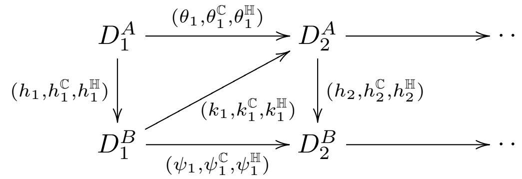
The combination of the existence and uniqueness theorems to produce a classification result proceeds exactly as on pages 374-380 of [6], using the notation of approximately commuting diagrams originally introduced in [5]. The first step establishes a commutative diagram of K_0 maps and an approximately commutative diagram of tracial state spaces.

Lemma 5.1. *Let A, B be direct limits of unital sequences $A_1 \to A_2 \to \cdots, B_1 \to B_2 \to \cdots$ of direct sums of basic building blocks with injective connecting maps, let*

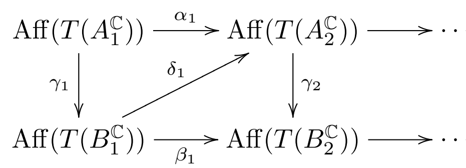
$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A \otimes_{\mathbb{R}} \mathbb{C}) & \longrightarrow & K_0(A \otimes_{\mathbb{R}} \mathbb{H}) \\ \phi_0 \downarrow & & \phi_0^{\mathbb{C}} \downarrow & & \phi_0^{\mathbb{H}} \downarrow \\ K_0(B) & \longrightarrow & K_0(B \otimes_{\mathbb{R}} \mathbb{C}) & \longrightarrow & K_0(B \otimes_{\mathbb{R}} \mathbb{H}) \end{array}$$

be a system of ordered group isomorphisms $\phi_0, \phi_0^{\mathbb{C}}, \phi_0^{\mathbb{H}}$ each preserving the class of the identity and let $\phi_T : T(B^{\mathbb{C}}) \to T(A^{\mathbb{C}})$ be a continuous affine isomorphism with $\phi_T \Phi_B^* = \Phi_A^* \phi_T$ such that $\langle \phi_0^{\mathbb{C}} g, \tau \rangle = \langle g, \phi_T \tau \rangle$ for each $g \in K_0(A^{\mathbb{C}})$ and each $\tau \in T(B^{\mathbb{C}})$.

For each i let D_i^A be the triple $K_0(A_i) \to K_0(A_i \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(A_i \otimes_{\mathbb{R}} \mathbb{H})$ and let D_i^B be the triple $K_0(B_i) \to K_0(B_i \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(B_i \otimes_{\mathbb{R}} \mathbb{H})$. After passing to subsequences *there exists a commutative diagram of positive unital group homomorphisms*



producing the given triple $(\phi_0, \phi_0^{\mathbb{C}}, \phi_0^{\mathbb{H}})$. *After further passing to subsequences there exists an approximately commutative system, in which each map commutes with the maps resulting from the natural involutory antiautomorphisms,*

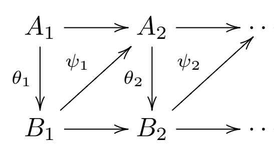


giving rise to $\phi_T^* : \mathrm{Aff}(T(A^{\mathbb{C}})) \to \mathrm{Aff}(T(B^{\mathbb{C}}))$ *and satisfying* $\langle h_i g, \tau \rangle = \langle g, \gamma_i^* \tau \rangle$ *and* $\langle k_i g', \tau' \rangle = \langle g', \delta_i^* \tau' \rangle$ *for each* i, *for each* $g \in K_0(A_i^{\mathbb{C}})$, *each* $\tau \in T(B_i^{\mathbb{C}})$, *each* $g' \in K_0(B_i^{\mathbb{C}})$ *and each* $\tau' \in T(A_{i+1}^{\mathbb{C}})$.

Proof. The argument on pages 374–376 of [6] applies directly to the current situation: by suitably choosing the finite-dimensional approximants to the affine function spaces they can be given involutions compatible with all the relevant maps and this gives rise to the compatibility in the diagram of affine function spaces. □

The next step is to produce a diagram of algebras and unital C^*-homomorphisms as on pages 376–378 of [6].

Lemma 5.2. *Let* $A, B, \phi_0, \phi_0^{\mathbb{C}}, \phi_0^{\mathbb{H}}, \phi_T$ *be as in Lemma* 5.1. *Then, after passing to subsequences, there exists a diagram of unital* C^**-homomorphisms*



such that:

(a) *The induced diagram of sequences of* K_0 *triples* $K_0(A_i) \to K_0(A_i \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(A_i \otimes_{\mathbb{R}} \mathbb{H})$ *and* $K_0(B_i) \to K_0(B_i \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(B_i \otimes_{\mathbb{R}} \mathbb{H})$ *commutes and gives rise to the triple* $(\phi_0, \phi_0^{\mathbb{C}}, \phi_0^{\mathbb{H}})$.

(b) *The induced diagram of affine function spaces* $\mathrm{Aff}(T(A_i^{\mathbb{C}})), \mathrm{Aff}(T(B_i^{\mathbb{C}}))$ *approximately commutes and gives rise to* ϕ_T^*.

(c) *The* K_0 *and trace mappings at each stage are compatible.*

Proof. Firstly construct the K_0 and affine function map sequences of Lemma 5.1. Then, given i, a finite subset F of $A_i^{\mathbb{C}}$ and $\epsilon > 0$, apply Proposition 4.4 to obtain $T \in \mathbb{N}$ such that, for $H = M_{\ell_1}(\mathbb{R}) \oplus \cdots \oplus M_{\ell_R}(\mathbb{R})$ with $\ell_j \geq T$ for each j, there exists a unital $*$-homomorphism $\psi : A_i \to B_i \otimes_{\mathbb{R}} H$ giving rise to the appropriate

K-theory maps and approximately giving rise to the given affine function space maps. Finally apply Proposition 3.6 to obtain $j \geq i$ and $\psi' : B_i \to B_j$, such that ${\psi'}^{\mathbb{C}}$ agrees to within ϵ on F with the original map $B_i \to B_j$ and such that there is a subalgebra H as above in $\psi'(B_i)' \cap B_j$. Then $\psi : A_i \to B_i \otimes_{\mathbb{R}} H$ gives rise to a $*$-homomorphism θ_i from A_i to B_j. Relabel B_j as B_i. A similar argument produces $\psi_i : B_i \to A_{i+1}$, which have been constructed to have the required properties. □

The classification result can now be obtained as on pages 378–380 of [6].

Theorem 5.3. *Let A, B be separable simple unital real C^*-algebras, each of which is the inductive limit of a sequence of direct sums of the basic building blocks $C([0,1], M_q(\mathbb{R}))$, $C([0,1], M_q(\mathbb{C}))$, $C([0,1], M_q(\mathbb{H}))$ and $A(1,\mathbb{R}) \otimes_{\mathbb{R}} M_q(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, where $A(1,\mathbb{R}) = \{f \in C([0,1],\mathbb{C}) : f(t) = \overline{f(1-t)}$ for all $0 \leq t \leq 1\}$. Let Φ_A, Φ_B be the associated involutory antiautomorphisms of $A^{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ and $B^{\mathbb{C}} = B \otimes_{\mathbb{R}} \mathbb{C}$. Suppose that there exists a triple of unital ordered group isomorphisms $(\phi_0, \phi_0^{\mathbb{C}}, \phi_0^{\mathbb{H}})$ from the triple $K_0(A) \to K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(A \otimes_{\mathbb{R}} \mathbb{H})$ to the triple $K_0(B) \to K_0(B \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(B \otimes_{\mathbb{R}} \mathbb{H})$ and that there exists a continuous affine isomorphism $\phi_T : T(B^{\mathbb{C}}) \to T(A^{\mathbb{C}})$ from the tracial state space of $B^{\mathbb{C}}$ to that of $A^{\mathbb{C}}$, with $\phi_T \Phi_B^* = \Phi_A^* \phi_T$, and that ϕ^T and $\phi_0^{\mathbb{C}}$ are compatible. Then there exists a $*$-isomorphism $\phi : A \to B$ which gives rise to the map ϕ_T and the triple $(\phi_0, \phi_0^{\mathbb{C}}, \phi_0^{\mathbb{H}})$.*

Proof. From the diagram of C^*-homomorphisms given by Lemma 5.2 there is a diagram of complexifications, where each map respects the involutory antiautomorphisms given by the real algebras. The argument on pages 378–380 of [6] shows that by a passage to subsequences the hypotheses of Proposition 2.6 are satisfied. (The only extra ingredient from Theorem 6 of [6] is the condition on the unitary which ensures that the corresponding inner automorphism respects the relevant involutions.) As on page 380 of [6] the diagram can be amended by composing with inner automorphisms to give the required result. □

References

[1] B. Blackadar, *K-theory for operator algebras* (Mathematical Sciences Research Institute Publications: 5), Springer-Verlag, New York, 1986, MR 0859867 (88g:46082), Zbl 0597.46072.

[2] O. Bratteli, *Inductive limits of finite-dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234, MR 0312282 (47 #844), Zbl 0264.46057.

[3] M-D Choi and G.A. Elliott, *Density of the self-adjoint elements with finite spectrum in an irrational rotation C^*-algebra*, Math. Scand. **67** (1990), 73–86, MR 1081290 (92a:46062), Zbl 0743.46070.

[4] G.A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44, MR 0397420 (53 #1279), Zbl 0323.46063.

[5] G.A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179–219, MR 1241132 (94i:46074), Zbl 0809.46067.

[6] G.A. Elliott, *A classification of certain simple C^*-algebras*, Quantum and Noncommutative Analysis (editors H. Araki et al.) Kluwer, Dordrecht, 1993, 373–385, MR 1276305 (95h:46089), Zbl 0843.46045.

[7] G.A. Elliott, *The classification problem for amenable C^*-algebras*, Proceedings of the International Congress of Mathematicians, Vol. 2 (Zurich, 1994), 922–932, Birkhauser, Basel, 1995, MR 1403992 (97g:46072), Zbl 0946.46050.

[8] E.G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conference Series in Mathematics, **46**, Conference Board of the Mathematical Sciences, Washington, D.C., 1981, MR 0623762 (84k:46042), Zbl 0475.46050.

[9] T. Giordano, *A classification of approximately finite real C^*-algebras*, J. Reine Angew. Math. **385** (1988), 161–194, MR 0931219 (89h:46078), Zbl 0635.46055.

[10] J.G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340, MR 0112057 (22 #2915), Zbl 0094.09701.

[11] K.R. Goodearl, *Notes on real and complex C^*-algebras*, Shiva, Natwich, Cheshire, 1982, MR 0677280 (85d:46079), Zbl 0495.46039.

[12] K.R. Goodearl and D. Handelman, *Classification of ring and C^*-algebra direct limits of finite-dimensional semisimple real algebras*, Mem. Amer. Math. Soc. **69**(**372**) (1987), MR 0904013 (88k:46067), Zbl 0629.46053.

[13] H. Lin, *An introduction to the classification of amenable C^*-algebras*, World Scientific, Singapore, 2001, MR 1884366 (2002k:46141), Zbl 1013.46055.

[14] K.E. Nielsen and K. Thomsen, *Limits of circle algebras*, Exposition. Math. **14** (1996), 17–56, MR 1382013 (97e:46097), Zbl 0865.46037.

[15] J. Phillips and I. Raeburn, *Automorphisms of C^*-algebras and second Čech cohomology*, Indiana Univ. Math. Journal **29** (1980), 799–822, MR 0589649 (82b:46089), Zbl 0475.46047.

[16] M. Rørdam and E. Størmer, *Classification of nuclear C^*-algebras. Entropy in operator algebras*, Encylopaedia of Mathematical Sciences VII, Springer-Verlag, Berlin, Heidelberg, New York, 2002, MR 1878881 (2002i:46047), Zbl 0985.00012.

[17] J. Rosenberg, *Continuous trace algebras from the bundle theoretic point of view*, J. Austral. Math. Soc. **47** (1989), 368–381, MR 1018964 (91d:46090), Zbl 0695.46031.

[18] P.J. Stacey, *Involutory $*$-antiautomorphisms in a direct limit of matrix algebras*, J. London Math. Soc. (2) **30** (1984), 486–500, MR 0810958 (88f:46118), Zbl 0525.46028.

[19] P.J. Stacey, *Real structure in direct limits of finite-dimensional C^*-algebras*, J. London Math. Soc. (2) **35** (1987), 339–352, MR 0881522 (88g:46083), Zbl 0591.46049.

[20] K. Thomsen, *Limits of certain subhomogeneous C^*-algebras*, Mém Soc. Math.Fr. (N.S.) No 71 (1997), vi + 125pp (1998), MR 1649315 (2000c:46110), Zbl 0922.46055.

[21] K. Thomsen, *Inductive limits of interval algebras: the tracial state space*, Amer. J. Math. **116** (1994), 605–620, MR 1277448 (95f:46099), Zbl 0814.46050.

DEPARTMENT OF MATHEMATICS, LA TROBE UNIVERSITY, VICTORIA 3086, AUSTRALIA
P.Stacey@latrobe.edu.au http://www.latrobe.edu.au/mathstats/staff/stacey.html

This paper is available via http://nyjm.albany.edu:8000/j/2004/10-13.html.