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Modules with Injective Cohomology, and Local Duality for a Finite Group

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Abstract. Let G be a finite group and k a field of characteristic p dividing $|G|$. Then Greenlees has developed a spectral sequence whose E_2 term is the local cohomology of $H^*(G,k)$ with respect to the maximal ideal, and which converges to $H_*(G,k)$. Greenlees and Lyubeznik have used Grothendieck's dual localization to provide a localized form of this spectral sequence with respect to a homogeneous prime ideal $\mathfrak{p}$ in $H^*(G,k)$, and converging to the injective hull $I_{\mathfrak{p}}$ of $H^*(G,k)/\mathfrak{p}$.

The purpose of this paper is give a representation theoretic interpretation of these local cohomology spectral sequences. We construct a double complex based on Rickard's idempotent kG-modules, and agreeing with the Greenlees spectral sequence from the E_2 page onwards. We do the same for the Greenlees-Lyubeznik spectral sequence, except that we can only prove that the E_2 pages are isomorphic, not that the spectral sequences are. Ours converges to the Tate cohomology of the certain modules $\kappa_{\mathfrak{p}}$ introduced in a paper of Benson, Carlson and Rickard. This leads us to conjecture that $\hat{H}^*(G,\kappa_{\mathfrak{p}}) \cong I_{\mathfrak{p}}$, after a suitable shift in degree. We draw some consequences of this conjecture, including the statement that $\kappa_{\mathfrak{p}}$ is a pure injective module. We are able to prove the conjecture in some cases, including the case where $H^*(G,k)_{\mathfrak{p}}$ is Cohen–Macaulay.

Contents

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1. Introduction

Let G be a finite group and k be a field of characteristic p. By a theorem of Evens and Venkov [10, 23], the cohomology ring $H^*(G,k)$ is a finitely generated graded commutative k-algebra. Further restrictions on the structure of $H^*(G,k)$ can be expressed in terms of the existence of certain spectral sequences. The first of these was constructed by Benson and Carlson [2] using multiple complexes and related finite Poincaré duality complexes of projective kG-modules. It follows from the existence of this spectral sequence that if $H^*(G,k)$ is Cohen–Macaulay, then it is Gorenstein. Even if $H^*(G,k)$ is not Cohen–Macaulay, the existence of the spectral sequence gives severe restrictions on the possibilities for the ring structure. An example of an application of this spectral sequence is the statement that if $H^*(G,k)$ is a polynomial ring, then it follows that $p = 2$, the generators are in degree one, and $G/O(G)$ is an elementary abelian 2-group.

Greenlees [11] found a way of using the same techniques coming from multiple complexes to construct a cleaner spectral sequence expressing much of the same information. This spectral sequence can be written as

$$H^*_{\mathfrak{m}}\, H^*(G,k) \Rightarrow H_*(G,k).$$

Here, $\mathfrak{m}$ denotes the maximal ideal generated by the elements of positive degree in $H^*(G,k)$, and $H^*_{\mathfrak{m}}$ denotes local cohomology in the sense of Grothendieck [13, 14]. Greenlees and Lyubeznik [12] showed how to apply dual localization to this spectral sequence, with respect to a homogeneous prime ideal $\mathfrak{p}$ of $H^*(G,k)$. This leads to a spectral sequence whose E_2 page is

$$H^*_{\mathfrak{p}}\, H^*(G,k)_{\mathfrak{p}}$$

and which converges to the injective hull, in the graded sense, of $H^*(G,k)/\mathfrak{p}$ as a $H^*(G,k)$-module, suitably shifted in degree.

The first purpose of this paper is to provide a construction of the Greenlees spectral sequence using Rickard's idempotent modules [22]. This construction is essentially a stabilized version of a construction which Carlson and Wheeler [8] introduced in the context of complexity quotient categories. The construction is given in Section 3, where the following theorem is made more precise.

Theorem 1.1. *There is a double complex consisting of homomorphisms from projective kG-modules to Rickard idempotent modules, whose spectral sequence is isomorphic to the Greenlees spectral sequence, from the E_1 page onwards.*

We give an interpretation in terms of this spectral sequence of the "last survivor" method of Section 7 of [2]. The result is a slightly stronger theorem, the proof of which can be found in Section 4.

Theorem 1.2. *Let $\zeta_1, \dots, \zeta_r$ be a homogeneous set of parameters for $H^*(G,k)$, and let $a_i = \deg(\zeta_i)$ $(1 \le i \le r)$. Then there is an element of degree $\sum_{i=1}^r (a_i - 1)$ in $H^*(G,k)$ which is not in the ideal $(\zeta_1, \dots, \zeta_r)$ and is not annihilated by any nonzero element of $k[\zeta_1, \dots, \zeta_r]$.*

In Section 5, we provide an analagous construction of a spectral sequence with the same E_2 page as the Greenlees–Lyubeznik dual localized spectral sequence. We have been unable to prove that these two spectral sequences are isomorphic, although this seems very likely. Our spectral sequence converges to the Tate cohomology $\hat{H}^*(G,\kappa_\mathfrak{p})$ with coefficients in the module $\kappa_\mathfrak{p}$ introduced in introduced by Benson, Carlson and Rickard [3] (in that paper, the module is called $\kappa(V)$, where V is the closed homogeneous irreducible subvariety of the cohomology variety V_G corresponding to $\mathfrak{p}$).

The modules $\kappa_\mathfrak{p}$ are interesting because they are idempotent, in the sense that $\kappa_\mathfrak{p} \otimes \kappa_\mathfrak{p} \cong \kappa_\mathfrak{p}$ in the stable module category, and tensoring with $\kappa_\mathfrak{p}$ picks out the "layer" in the stable module category corresponding to the prime $\mathfrak{p} \subseteq H^*(G,k)$.

If it could be proved that the two spectral sequences are isomorphic, then it would follow that $\hat{H}^*(G,\kappa_\mathfrak{p})$ is an injective $H^*(G,k)$-module. This would allow the modules $\kappa_\mathfrak{p}$ to be compared with the modules $T(I_\mathfrak{p})$ introduced by Benson and Krause [5]. The second purpose of this paper is to investigate some special cases in which we are able to make this comparison. We give evidence for the following conjecture.

Conjecture 1.3. *Let $\mathfrak{p} \neq \mathfrak{m}$ be a homogeneous prime ideal of dimension d in $H^*(G,k)$. Then $T(I_\mathfrak{p})$ and $\Omega^{-d}\kappa_\mathfrak{p}$ are stably isomorphic.*

The modules $T(I_\mathfrak{p})$ are the representing objects for the contravariant exact functor which takes a module to the Matlis dual of its Tate cohomology, in the sense that there are functorial isomorphisms

$$D_\mathfrak{p}\hat{H}^*(G,M) \cong \widehat{\mathrm{Ext}}^*_{kG}(M,T(I_\mathfrak{p})).$$

In particular, the $T(I_\mathfrak{p})$ are pure injective modules, and their cohomology is given by $\hat{H}^*(G,T(I_\mathfrak{p})) \cong I_\mathfrak{p}$, the injective hull of $H^*(G,k)/\mathfrak{p}$.

The reason for the interest in the above conjecture is that what we know about the properties of $T(I_\mathfrak{p})$ and of $\Omega^{-d}\kappa_\mathfrak{p}$ have little overlap. That a single module should have all these properties seems to have be significant. It seems likely that there are many consequences for the interactions between commutative algebra and modular representation theory.

So for example, it would follow from this conjecture that $\kappa_\mathfrak{p}$ is pure injective, so that there are no phantom maps into $\kappa_\mathfrak{p}$. It also follows that $\hat{H}^*(G,\kappa_p) \cong I_\mathfrak{p}[d]$, and that $\mathrm{End}_k(\kappa_\mathfrak{p})$ is a kG-module whose Tate cohomology is the completion $H^*(G,k)^\wedge_\mathfrak{p}$ of the localized cohomology ring with respect to the powers of its maximal ideal. Another consequence of the conjecture is the formula

$$T(I_\mathfrak{p}) \otimes_k T(I_\mathfrak{p}) \cong \Omega^{-d}T(I_\mathfrak{p})$$

in the stable module category.

We prove the above conjecture in the case where $H^*(G,k)_\mathfrak{p}$ is Cohen–Macaulay of Krull dimension $r-d$, and also in the case where the depth is $r-d-1$ and the Krull dimension is $r-d$. In particular, since any localization of a Cohen–Macaulay ring is Cohen–Macaulay, the conjecture holds whenever $H^*(G,k)$ is Cohen–Macaulay. The proof of the following theorem can be found in Section 7.

Theorem 1.4. *Let G be a finite group of p-rank r, and let k be a field of characteristic p. Suppose that $\mathfrak{p} \neq \mathfrak{m}$ is a homogeneous prime ideal in $H^*(G,k)$ of Krull dimension d, such that the localization $H^*(G,k)_\mathfrak{p}$ satisfies one of the following conditions:*

(∗) *$H^*(G,k)_{\mathfrak{p}}$ is Cohen–Macaulay of Krull dimension $r-d$ (i.e., its depth is also equal to $r-d$), or*
(∗∗) *$H^*(G,k)_{\mathfrak{p}}$ has depth $r-d-1$ and Krull dimension $r-d$.*

Then the following hold:

(i) *The cohomology $\hat{H}^*(G,\kappa_{\mathfrak{p}})$ is injective in the category of graded $H^*(G,k)$-modules; in particular, it is isomorphic to $I_{\mathfrak{p}}[d]$.*
(ii) *The modules $T(I_{\mathfrak{p}})$ and $\Omega^{-d}\kappa_{\mathfrak{p}}$ are stably isomorphic.*
(iii) *The modules $\kappa_{\mathfrak{p}}$ are pure injective.*
(iv) *There are no phantom maps into $\kappa_{\mathfrak{p}}$.*
(v) *The ring $\widehat{\mathrm{Ext}}^*_{kG}(\kappa_{\mathfrak{p}},\kappa_{\mathfrak{p}}) \cong \hat{H}^*(G,\mathrm{End}_k(\kappa_{\mathfrak{p}}))$ is isomorphic to the completion $H^*(G,k)^{\wedge}_{\mathfrak{p}}$ of the localized cohomology $H^*(G,k)_{\mathfrak{p}}$ with respect to the maximal ideal $\mathfrak{p}_{\mathfrak{p}}$.*
(vi) *The formula $T(I_{\mathfrak{p}}) \otimes T(I_{\mathfrak{p}}) \cong \Omega^{-d}T(I_{\mathfrak{p}})$ holds in the stable module category.*

If $H^*(G,k)$ is Cohen–Macaulay, then (∗) holds for $H^*(G,k)_{\mathfrak{p}}$. Furthermore, $H^*(G,k)$ is Cohen–Macaulay in (at least) the following cases:

(a) *G has abelian Sylow p-subgroups,*
(b) *$G = GL_n(\mathbb{F}_q)$ with q a power of a prime different from p,*
(c) *$p = 2$ and G has extraspecial Sylow 2-subgroups,*
(d) *$p = 2$ and G is a finite simple group of 2-rank at most three, or isomorphic to $U_3(2^n)$ or $Sz(2^{2n+1})$,*
(e) *$p = 3$ and G has extraspecial Sylow 3-subgroups of order 3^{1+2} and exponent 3.*

Finally, if G has p-rank at most two, then either (∗) or (∗∗) is satisfied for every prime ideal $\mathfrak{p}$.

Background material on commutative algebra can be found in Matsumura [16]. References on local cohomology include Brodmann and Sharp [6], Bruns and Herzog [7], and Grothendieck [13, 14].

Conventions and notations. We assume throughout that G is a finite group of order divisible by p, and that k is a field of characteristic p. When we talk about modules and maps over a $\mathbb{Z}$-graded ring, we always mean graded modules unless otherwise specified. So for example when we talk about injective hulls, these are taken in the category of graded modules. This is not the same as injective hulls in the category of all modules. Also, when we use the term “ideal”, we mean graded ideal unless otherwise specified. If M is a graded module, we write $M[n]$ for the graded module with $M[n]_i = M_{i+n}$.

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2. Depth

Much of this paper is concerned with the depth and the Krull dimension of the cohomology ring, and various localizations of it. We begin with a statement that must be well known, but for which it does not seem easy to find a reference in the literature.

Theorem 2.1. *Let P be a Sylow p-subgroup of G, and k a field of characteristic p. Then the depth of $H^*(G,k)$ is at least as big as the depth of $H^*(P,k)$.*

Proof. There are several ways to prove this theorem. One way is to notice that because of the existence of the transfer map, $H^*(G,k)$ is a direct summand of $H^*(P,k)$ as an $H^*(G,k)$-module. Since $H^*(P,k)$ is finitely generated as an $H^*(G,k)$-module, the depth of $H^*(P,k)$ as a ring is equal to its depth as an $H^*(G,k)$-module. Its depth is therefore at most as big as the depth of the summand $H^*(G,k)$. □

A localized version of the same theorem is also true.

Theorem 2.2. *Let $\mathfrak{p}$ be a prime ideal of $H^*(P,k)$, and let $\mathfrak{q} = \mathfrak{p} \cap H^*(G,k)$ be the prime ideal of $H^*(G,k)$ over which it lies. Then the depth of the localization $H^*(G,k)_\mathfrak{q}$ is at least as big as the depth of $H^*(P,k)_\mathfrak{p}$.*

Proof. The proof is essentially the same as the proof of Theorem 2.1, but this time, we need to observe that $H^*(G,k)_\mathfrak{q}$ is a direct summand of $H^*(P,k)_\mathfrak{p}$ as an $H^*(G,k)_\mathfrak{q}$-module. □

3. The Greenlees spectral sequence

We begin by recalling what we need from Rickard's theory of idempotent modules [22]. If $\mathcal{C}$ is a thick subcategory of the stable category $\mathsf{stmod}(kG)$ of finitely generated kG-modules, then there is a triangle in $\mathsf{StMod}(kG)$ of the form

$$E_\mathcal{C} \to k \to F_\mathcal{C} \to \Omega^{-1} E_\mathcal{C}$$

where $E_\mathcal{C}$ is a direct limit of modules in $\mathcal{C}$, and $F_\mathcal{C}$ is $\mathcal{C}$-local in the sense that $\underline{\mathrm{Hom}}_{kG}(X, F_\mathcal{C}) = 0$ for all modules X in $\mathcal{C}$. The modules $E_\mathcal{C}$ and $F_\mathcal{C}$ are idempotent modules in the sense that each is stably isomorphic to its tensor square.

If $\mathcal{V}$ is a collection of closed homogeneous irreducible subvarieties of the cohomology variety V_G, closed under specialization (i.e., if $V \in \mathcal{V}$ and $W \subseteq V$ then $W \in \mathcal{V}$), let $\mathcal{C}_\mathcal{V}$ be the thick subcategory consisting of the modules M such that each irreducible component of $V_G(M)$ is in $\mathcal{V}$. In this case, we write $E_\mathcal{V}$ and $F_\mathcal{V}$ for the corresponding idempotent modules. Finally, if $\zeta \in H^n(G,k)$, write E_ζ and F_ζ for the idempotent modules corresponding to the hypersurface in V_G determined by ζ. The module F_ζ can be described as the colimit of

$$k \xrightarrow{\hat{\zeta}} \Omega^{-n} k \xrightarrow{\hat{\zeta}} \Omega^{-2n} k \to \dots$$

where the maps $\hat{\zeta}$ are arbitrary representatives of ζ. In particular, $\hat{H}^*(G, F_\zeta)$ is the localization of $H^*(G,k)$ (or equivalently of $\hat{H}^*(G,k)$) obtained by inverting ζ. More generally, if $\mathcal{V}$ is any nonempty collection generated by hypersurfaces, in the sense that each element of $\mathcal{V}$ is contained in a hypersurface in $\mathcal{V}$, then $\hat{H}^*(G, F_\mathcal{V})$ is the localization of $H^*(G,k)$ (or of $\hat{H}^*(G,k)$, it doesn't matter) obtained by inverting the elements corresponding to the hypersurfaces. Furthermore, for any kG-module M, $\hat{H}^*(G, F_\mathcal{V} \otimes_k M)$ is the localization of $\hat{H}^*(G,M)$ obtained by inverting the same elements.

Choose a homogeneous set of parameters $\zeta_1, \dots, \zeta_r$ for $H^*(G,k)$. For each ζ_i, we have a cochain complex

$$0 \to k \to F_{\zeta_i} \to 0$$

where k is taken to be in degree zero and F_{ζ_i} in degree one. The cohomology of this complex is $\Omega^{-1}E_{\zeta_i}$, concentrated in degree one.

Tensoring these complexes together (over k, with diagonal G-action), we obtain a complex

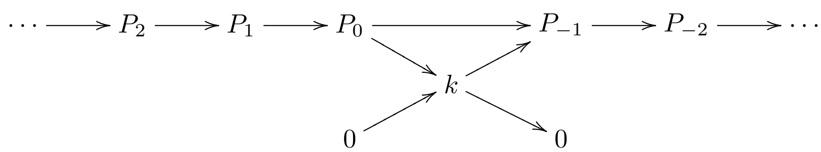
$$\Lambda^* = \Lambda^*(\zeta_1, \dots, \zeta_r) :$$
$$0 \to k \to \bigoplus_{1 \le i \le r} F_{\zeta_i} \to \bigoplus_{1 \le i < j \le r} F_{\zeta_i} \otimes_k F_{\zeta_j} \to \dots \to \bigotimes_{1 \le i \le r} F_{\zeta_i} \to 0.$$

By the Künneth theorem, this is exact except in degree r, where its cohomology is

$$\Omega^{-1}E_{\zeta_1} \otimes_k \cdots \otimes_k \Omega^{-1}E_{\zeta_r}$$

which is a projective kG-module.

Now let $\hat{P}_*$ be a Tate resolution of the trivial kG-module. In other words, $\hat{P}_*$ is a complex of the form



obtained by splicing together a projective resolution and an injective resolution of the trivial module.

Consider the double complex $\hat{E}_0^{*,*} = \operatorname{Hom}_{kG}(\hat{P}_*, \Lambda^*)$. This gives rise to two spectral sequences. The spectral sequence where we take cohomology with respect to the differential coming from Λ^* first has its E_1 page concentrated on a single line, where it consists of

$$\operatorname{Hom}_{kG}(\hat{P}_*, \Omega^{-1}E_{\zeta_1} \otimes_k \cdots \otimes_k \Omega^{-1}E_{\zeta_r}).$$

The E_2 page is then the Tate cohomology of this projective module, which is zero. It follows that the cohomology of the total complex $\operatorname{Tot} \hat{E}_0^{*,*}$ is zero.

On the other hand, if we take cohomology with respect to the differential coming from $\hat{P}_*$ first, then the E_1 page is

$$\hat{E}_1^{s,t} = \hat{H}^t(G, \Lambda^s).$$

Now each Λ^s is a direct sum of modules of the form F_ζ, where ζ is a product of a subset of size s of $\zeta_1, \dots, \zeta_r$. The cohomology of this module is the localization $\hat{H}^*(G,k)_\zeta$. The maps in the complex are the signed inclusions. So this is the stable Koszul complex

$$\hat{E}_1^{*,*} = C^*(\hat{H}^*(G,k); \zeta_1, \dots, \zeta_r).$$

See for example Section 3.5 of Bruns and Herzog [7] for a description of the stable Koszul complex and the fact that it calculates local cohomology. Thus we have

$$\hat{E}_2^{s,t} = H_{\mathfrak{m}}^{s,t} \hat{H}^*(G,k) \Rightarrow 0.$$

Here, we have given local cohomology a double grading. The first grading is the local cohomological grading, and the second is the internal grading coming from the fact that $\hat{H}^*(G,k)$ is a graded ring. The differentials in this spectral sequence are

$$d_j \colon \hat{E}_j^{s,t} \to \hat{E}_j^{s+j, t-j+1}.$$

Now, this is not quite the spectral sequence which Greenlees [11] constructed, so we make the following modification. Let $\hat{P}_*^-$ be the subcomplex

$$0 \to P_{-1} \to P_{-2} \to \dots$$

of $\hat{P}_*$, and let P_* be the quotient complex

$$\dots \to P_1 \to P_0 \to 0.$$

Then we can consider $\operatorname{Hom}_{kG}(\hat{P}_*^-, k)$ as a quotient complex of $\hat{E}_0^{*,*}$ by a subcomplex which we call $E_0^{*,*}$ of the following form.

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}_{kG}(P_{-2}, \Lambda^1) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Hom}_{kG}(P_{-2}, \Lambda^r) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}_{kG}(P_{-1}, \Lambda^1) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Hom}_{kG}(P_{-1}, \Lambda^r) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_{kG}(P_0, k) & \longrightarrow & \operatorname{Hom}_{kG}(P_0, \Lambda^1) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Hom}_{kG}(P_0, \Lambda^r) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_{kG}(P_1, k) & \longrightarrow & \operatorname{Hom}_{kG}(P_1, \Lambda^1) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Hom}_{kG}(P_1, \Lambda^r) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_{kG}(P_2, k) & \longrightarrow & \operatorname{Hom}_{kG}(P_2, \Lambda^1) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Hom}_{kG}(P_2, \Lambda^r) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & & & \vdots & &
\end{array}$$

We have a short exact sequence of complexes

$$0 \to \operatorname{Tot} E_0^{*,*} \to \operatorname{Tot} \hat{E}_0^{*,*} \to \operatorname{Hom}_{kG}(\hat{P}_*^-, k) \to 0.$$

Since $\operatorname{Tot} \hat{E}_0^{*,*}$ is exact, this gives

$$H^n(\operatorname{Tot} E_0^{*,*}) \cong H^{n+1}(\operatorname{Hom}_{kG}(\hat{P}_*^-, k)).$$

The cohomology of $\operatorname{Hom}_{kG}(\hat{P}_*^-, k)$ is the negative part of Tate cohomology, shifted in degree by one, $\hat{H}^-(G, k)[-1]$, which is the same as ordinary homology with degrees negated. So we have

$$H^n(\operatorname{Tot} E_0^{*,*}) \cong \begin{cases} H_{-n}(G, k) & n \leq 0 \\ 0 & n > 0. \end{cases}$$

Now, localizing Tate cohomology at any nonzero element kills all negative degree elements, and so the result is the same as the localization of ordinary cohomology. It follows that if we take cohomology with respect to the vertical differential first, the E_1 page of the spectral sequence is the stable Koszul complex for ordinary cohomology,

$$E_1^{*,*} = C^*(H^*(G,k);\zeta_1,\dots,\zeta_r).$$

Thus we have

$$E_2^{s,t} = H^{s,t}_{\mathfrak{m}}\, H^*(G,k) \Rightarrow H_{-s-t}(G,k).$$

This is the required module theoretic construction of the Greenlees spectral sequence.

Note that as a module over $H^*(G,k)$, the homology $H_*(G,k)$ is just the injective hull $I_{\mathfrak{m}}$ of $k = H^*(G,k)/\mathfrak{m}$. So another way of writing this spectral sequence is

$$E_2^{*,*} = H^*_{\mathfrak{m}}\, H^*(G,k) \Rightarrow I_{\mathfrak{m}}. \tag{3.1}$$

4. The last survivor

In this section, we give an interpretation of the last survivor of Section 7 of [2] in terms of the Greenlees spectral sequence (3.1).

Since $H^*(G,k)$ and $H_*(G,k)$ are vector space duals, the identity element of $H^0(G,k)$ corresponds to an isomorphism $\alpha_G : H_0(G,k) \to k$ with the property that if $i_{H,G} : H \to G$ is the inclusion of a subgroup H of G then $\alpha_G \circ (i_{H,G})_* = \alpha_H$.

Now suppose that G has p-rank r, so that by a theorem of Quillen [19, 20], $H^*(G,k)$ has Krull dimension r. Then $H^i_{\mathfrak{m}}\, H^*(G,k) = 0$ for $i > r$, and so there is a right-hand edge homomorphism for the Greenlees spectral sequence (3.1)

$$H^{r,t}_{\mathfrak{m}}\, H^*(G,k) \to H_{-r-t}(G,k).$$

This is an isomorphism precisely when $H^*(G,k)$ is Cohen–Macaulay. Composing with α_G, we obtain a homomorphism

$$\gamma_G : H^{r,-r}_{\mathfrak{m}}\, H^*(G,k) \to k.$$

If H is a subgroup of G then transfer in negative Tate cohomology $\hat{H}^{-n}(H,k) \to \hat{H}^{-n}(G,k)$ is the same as the map $(i_{H,G})_* : H_{n-1}(H,k) \to H_{n-1}(G,k)$ induced by the inclusion $i_{H,G} : H \to G$. It follows that transfer induces a homomorphism of spectral sequences

$$\begin{array}{ccc} H^{s,t}_{\mathfrak{m}}\, H^*(H,k) & \Longrightarrow & H_{-s-t}(H,k) \\ \Big\downarrow{\scriptstyle (\mathrm{Tr}_{H,G})_*} & & \Big\downarrow{\scriptstyle (i_{H,G})_*} \\ H^{s,t}_{\mathfrak{m}}\, H^*(G,k) & \Longrightarrow & H_{-s-t}(G,k) \end{array} \tag{4.1}$$

Now let E be an elementary abelian p-subgroup of G of maximal rank r. Then $H^*(E,k)$ is Cohen–Macaulay of Krull dimension r, and so the edge homomorphism is an isomorphism in this case. So the edge homomorphisms fit into a commutative diagram

$$\begin{array}{ccccc} H^{r,-r}_{\mathfrak{m}} H^*(E,k) & \xrightarrow{\cong} & H_0(E,k) & \xrightarrow{\alpha_E} & k \\ \Big\downarrow{\scriptstyle (\mathrm{Tr}_{E,G})_*} & & \Big\downarrow{\scriptstyle (i_{E,G})_*} & & \Big\| \\ H^{r,-r}_{\mathfrak{m}} H^*(G,k) & \longrightarrow & H_0(G,k) & \xrightarrow{\alpha_G} & k \end{array}$$

It follows that

$$\gamma_G \circ (\mathrm{Tr}_{E,G})_* = \gamma_E. \tag{4.2}$$

Since the top horizontal map in the above commutative square is an isomorphism, it follows that γ_E is nonzero, and hence also γ_G is nonzero.

Theorem 4.1. *There is a nonzero canonical homomorphism*

$$\gamma_G \colon H^{r,-r}_{\mathfrak{m}} H^*(G,k) \to k. \qquad \square$$

A good way of interpreting this theorem is in terms of Grothendieck duality [13, 14]. Let R be a polynomial subring $k[\zeta_1, \dots, \zeta_r]$ of $H^*(G,k)$ over which the latter is finitely generated as a module (i.e., a Noether normalization of $H^*(G,k)$). Set $a_i = \deg(\zeta_i)$, and $a = \sum_{i=1}^r a_i$. Then the dualizing module for R is $R[-a]$. So Grothendieck duality for R states that for any finitely generated R-module M, the graded vector space dual of $H^{i,*}_{\mathfrak{m}}(M)$ is isomorphic to $\mathrm{Ext}^{r-i,-*}_R(M, R[-a])$. So for example, dualizing the spectral sequence (3.1) gives a spectral sequence

$$\mathrm{Ext}^{r-s,-t}_R(H^*(G,k), R[-a]) \Rightarrow H^{-s-t}(G,k).$$

Reindexing gives

$$\mathrm{Ext}^{s,t}_R(H^*(G,k), R[-a+r]) \Rightarrow H^{s+t}(G,k).$$

Similarly, dualizing the transfer diagram of spectral sequences (4.1) gives

$$\begin{array}{ccc} \mathrm{Ext}^{s,t}_R(H^*(G,k), R[-a+r]) & \Longrightarrow & H^{s+t}(G,k) \\ \big\downarrow {\scriptstyle \mathrm{Tr}^*_{H,G}} & & \big\downarrow {\scriptstyle \mathrm{res}_{H,G}} \\ \mathrm{Ext}^{s,t}_R(H^*(H,k), R[-a+r]) & \Longrightarrow & H^{s+t}(H,k), \end{array}$$

provided that R is also a Noether normalization of $H^*(H,k)$ via restriction, which happens precisely when H has p-rank r. Since $\mathrm{Ext}^0 = \mathrm{Hom}$, this means that the element γ_G can be interpreted as an element of

$$\mathrm{Hom}^*_R(H^*(G,k), R[-a])$$

of degree r (dualizing negates the degree), or in other words as a degree preserving R-module homomorphism

$$\widetilde{\gamma}_G \colon H^*(G,k) \to R[-a+r].$$

In this guise, the transfer formula (4.2) now reads

$$\widetilde{\gamma}_G \circ \mathrm{Tr}_{E,G} = \widetilde{\gamma}_E. \tag{4.3}$$

The homomorphism $\widetilde{\gamma}_E$ is easy to interpret. As an R-module, $H^*(E,k)$ is free on a finite set of homogeneous generators. These generators lie in degrees zero through $a - r = \sum_{i=1}^r (a_i - 1)$, and exactly one generator, say u, lies in the largest degree $a-r$. It follows that $\widetilde{\gamma}_E$ must vanish on all the generators except u, and must take u to some scalar in k times the identity element of $R[-a+r]$. In particular, $\widetilde{\gamma}_E$ is surjective. It follows from the transfer formula that $\widetilde{\gamma}_G$ is also surjective.

Theorem 4.2. *There is a surjective R-module homomorphism $\widetilde{\gamma}_G : H^*(G,k) \to R[-a+r]$.*

In particular, choosing an inverse image under $\widetilde{\gamma}_G$ for the identity element of R, we see that $H^*(G,k)$ has a nonzero element in degree $a-r$ which is not in the ideal $(\zeta_1,\dots,\zeta_r)$, and is not annihilated by multiplication by any nonzero element of R. Such an element can be chosen to be a transfer from E. It is not unique, but it is called the "last survivor".

In fact, this interpretation of the last survivor gives more information than was given in Section 7 of [2], because in that context it was not at all clear that the last survivor is not annihilated by any nonzero element of R.

5. The kappa modules

Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal in $H^*(G,k)$, corresponding to an irreducible subvariety V of the cohomology variety V_G. Let $\mathcal{V}$ denote the set of subvarieties W satisfying $W \not\supseteq V$. Then the kappa module $\kappa_\mathfrak{p}$ is defined to be $E_V \otimes F_\mathcal{V}$. Actually, this is not quite the definition given in [3] (where the notation $\kappa(V)$ is used), but is easily seen to be equivalent. This definition is better suited to our purposes, because $\mathcal{V}$ is determined by the hypersurfaces in it, and this makes $\hat{H}^*(G,F_\mathcal{V})$ easier to calculate. Namely, we have

$$\hat{H}^*(G,F_\mathcal{V}) = H^*(G,k)_\mathfrak{p},$$

the localization of $H^*(G,k)$ obtained by formally inverting all the elements not in $\mathfrak{p}$.

Now let h be the height of $\mathfrak{p}$, which is equal to the Krull dimension of the localization $H^*(G,k)_\mathfrak{p}$. Using the form of Noether normalization proved in Nagata [17], we can choose a homogeneous system of parameters $\zeta_1,\dots,\zeta_r$ for $H^*(G,k)$ in such a way that $\zeta_1,\dots,\zeta_h$ lie in $\mathfrak{p}$. This means that the images of $\zeta_1,\dots,\zeta_h$ in $H^*(G,k)_\mathfrak{p}$ form a system of parameters there. Consider the complex

$$\Lambda^*(\zeta_1,\dots,\zeta_h): \quad 0 \to k \to \bigoplus_{1\le i\le h} F_{\zeta_i} \to \bigoplus_{1\le i<j\le h} F_{\zeta_i}\otimes_k F_{\zeta_j} \to \dots \to \bigotimes_{1\le i\le h} F_{\zeta_i} \to 0$$

obtained by tensoring together the complexes

$$0 \to k \to F_{\zeta_i} \to 0$$

$(1 \le i \le h)$ as in Section 3. The cohomology of this complex is

$$\begin{aligned}\Omega^{-1}E_{\zeta_1}\otimes_k \dots \otimes_k \Omega^{-1}E_{\zeta_h} &\cong \Omega^{-h}(E_{\zeta_1}\otimes_k\dots\otimes_k E_{\zeta_h}) \oplus (\text{projective})\\ &\cong \Omega^{-h}E_V \oplus (\text{projective}),\end{aligned}$$

concentrated in degree h. So defining

$$\Lambda^*_\mathfrak{p} = \Lambda^*(\zeta_1,\dots,\zeta_h)\otimes F_\mathcal{V},$$

the cohomology of $\Lambda^*_\mathfrak{p}$ is

$$\Omega^{-h}E_V \otimes_k F_\mathcal{V} \cong \Omega^{-h}\kappa_\mathfrak{p} \oplus (\text{projective})$$

concentrated in degree h.

Now consider the double complex obtained by taking homomorphisms from a Tate resolution of k to $\Lambda^*_\mathfrak{p}$,

$$E_0^{*,*}(\mathfrak{p}) = \mathrm{Hom}_{kG}(\hat{P}_*, \Lambda^*_\mathfrak{p}).$$

If we take cohomology with respect to the differential coming from $\Lambda^*_{\mathfrak{p}}$ first, the E_1 page is concentrated in horizontal degree h, and is equal to $\mathrm{Hom}_{kG}(\hat{P}_*, \Omega^{-h}\kappa_{\mathfrak{p}})$. So the E_2 page is also concentrated in horizontal degree h, and is equal to $\hat{H}^*(G, \Omega^{-h}\kappa_{\mathfrak{p}}) = \hat{H}^*(G, \kappa_{\mathfrak{p}})[h]$. So we have

$$H^*(\mathrm{Tot}\, E_0^{*,*}(\mathfrak{p})) \cong \hat{H}^*(G, \kappa_{\mathfrak{p}}).$$

On the other hand, if we take cohomology with respect to the differential coming from $\hat{P}_*$ first, then we have

$$E_1^{s,t}(\mathfrak{p}) = \hat{H}^t(G, \Lambda^s_{\mathfrak{p}}).$$

This is the stable Koszul complex

$$E_1^{*,*}(\mathfrak{p}) = C^*(H^*(G,k)_{\mathfrak{p}}; \zeta_1, \dots, \zeta_h).$$

So doing the differential coming from $\Lambda^*_{\mathfrak{p}}$ gives us the E_2 page as local cohomology with respect to $\mathfrak{p}$ of the localized cohomology ring. So we have

$$E_2^{*,*}(\mathfrak{p}) = H^*_{\mathfrak{p}}\, H^*(G,k)_{\mathfrak{p}} \Rightarrow \hat{H}^*(G, \kappa_{\mathfrak{p}}). \tag{5.1}$$

6. Dual localization

There is another way to obtain a spectral sequence with E_2 page isomorphic to $H^*_{\mathfrak{p}}\, H^*(G,k)_{\mathfrak{p}}$, due to Greenlees and Lyubeznik [12]. Namely, they use Matlis duality twice, in a process which they call "dual localization". If $\mathfrak{p}$ is a prime ideal in $H^*(G,k)$, and X is a module over $H^*(G,k)_{\mathfrak{p}}$, then we write $D_{\mathfrak{p}}(X)$ for the Matlis dual of X (Matlis [15]):

$$D_{\mathfrak{p}}(X) = \mathrm{Hom}^*_{H^*(G,k)_{\mathfrak{p}}}(X, I_{\mathfrak{p}}).$$

Here, $I_{\mathfrak{p}}$ denotes the injective hull of $H^*(G,k)/\mathfrak{p}$ as an $H^*(G,k)$-module. The module $I_{\mathfrak{p}}$ is in a natural way a module for the completion $H^*(G,k)^{\wedge}_{\mathfrak{p}}$ of $H^*(G,k)_{\mathfrak{p}}$ with respect to the powers of the maximal ideal $\mathfrak{p}_{\mathfrak{p}}$. So $D_{\mathfrak{p}}$ takes $H^*(G,k)_{\mathfrak{p}}$-modules to $H^*(G,k)^{\wedge}_{\mathfrak{p}}$-modules. It takes Artinian modules to Noetherian modules and vice-versa. Applying $D_{\mathfrak{p}}$ twice to an Artinian module returns the same module, whereas applying $D_{\mathfrak{p}}$ twice to a Noetherian module returns its $\mathfrak{p}$-completion.

The functor T from injective $H^*(G,k)$-modules to the stable category of kG-modules, defined in Benson and Krause [5], is designed to deal with Matlis duality in this setting. If M is a kG-module, it gives an isomorphism (Lemma 3.2 of [5]):

$$D_{\mathfrak{p}}\hat{H}^*(G,M) \cong \widehat{\mathrm{Ext}}^*_{kG}(M, T(I_{\mathfrak{p}})) \tag{6.1}$$

which may be viewed as a generalization of Tate duality. Namely, in case $\mathfrak{p} = \mathfrak{m}$, the ideal generated by the homogeneous elements of positive degree, $D_{\mathfrak{m}}$ is the same as (graded) vector space duality over k, and $T(I_{\mathfrak{m}})$ is just $\Omega(k)$ (Lemma 3.1 of [5]).

Grothendieck duality [13, 14] implies that applying $D_{\mathfrak{m}}$ to the Greenlees spectral sequence (3.1) gives a spectral sequence

$$\mathrm{Ext}_R^{r-*,-*}(H^*(G,k), R[-a]) \Rightarrow H^*(G,k).$$

Localization with respect to $\mathfrak{p}$ is exact, and so we obtain a spectral sequence

$$\mathrm{Ext}_{R_{\mathfrak{q}}}^{r-*,-*}(H^*(G,k)_{\mathfrak{p}}, R_{\mathfrak{q}}[-a]) \Rightarrow H^*(G,k)_{\mathfrak{p}},$$

where $\mathfrak{q} = \mathfrak{p} \cap R$, which is a prime ideal in R. Now if d is the Krull dimension of $H^*(G,k)/\mathfrak{p}$, then $H^*(G,k)_\mathfrak{p}$ has Krull dimension $r - d$. So Grothendieck duality implies that applying $D_\mathfrak{p}$ to the above spectral sequence gives

$$H^{*-d,*}_\mathfrak{p} H^*(G,k)_\mathfrak{p} \Rightarrow I_\mathfrak{p}.$$

Regrading, we obtain

$$H^{*,*}_\mathfrak{p} H^*(G,k)_\mathfrak{p} \Rightarrow I_\mathfrak{p}[d]. \tag{6.2}$$

This is the Greenlees–Lyubeznik spectral sequence.

In the Cohen–Macaulay case, this immediately calculates $\hat{H}^*(G,\kappa_\mathfrak{p})$ for us, by comparing the spectral sequences (5.1) and (6.2). In the case where $H^*(G,k)_\mathfrak{p}$ is not Cohen–Macaulay, it would be desirable to have a more direct way to compare these spectral sequences.

Theorem 6.1. *Suppose that $H^*(G,k)_\mathfrak{p}$ is Cohen–Macaulay. Then $\hat{H}^*(G,\kappa_\mathfrak{p}) \cong I_\mathfrak{p}[d]$.*

Proof. If $H^*(G,k)_\mathfrak{p}$ is Cohen–Macaulay then local cohomology is concentrated in a single degree. Namely, $H^{s,t}_\mathfrak{p} H^*(G,k)_\mathfrak{p} = 0$ for $s \neq d$. So the spectral sequences (5.1) and (6.2) are each concentrated in a single column, and there are no differentials or ungrading problems. Thus the two spectral sequences give isomorphisms

$$\hat{H}^t(G,\kappa_\mathfrak{p}) \cong H^{d,t-d}_\mathfrak{p} H^*(G,k)_\mathfrak{p} \cong I_\mathfrak{p}[d]_t. \qquad \square$$

7. Comparison

We would like to compare the spectral sequences (5.1) and (6.2) and prove that they are isomorphic, so that $\hat{H}^*(G,\kappa_\mathfrak{p}) \cong I_\mathfrak{p}[d]$. Once this is achieved, the next stage would be to prove that if M is any kG-module satisfying $\hat{H}^*(G,M) = 0$, then $\underline{\mathrm{Hom}}_{kG}(M,\kappa_\mathfrak{p}) = 0$. This would enable us to apply Proposition 4.2 of Benson and Krause [5] to deduce that $T(I_\mathfrak{p}) \cong \Omega^{-d}\kappa_\mathfrak{p}$, and Conjecture 1.3 would be proved.

We have been unable to carry out this program in general. Instead, we explain an alternative route which proves Conjecture 1.3 under some extra hypotheses.

Applying $D_\mathfrak{p}$ to the spectral sequence (5.1) and using (6.1), we obtain a spectral sequence

$$\mathrm{Ext}^{r-d-s,-t}_{R^\wedge_\mathfrak{q}}(H^*(G,k)^\wedge_\mathfrak{p}, R^\wedge_\mathfrak{q}[-a]) \Rightarrow \widehat{\mathrm{Ext}}^{-s-t}_{kG}(\kappa_\mathfrak{p}, T(I_\mathfrak{p})).$$

Reindexing gives

$$\mathrm{Ext}^{s,t}_{R^\wedge_\mathfrak{q}}(H^*(G,k)^\wedge_\mathfrak{p}, R^\wedge_\mathfrak{q}[-a+r]) \Rightarrow \widehat{\mathrm{Ext}}^{s+t}_{kG}(\Omega^{-d}\kappa_\mathfrak{p}, T(I_\mathfrak{p})). \tag{7.1}$$

The desired isomorphism between $\Omega^{-d}\kappa_\mathfrak{p}$ and $T(I_\mathfrak{p})$ should be an element of degree zero in the right hand side of (7.1). There is an obvious candidate in the E_2 page, namely the image $\widetilde{\gamma}_{G,\mathfrak{p}}$ of the element $\widetilde{\gamma}_G$ of Section 4 under $\mathfrak{p}$-completion. The problem is that it is not at all obvious that this element should be a universal cycle, without a more direct comparison of the spectral sequences.

Theorem 7.1. *Let $\mathfrak{p}$ be a prime ideal in $H^*(G,k)$ corresponding to a homogeneous irreducible subvariety V which is contained in an irreducible component of V_G of maximal dimension (equal to the p-rank of G).*

Suppose that $\widetilde{\gamma}_{G,\mathfrak{p}}$ *is a universal cycle in the spectral sequence* (7.1). *Then any representative in*

$$\widehat{\operatorname{Ext}}^0_{kG}(\Omega^{-d}\kappa_{\mathfrak{p}}, T(I_{\mathfrak{p}})) = \underline{\operatorname{Hom}}_{kG}(\Omega^{-d}\kappa_{\mathfrak{p}}, T(I_{\mathfrak{p}}))$$

of $\widetilde{\gamma}_{G,\mathfrak{p}}$ *is a stable isomorphism.*

Proof. Write $\rho : \Omega^{-d}\kappa_{\mathfrak{p}} \to T(I_{\mathfrak{p}})$ for a representative of $\widetilde{\gamma}_{G,\mathfrak{p}}$, and complete ρ to a triangle

$$\Omega^{-d}\kappa_{\mathfrak{p}} \to T(I_{\mathfrak{p}}) \to U \to \Omega^{-d-1}\kappa_{\mathfrak{p}}$$

in $\mathsf{StMod}(kG)$. By Theorem 7.3 of [5], the variety of $T(I_{\mathfrak{p}})$ is

$$\mathcal{V}_G(T(I_{\mathfrak{p}})) = \{V\}.$$

By Lemma 10.4 of [3], we also have

$$\mathcal{V}_G(\Omega^{-d}\kappa_{\mathfrak{p}}) = \{V\}.$$

So it follows that $\mathcal{V}_G(U) \subseteq \{V\}$.

Let E be a maximal elementary abelian subgroup of G such that the image $V_{G,E}$ of $V_E \to V_G$ is an irreducible component of maximal dimension (equal to the p-rank of G) containing V. Then we have a diagram of spectral sequences

$$\begin{array}{ccc}
\operatorname{Ext}^{s,t}_{R^\wedge_{\mathfrak{q}}}(H^*(G,k)^\wedge_{\mathfrak{p}}, R^\wedge_{\mathfrak{q}}[-a+r]) & \Longrightarrow & \widehat{\operatorname{Ext}}^{s+t}_{kG}(\Omega^{-d}\kappa_{\mathfrak{p}}, T(I_{\mathfrak{p}})) \\
\Big\downarrow {\scriptstyle \operatorname{Tr}^*_{E,G}} & & \Big\downarrow {\scriptstyle \operatorname{res}_{G,E}} \\
\operatorname{Ext}^{s,t}_{R^\wedge_{\mathfrak{q}}}(H^*(E,k)^\wedge_{\mathfrak{p}}, R^\wedge_{\mathfrak{q}}[-a+r]) & \Longrightarrow & \widehat{\operatorname{Ext}}^{s+t}_{kE}(\Omega^{-d}\kappa_{\mathfrak{p}}{\downarrow_E}, T(I_{\mathfrak{p}}){\downarrow_E}).
\end{array}$$

Now by Lemma 8.2 of [3], we have

$$\kappa_{\mathfrak{p}}{\downarrow_E} \cong \bigoplus_{\operatorname{res}^{-1}_{G,E}(\mathfrak{P})=\mathfrak{p}} \kappa_{\mathfrak{P}}.$$

Here, $\mathfrak{P}$ runs over the (finite) set of prime ideals in $H^*(E,k)$ whose inverse image under restriction are equal to $\mathfrak{p}$. This set corresponds to the set of irreducible components in V_E of the inverse image of V under $\operatorname{res}^*_{G,E}$. Similarly, by Proposition 7.1 of [5], we have

$$T(I_{\mathfrak{p}}){\downarrow_E} \cong T(r_{G,E}(I_{\mathfrak{p}})) \cong \bigoplus_{\operatorname{res}^{-1}_{G,E}(\mathfrak{P})=\mathfrak{p}} T(I_{\mathfrak{P}}).$$

Now $H^*(E,k)$ is Cohen–Macaulay, so the spectral sequence

$$\operatorname{Ext}^{s,t}_{R^\wedge_{\mathfrak{q}}}(H^*(E,k)^\wedge_{\mathfrak{p}}, R^\wedge_{\mathfrak{q}}[-a+r]) \Rightarrow \widehat{\operatorname{Ext}}^{s+t}_{kE}(\Omega^{-d}\kappa_{\mathfrak{p}}{\downarrow_E}, T(I_{\mathfrak{p}}){\downarrow_E})$$

degenerates to an isomorphism

$$\begin{aligned}
\operatorname{Hom}^t_{R^\wedge_{\mathfrak{q}}}(H^*(E,k)^\wedge_{\mathfrak{p}}, R^\wedge_{\mathfrak{q}}[-a+r]) &\cong \widehat{\operatorname{Ext}}^t_{kE}(\Omega^{-d}\kappa_{\mathfrak{p}}{\downarrow_E}, T(I_{\mathfrak{p}}){\downarrow_E}) \\
&\cong \bigoplus_{\operatorname{res}^{-1}_{G,E}(\mathfrak{P})=\mathfrak{p}} \widehat{\operatorname{Ext}}^t_{kE}(\Omega^{-d}\kappa_{\mathfrak{P}}, T(I_{\mathfrak{P}})).
\end{aligned}$$

By Equation (4.3) we have $\widetilde{\gamma}_G \circ \operatorname{Tr}_{E,G} = \widetilde{\gamma}_E$, or $\operatorname{Tr}^*_{E,G}(\widetilde{\gamma}_G) = \widetilde{\gamma}_E$. So the element $\operatorname{Tr}^*_{G,E}(\widetilde{\gamma}_{G,\mathfrak{p}})$ is the sum of the elements $\widetilde{\gamma}_{E,\mathfrak{P}}$, each of which, regarded as an element of $\underline{\mathrm{Hom}}_{kE}(\Omega^{-d}\kappa_{\mathfrak{P}}, T(I_{\mathfrak{P}}))$, is a stable isomorphism. So the restriction of ρ, which represents $\mathrm{Tr}^*_{E,G}(\widetilde{\gamma}_G)$, is a stable isomorphism $\Omega^{-d}\kappa_{\mathfrak{p}}\downarrow_E \to T(I_{\mathfrak{p}})\downarrow_E$. It follows that $U\downarrow_E$ is projective. Since $\mathcal{V}_G(U) \subseteq \{V\}$, this forces $\mathcal{V}_G(U) = \emptyset$ by Theorem 10.6 of [3]. It then follows, using Section 10 of [3], that U is projective as a kG-module, so that ρ is a stable isomorphism. □

We now turn to the proof of Theorem 1.4. If $H^*(G,k)_{\mathfrak{p}}$ is Cohen–Macaulay, then it is equidimensional, and the spectral sequence (7.1) is concentrated on a single column. In this case, the conditions of the above theorem are satisfied: Every irreducible subvariety is contained in a component of maximal dimension, and $\widetilde{\gamma}_{G,\mathfrak{p}}$ is always a universal cycle.

If the depth and Krull dimension of $H^*(G,k)_{\mathfrak{p}}$ differ by one, and if $\mathfrak{p}$ also contains a minimal prime of dimension r, then it still follows that $\widetilde{\gamma}_{G,\mathfrak{p}}$ is a universal cycle, since the spectral sequence is concentrated in two adjacent columns and there is no room for nonzero differentials. So the conditions of the theorem are satisfied in case $(**)$ of Theorem 1.4.

We have now shown that if condition $(*)$ or $(**)$ of Theorem 1.4 holds then any representative $\rho : \Omega^{-d}\kappa_{\mathfrak{p}} \to T(I_{\mathfrak{p}})$ of $\widetilde{\gamma}_{G,\mathfrak{p}}$ is a stable isomorphism, thereby proving that part (ii) of the theorem holds. Since $\hat{H}^*(G, T(I_{\mathfrak{p}}))$ is injective in the category of stable $H^*(G,k)$-modules, a degree shift shows that part (i) of the theorem holds. By Theorem 5.1 of [5], the modules $T(I_{\mathfrak{p}})$ are pure injective. They are even Σ-pure injective. So part (iii) of the theorem holds. By theorem 1.1.4 of [4], this implies that part (iv) holds. Finally, part (v) of the theorem follows from Corollary 3.7 of [5] and part (vi) follows from part (ii) and the idempotent property for $\kappa_{\mathfrak{p}}$.

The fact that $H^*(G,k)$ is Cohen–Macaulay in the cases (a)–(e) follows from the work of a number of people [1, 9, 18, 21]. Finally, if G has p-rank two, then every maximal elementary abelian p-subgroup has p-rank two, and so $H^*(G,k)$ is equidimensional with depth one or two. It follows that $(**)$ holds in this case.

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