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On Classes of *p*-adic Lie Groups

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Abstract. We consider non-contracting p-adic Lie groups and we establish equivalence relations and connections among the following classes of p-adic Lie groups: (1) non-contracting; (2) type R; (3) distal and (4) Tortrat. We also deduce that non-contracting p-adic Lie groups are unimodular and IN p-adic Lie groups are non-contracting.

In this note we prove p-adic analogue of results in [DR2] and [Ro].

Let G be a locally compact group and e denote the identity of G. Let $\mathcal{P}(G)$ be the space of all regular Borel probability measures on G, equipped with the weak* topology with respect to all bounded continuous functions on G: see [H] for more details on probability measures on locally compact groups. A locally compact group G is called *non-contracting* if e is not a limit point of $\{x^n g x^{-n} \mid n \in \mathbb{Z}\}$ for any $g \in G \setminus (e)$ and any $x \in G$.

Let V be a finite-dimensional vector space over $\mathbb{Q}_p$ and T be a group of linear transformations on V. Then we say that V is of *type* R_T if all eigenvalues of T are of absolute value one. A p-adic Lie group G is called *type* R if $L(G)$ is type $\mathrm{R}_{\mathrm{Ad}(G)}$ where $L(G)$ is the Lie algebra of G. Let $\mathrm{Aut}(L(G))$ be the group of all Lie algebra automorphisms of $L(G)$. It should be noted that $\mathrm{Aut}(L(G))$ is an algebraic subgroup of $GL(L(G))$, the general linear group on $L(G)$. We now prove the following using methods in [Wa2].

Theorem 1. *Let G be a p-adic Lie group and $L(G)$ be the Lie algebra of G. Then the following are equivalent.*

(1) *G is non-contracting.*
(2) *For each $x \in G$ there exists an open subgroup $U(x)$ invariant under the conjucation of x and for any compact subset C of $U(x)$ the orbit $\{x^n C x^{-n} \mid n \in \mathbb{Z}\}$ is relatively compact.*
(3) *The closed subgroup generated by $\mathrm{Ad}(x)$ is compact in $\mathrm{Aut}(L(G))$ for any $x \in G$.*
(4) *G is of type R.*

Proof. Let $x \in G$ and $\alpha\colon G \to G$ be $\alpha(g) = xgx^{-1}$ for all $g \in G$. Suppose G is non-contracting. Then Theorem 3.6 of [Wa2] implies (2).

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We now prove (2) $\Rightarrow$ (3). Suppose there is an open subgroup U invariant under α and orbits of the cyclic group generated by α in U are all relatively compact. Then let $\alpha = \alpha_u \alpha_s$ be the Jordan decomposition of α in $\mathrm{Aut}(L(G))$ where α_s and α_u are semisimple and unipotent parts of α respectively. Let T be the torus generated by α_s and T_a and T_d be the anisotropic and split parts of T. Then $T_a(\mathbb{Q}_p)T_d(\mathbb{Q}_p)$ is of finite index in $T(\mathbb{Q}_p)$ (see [Wa2]). By considering a power of α_s we may assume that α_s is in $T_a(\mathbb{Q}_p)T_d(\mathbb{Q}_p)$ and let α_d be the split part of α. Since the closed subgroup generated by α_u and the subgroup $T_a(\mathbb{Q}_p)$ are compact, to prove the closed subgroup generated by α is compact it is enough to prove that the subgroup generated by α_d is relatively compact. Since orbits in U for the cyclic group generated by α are relatively compact the eigenvalues of α_d are of p-adic absolute value one and hence the closed subgroup generated by α_d is compact.

It is easy to see that (3) $\Rightarrow$ (4). We now claim that (4) $\Rightarrow$ (1). Suppose there exits a $g \in G$ and $h \in G$ such that e is a limit point $(g^n h g^{-n})$. Let $V_x = \{v \in L(G) \mid \mathrm{Ad}(x)^n(v) \to 0\}$, for $x \in G$. Suppose G is of type R. Then both V_g and $V_{g^{-1}}$ are of dimension zero. Then by Theorem 3.6 of [Wa2], there exists a closed open subgroup M of G such that $gMg^{-1} = M$ and $\{g^n x g^{-n} \mid n \in \mathbb{Z}\}$ is relatively compact for all $x \in M$. Since e is a limit point of $(g^n h g^{-n})$, h belongs to any neighbourhood of e in G that is invariant under the conjugation by g. By Corollary 1.4 of [Wa2], M has arbitrarily small open subgroups invariant under the conjugation by g and hence $h = e$. This proves that G is non-contracting. □

Corollary 1. *Let G be a p-adic Lie group. Suppose G is non-contracting. Then G is unimodular.*

Proof. Let m be the left Haar measure on G and Δ be the unimodular homomorphism on G, that is $m(Ex) = \Delta(x)m(E)$ for all $x \in G$ and for all Borel sets E of G. Let $x \in G$. Then by Theorem 1, there exists an open subgroup U such that $xUx^{-1} = U$. Since G is totally disconnected, there exists a compact open subgroup K of U. Again by Theorem 1, we get that $\cup x^n K x^{-n} = L$, say is a relatively compact open subgroup of G and $xLx^{-1} = L$. Thus, $m(L) = m(xLx^{-1}) = m(Lx^{-1}) = \Delta(x^{-1})m(L)$. Since L is a relatively compact open subgroup, we have $0 < m(L) < \infty$. This implies that $\Delta(x) = 1$. Thus, G is unimodular. □

Proposition 1. *Let G be a Zariski-connected p-adic algebraic group. Suppose G is non-contracting. Then G is a compact extension of its nilradical.*

Proof. Let G be connected algebraic group that is non-contracting. Let us first consider the case when G is semisimple. Let T be a maximal $\mathbb{Q}_p$-split torus of G. Then $\mathrm{Ad}T$ is isomorphic to $(\mathbb{Q}_p^*)^n$ for some n where $\mathbb{Q}_p^*$ is the multiplicative group of units in $\mathbb{Q}_p$. By Theorem 1, every element of $\mathrm{Ad}T$ generates a relatively compact subgroup. This implies that T is central and hence since G is semisimple, T is trivial. Now by Theorem 3.1 of [PR], G is compact.

We now consider the case when G is solvable. Let U be the unipotent radical of G and T be a torus such that G is the semidirect product of U and T. Let T_s be the $\mathbb{Q}_p$-split part of T. Then as in the previous case, we may prove that T_s centralizes U. This implies that G is a compact extension of a nilpotent normal subgroup.

Now let G be any connected algebraic group. Let S and U be the solvable and unipotent radicals of G respectively. By a result of G. D. Mostow, there exists a reductive Levi subgroup L of G such that G is the semidirect product of L and U

and the connected component of identity in the center of L, say T is a maximal torus of S (see 11.22 and Theorem 11.23 of [B]). Also, by Theorem 2.4 of [PR], $L = RT$ where R is a connected semisimple subgroup of G and hence since $S = TU$ (see Theorem 10.6 of [B]), we have $G = LU = RTU = RS$. Since G is non-contracting, R is also non-contracting. This implies that R is compact. Since S is a solvable connected that is non-contracting, we have S is a compact extension of its nilradical. Since the nilradical of S is same as the nilradical of G, we get that G is a compact extension of its nilradical. □

A locally compact group G is called *distal* if e is not a limit point of $\{gxg^{-1} \mid g \in G\}$ for any $x \in G \setminus (e)$.

A locally compact group G is said to be a *IN-group* if there exists a compact invariant neighbourhood of e. See [GM] and [P] for more details on IN-groups.

A locally compact group G is called *Tortrat* if a sequence of the form $(g_n \lambda g_n^{-1})$, where $\lambda \in \mathcal{P}(G)$ and (g_n) is a sequence in G, has an idempotent limit point only if λ is an idempotent. See [Ra] for more details on Tortrat groups.

A *local field* $\mathbb{K}$ is a commutative non-discrete locally compact field (see [We]). A locally compact group G is said to be a *linear group* if G is a closed subgroup of $GL(V)$, the general linear group on a finite-dimensional vector space V over a local field $\mathbb{K}$.

It is proved in [Ro], that compact extensions of nilpotent normal subgroups are distal. Here, we prove that compact extensions of (not necessarily normal) unipotent groups are distal.

Proposition 2. *Let G be a linear group. Suppose there exist an unipotent algebraic (not necessarily normal) subgroup U of G such that G/U is compact. Then G is distal.*

Proof. Let V be a finite-dimensional vector space over a local field such that G is a closed subgrup of $GL(V)$. Let W be the algebra of all linear endomorphisms on V. Now, for $g \in GL(V)$, define $\phi_g \colon W \to W$ by $\phi_g(w) = gwg^{-1}$ for all $w \in W$. Let (g_n) be a sequence in G such that $g_n x g_n^{-1} \to e$ for some $x \in G$. Since G/U is compact, by passing to a subsequence of (g_n), we may assume that there exists a sequence (h_n) in G such that $u_n = h_n^{-1} g_n \in U$ and $h_n \to h \in G$. This implies that $u_n x u_n^{-1} \to e$. Let $\phi_n = \phi_{u_n}$. Then by Lemma 2.2 of [DR1], there exist sequences (a_n) and (b_n) in U such that $u_n = a_n b_n$, $a_n \to a$ in U and $b_n w b_n^{-1} = w$ for all w such that $(u_n w u_n^{-1})$ converges. Since $u_n x u_n^{-1} \to e$, we have $b_n x b_n^{-1} \to e$ and hence $x = e$. This proves that G is distal. □

A p-adic Lie group G is called *Ad-regular* if $Z(G)$ is the kernel of the adjoint representation of G.

Theorem 2. *Let G be a Ad-regular p-adic Lie group. Then the following are equivalent:*

(1) *G is non-contracting;*
(2) *G is distal;*
(3) *G is of type R.*

In addition, if G is a p-adic linear group, then (1), (2) *and* (3) *are equivalent to*

(4) *G is Tortrat.*

Proof. In view of Theorem 1, it is enough to prove that (1) is equivalent to (2). Let G be a Ad-regular p-adic Lie group. Suppose G is non-contracting. Let H be the algebraic closure of $\mathrm{Ad}(G)$. Then any $\mathbb{Q}_p$-split semisimple element occurring in the Jordan decomposition of any element of Inn (G) generates a relatively compact subgroup and hence $\mathrm{Ad}(G)$ is contained in a compact extension of an unipotent subgroup of H. By Proposition 2, we get that $\mathrm{Ad}(G)$ is distal. Since the kernel of the adjoint representation is the center of G, G is distal. This proves that (1) $\Rightarrow$ (2). That (2) $\Rightarrow$ (1) is obvious. The second part of the theorem is proved in Theorem 2 of [Ra]. □

Remark 1. Let G be a p-adic Lie group. Suppose for each $g \in G$ there exists a compact neighbourhood $K(g)$ of e such that $gK(g)g^{-1} = K(g)$. Then G is non-contracting, which may be seen as follows: Let x be a point in G and $C(x) = \{g \in G \mid x^n g x^{-n} \to e\}$. Then it is easy to see that $C(x) \subset K(x)$. By Theorem 3.6 of [Wa2], $C(x)$ is a closed subgroup of G. Thus, $C(x)$ is compact and hence it is trivial (see Theorem 3.5 of [Wa2]). By Theorem 1, we have G is non-contracting. In particular, IN p-adic Lie groups are non-contracting. In fact, a similar argument proves that IN p-adic Lie groups are distal without using Theorem 2.

A compactly generated locally compact group is said to be of *polynomial growth* if for every compact neighbourhood U of e, $m(U^n) \leq Kn^l$ for all n and for some constant K and an integer l where m is a Haar measure on G. See [Gu], [L] and [P] for more details on the theory of polynomial growth. Since only reductive p-adic algebraic groups are compactly generated (see Proposition 3.15 of [PR]), we have the following.

Corollary 2. *Let G be a Zariski-connected p-adic reductive algebraic group. Then* (1)*,* (2)*,* (3) *and* (4) *of Theorem* 2 *are equivalent to either of the following conditions.*

(5) *G has polynomial growth.*
(6) *G is an IN-group.*

Proof. Suppose G has polynomial growth. Then by Theorem 2 of [L], there exists a compact normal subgroup H of G such that G/H is a real Lie group. Since G is totally disconnected, G/H is discrete. This implies that G has a compact open normal subgroup H. Thus, G is an IN-group. This proves (5) implies (6) and that (6) implies (1) follows from Remark 1.

Suppose G is non-contracting. By Proposition 1, G is a compact extension of its nilradical, say N. Since G is a reductive group, the connected component of the identity of the center of G, say Z, is a torus and Z is the solvable radical of G (see 11.21 of [B]). This implies that $N \subset Z$. Thus, G is a compact extension of its center and hence G has polynomial growth (see [P]). □

Remark 2. The results in this note may be proved for any linear algebraic group G defined over a non-Archimedean local field $\mathbb{K}$ provided G is connected and has a Levi-decomposition defined over $\mathbb{K}$. It may be mentioned that results in [Wa1] are used in the place of results in [Wa2] in the arguement.

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