# Power Weakly Mixing Infinite Transformations

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ABSTRACT. We construct a rank one infinite measure preserving transformation $T$ such that for all sequences of nonzero integers $\{k_1, \ldots, k_r\}$, $T^{k_1} \times \ldots \times T^{k_r}$ is ergodic.

CONTENTS

## 1. Introduction

It is well known that for the case of finite measure preserving transformations, if $T$ is weakly mixing then $T^{k_1} \times \ldots \times T^{k_r}$ is ergodic for any sequence of nonzero integers $\{k_1, \ldots, k_r\}$. Kakutani and Parry proved in [KP] that there exist infinite (measure preserving) transformations such that $T \times \cdots \times T$ ($r$ terms) is ergodic but $T \times \cdots \times T$ ($r+1$ terms) is not; in this case the transformation is said to have *ergodic index* $r$. $T$ is said to have *infinite ergodic index* if $T \times \cdots \times T$ ($r$ terms) is ergodic for all $r > 0$. In [KP], they also constructed infinite Markov shifts of infinite ergodic index. A finite or infinite measure preserving transformation $T$ is *weakly mixing* if for all finite measure preserving ergodic transformations $S$, the product $T \times S$ is ergodic. For the case of infinite transformations $T$, it was shown in [ALW] that ergodicity of $T \times T$ implies weak mixing but that there exist infinite weak mixing transformations with $T \times T$ not conservative, hence not ergodic. It was shown in [AFS] that an infinite transformation $T$ may be weakly mixing with $T \times T$ conservative but still not ergodic, and that there exist rank one infinite transformations of infinite ergodic index.

In this paper we introduce a condition stronger than infinite ergodic index. Define a transformation $T$ to be *power weakly mixing* if for all finite sequences of nonzero integers $\{k_1, \ldots, k_r\}$,

$$T^{k_1} \times \ldots \times T^{k_r}$$

is ergodic. Clearly, any power weakly mixing transformation has infinite ergodic index. Also, $T$ is weakly mixing, but it follows from [ALW] that there exists a conservative ergodic infinite measure preserving transformation $R$ such that $T \times R$ is not conservative, hence not ergodic. Finally, we mention that it has been shown recently that infinite ergodic index does not imply power weak mixing [AFS2].

In Section 2 we prove some preliminaries on approximation and in Section 3 we construct a rank one infinite measure preserving transformation which is power weakly mixing. We refer to [AFS] for terms not defined here.

## 2. Approximation Properties

In this section we prove an approximation lemma for transformations defined by cutting and stacking [F]. This idea has been used earlier in e.g., [AFS] to show that a specific transformation has infinite ergodic index. However, here we present it in greater generality that permits other applications such as in [AFS2]. Thus we first describe cutting and stacking constructions [F].

Let $X$ be a finite or infinite interval of real numbers and $\mu$ be Lebesgue measure. A *column* $\mathcal{C}$ consists of a collection of pairwise disjoint intervals in $X$ of the same measure denoted by $B^{(0)}, B^{(1)}, \ldots, B^{(h-1)}$, where $h > 0$. The elements of $\mathcal{C}$ are called *levels* and $h$ is the *height* of $\mathcal{C}$. The column $\mathcal{C}$ partially defines a transformation $T$ on levels $B^{(i)}$, $i = 0, \ldots, h - 2$, by the (unique orientation preserving) translation that takes interval $B^{(i)}$ to interval $B^{(i+1)}$. Thus we shall write $B^{(i)}$ as $T^i B^{(0)}$, $i = 0, \ldots, h - 1$.

A *cutting and stacking* construction for a measure preserving transformation $T : X \to X$ consists of a sequence of columns

$$\mathcal{C}_n = \{B_n, TB_n, \ldots, T^{h_n - 1} B_n\}$$

of height $h_n$ such that:

i) $\mathcal{C}_{n+1}$ is obtained by cutting $\mathcal{C}_n$ into $c_n$ equal-measure subcolumns (or *copies* of $\mathcal{C}_n$), putting a number of *spacers* (new levels of the same measure as any of the levels in the $c_n$ subcolumns) above each subcolumn, and stacking from left to right (i.e., the top (or top spacer if it exists) of each subcolumn is mapped by translation to the bottom subinterval of the adjacent column to its right). In this way $\mathcal{C}_{n+1}$ consists of $c_n$ copies of $\mathcal{C}_n$, possibly separated by spacers. We assume $c_n \geq 2$.

ii) $B_n$ is a union of elements from $\{B_{n+1}, TB_{n+1}, \ldots, T^{h_{n+1} - h_n} B_{n+1}\}$.

iii) $\bigcup_n \mathcal{C}_n$ generates the Borel sets, i.e., for all subsets $A$ in $X$, $\mu(A) > 0$, and for all $\epsilon > 0$, there exists $C$, a union of elements from $\mathcal{C}_n$, for some $n$, such that $\mu(A \triangle C) < \epsilon$.

Suppose $I = T^j B_\ell$ is in $\mathcal{C}_\ell$, for $j = 0, \dots, h_\ell - 1$. For any $n > \ell$, $I$ is the union of some elements in $\mathcal{C}_n = \{B_n, TB_n, \dots, T^{h_n - 1} B_n\}$. We call the elements in this union *sublevels* or *copies* of $I$.

Given a real number $0 < \epsilon < 1$, and a subset $A$ of $X$ with $\mu(A) > 0$, we say that a subset $I$ of $X$ is $(1 - \epsilon)$-*full* of A provided

$$\mu(I \cap A) > (1 - \epsilon)\mu(I).$$

A set $I$ in the product space $\Pi_{i=1}^r X$ is a *rectangle* if $I$ can be written as the Cartesian product of levels in some column $\mathcal{C}_k$. We let $\nu$ be the product measure $\mu^r$. Rectangles $I$ are defined to be $(1 - \epsilon)$-full of a set $A$ in a similar way as before.

**Lemma 2.1** (Double Approximation Lemma). *Suppose $A$ is a subset of the product space $\Pi_{i=1}^r X$ with $\nu(A) > 0$. Let $I = I_1 \times \dots \times I_r$ be a rectangle in $\mathcal{C}_\ell$ that is $(1 - \epsilon)$-full of A. For $n > \ell$, let $P_n = c_\ell \cdots c_{n-1}$, let $V_n$ index the $P_n$ copies of $C_\ell$ in $C_n$, and let $V = V(n, r) = V_n \times \dots \times V_n$ ($r$ times). Then for any $\delta$, $0 < \delta < 1$, and for any $\tau$, $0 < \tau < 100(1 - \epsilon)$, there exists an integer $N$ such that for all $n > N$, there is a set $V''$ of size at least $\tau$ percent of $V$ such that for all $v = (v_1, \dots, v_r) \in V''$, $I_v$ is $(1 - \delta)$-full of A and each $I_v$ is of the form $I_v = I_1'' \times \dots \times I_r''$ where $I_m''$ is a sublevel of $I_m$ in the $v_m$-copy of $\mathcal{C}_\ell$, $m = 1, \dots, r$.*

**Proof.** For convenience, let $A$ denote $I \cap A$ and let $t$ denote $\frac{\tau}{100}$. Then $\nu(I \triangle A) < \epsilon \nu(I)$. We have that $V_n = \{1, \dots, P_n\}$ and $V = \{(v_1, \dots, v_r) | v_i \in V_n\}$. Then $I = \cup_{v \in V} I_v$.

Choose $c > \frac{\delta + 1}{1 - t - \epsilon} > 0$. Next pick $N > \ell$ sufficiently large so that for any $n \geq N$ there exists $V'$ a subset of $V$ such that $I' = \cup_{v \in V'} I_v$ satisfies

$$\nu(I' \triangle A) < \frac{\delta}{c} \nu(I).$$

Thus,

$$\nu(I' \triangle I) < \frac{\delta}{c} \nu(I) + \epsilon \nu(I)$$
$$= (\frac{\delta}{c} + \epsilon) \nu(I).$$

Now let $V'' = \{v \in V' | \nu(I_v \setminus A) < \delta \nu(I_v)\}$ and set $I'' = \cup_{v \in V''} I_v$, the union of the $(1 - \delta)$-full $I_v$ subintervals. Then

$$\delta \nu(I' \triangle I'') = \sum_{v \in V' \triangle V''} \delta \nu(I_v)$$
$$\leq \sum_{v \in V' \triangle V''} \nu(I_v \setminus A)$$
$$\leq \nu(I' \triangle A).$$

So

$$\nu(I'' \bigtriangleup I) \leq \frac{1}{\delta}\nu(I' \bigtriangleup A) + \nu(I' \bigtriangleup I)$$
$$< \frac{1}{c}\nu(I) + (\frac{\delta}{c} + \epsilon)\nu(I)$$
$$< (1 - t)\nu(I).$$

Therefore, more than $\tau$ percent of the subrectangles contained in $I$ are in $I''$ and are thus $(1 - \delta)$-full of $A$.  □

## 3.  A Power Weakly Mixing $T$

In this section we construct a rank one infinite measure preserving transformation $T$ that is power weakly mixing; then we mention a family of such transformations for which the same proof applies. We start by defining inductively a sequence of columns $\{\mathcal{C}_n\}$. Let $\mathcal{C}_0$ have base $B_0 = [0, 1)$ and height $h_0 = 1$. Given a column $\mathcal{C}_\ell$ with base $B_\ell = [0, \frac{1}{4^{\ell-1}})$ and height $h_\ell$, $\mathcal{C}_{\ell+1}$ is formed by cutting $\mathcal{C}_\ell$ vertically three times so that $B_\ell$ is cut into the intervals $B_{\ell,1} = [0, \frac{1}{4^\ell})$, $B_{\ell,2} = [\frac{1}{4^\ell}, \frac{1}{2}(\frac{1}{4^{\ell-1}}))$, $B_{\ell,3} = [\frac{1}{2}(\frac{1}{4^{\ell-1}}), \frac{3}{4}(\frac{1}{4^{\ell-1}}))$, $B_{\ell,4} = [\frac{3}{4}(\frac{1}{4^{\ell-1}}), \frac{1}{4^{\ell-1}})$. We then add a column of spacers $h_\ell$ high to the top of the subcolumn whose base is $B_{\ell,2}$. Next we add one spacer to the top of the subcolumn whose base is $B_{\ell,4}$; this is called the *staircase spacer* of $\mathcal{C}_\ell$. Then stack from left to right, i.e., the top level on the left is sent to the bottom level on the right by the translation map. The resulting column $\mathcal{C}_{\ell+1}$ now has base $B_{\ell+1} = [0, \frac{1}{4^\ell})$ and height $h_{\ell+1} = 5h_\ell + 1$. The union of the columns is $X = [0, \infty)$. This defines a conservative ergodic rank one infinite measure preserving transformation $T$.

Any column $\mathcal{C}_n = \{B_n, \dots, T^{h_n - 1}B_n\}$ has four subcolumns

$$\mathcal{C}_{n,i} = \{B_{n,i}, \dots, T^{h_n - 1}B_{n,i}\}$$

for $i = 1, \dots, 4$. Using the new column of spacers that is added to $\mathcal{C}_n$, and ignoring the top level of $\mathcal{C}_{n+1}$, we can think of $\mathcal{C}_{n+1}$ as divided into five *sections*, numbered from bottom to top. In this way each level of $\mathcal{C}_n$ has a sublevel in the first, second, fourth and fifth section of $\mathcal{C}_{n+1}$.

**Lemma 3.1.** *Given subsets $A$ and $B$ of $\Pi_{i=1}^r X$ of positive measure there exist rectangles $I = I_1 \times \cdots \times I_r$ and $J = J_1 \times \cdots \times J_r$ with $I_1, \dots, I_r, J_1, \dots, J_r$ in a column $\mathcal{C}_\ell$ such that for all $m = 1, \dots, r$, $I_m$ may be chosen in the fifth section and $J_m$ in the second section of $\mathcal{C}_\ell$, or $I_m$ in the second section and $J_m$ in the fifth section of $\mathcal{C}_\ell$, and with $I$ and $J$ $\frac{3}{4}$-full of sets $A$ and $B$ respectively.*

**Proof.** Choose rectangles $I' = I'_1 \times \cdots \times I'_r$ and $J' = J'_1 \times \cdots \times J'_r$, with $I'_m$ and $J'_m$ in column $\mathcal{C}_{\ell-1}$, such that $I'$ and $J'$ are $(1 - \frac{1}{4^{r+1}})$-full of $A$ and $B$ respectively. Now look at the copies of $\mathcal{C}_{\ell-1}$ in $\mathcal{C}_\ell$ placed as second and fifth subcolumns in the above order. To have $I_m$ above $J_m$, let $I_m$ be the top copy of $I'_m$ in $\mathcal{C}_\ell$ and let $J_m$ be the bottom copy of $J'_m$ in $\mathcal{C}_\ell$. To have $I_m$ below $J_m$ make an analogous choice. Let $I = I_1 \times \cdots \times I_r$ and $J = J_1 \times \cdots \times J_r$. One verifies that $I$ and $J$ are $\frac{3}{4}$-full of $A$ and $B$.  □

Given a level $I$ in $\mathcal{C}_n$ and an integer $k > 0$, we will be interested in studying $T^{kh_n}I$ (a translation of $I$ through $\mathcal{C}_n$ $k$ times). The intersection of $T^{kh_n}I$ with
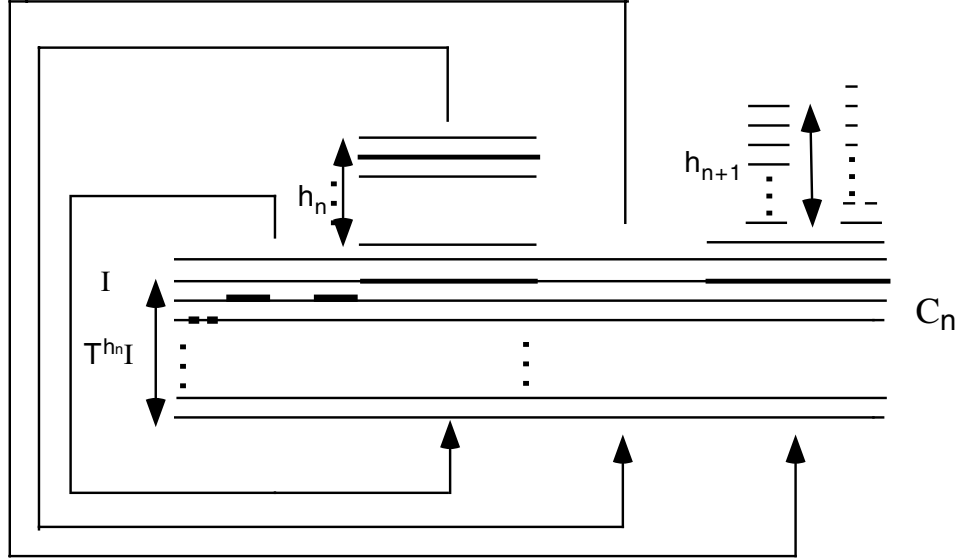
FIGURE 1. A $\mathcal{C}_n$ column.

the levels of $\mathcal{C}_n$ is called the $k$-*crescent* of $I$ in $\mathcal{C}_n$. The part of $T^{kh_n}I$ that is in $\mathcal{C}_{n,i}$ will be called the $(k,i)$-*subcrescent* of $I$ in $\mathcal{C}_n$, for $i = 1, \ldots, 4$, or simply a $k$-subcrescent. To simplify our estimates we will be mainly concerned with the $(k,1)$- and $(k,2)$-subcrescents. Figure 1 illustrates the 1-crescent in $\mathcal{C}_n$ of the interval in the top level of the fifth section of $\mathcal{C}_n$.

**Lemma 3.2.** *Let $I$ be a level in the fifth section of column $\mathcal{C}_n$, $n > 0$, let $J$ be any level in the second section of $\mathcal{C}_n$, and let $d$ be the distance that $J$ is below $I$, $h_n/5 < d < h_n$. If $k = 0, 1, \ldots, 4$ then $T^{kh_n}I$ contains a copy of $I$ in the fourth or fifth section of $\mathcal{C}_{n+1}$ that is at distance $d$ from a copy of $J$ in $\mathcal{C}_{n+1}$. Furthermore, for $k = 1, 2, \ldots, 4$ we have*

$$\mu(T^{kh_n}I \cap J) > \frac{1}{8^d}\mu(J).$$

**Proof.** The statement of the lemma is clear when $k = 0$. Next we observe that $T^{h_n}I$ contains a copy of $I$ in the fourth subcolumn of $\mathcal{C}_n$ that becomes a full level in the fifth section of $\mathcal{C}_{n+1}$; this will still be at distance $d$ from a copy of $J$ in the fifth section of $\mathcal{C}_{n+1}$. At the same time, the $(1,1)$-subcrescent of $I$ in $\mathcal{C}_n$ starts one level below $I$ (in $\mathcal{C}_n$) and consists, on each level at distance $j$ below $I$, $1 \le j < h_n$, of two intervals, contained in $T^{h_n}I$, each of length $(\frac{1}{16})^j$. Thus $\mu(T^{h_n}I \cap J) > \frac{1}{8^d}\mu(J)$.

For $T^{2h_n}I$ we observe that there is a copy of $I$ is in the third subcolumn of $\mathcal{C}_n$ hence fourth section of $\mathcal{C}_{n+1}$; in this case we choose a copy of $J$ in the same section thus at distance $d$. There is also a $(2,1)$-subcrescent of $I$ in the first subcolumn of $\mathcal{C}_n$. $T^{3h_n}I$ and $T^{4h_n}I$ also contain a copy of $I$ in the fourth subcolumn of $\mathcal{C}_n$. Also, $T^{3h_n}I$ contains a $(3,2)$-subcrescent in the second column of $\mathcal{C}_n$ that comes from the $(2,1)$-subcrescent mentioned above. $T^{4h_n}I$ also contains a $(4,1)$-subcrescent in the first subcolumn of $\mathcal{C}_n$. □

**Lemma 3.3.** *Let $I$ be a level in the fifth section of column $\mathcal{C}_n$, let $J$ be any level in the second section of $\mathcal{C}_n$, with $d$ the distance that $J$ is below $I$, $h_n/5 < d < h_n$. Let $k$ be any integer with $0 < k < d/25$. Then*

$$\mu(T^{kh_n} I \cap J) > \frac{1}{8^{k+d}} \mu(J).$$

**Proof.** Write

$$k = \sum_{j=0}^{k'} k_j 5^j$$

where $k_j = 0, 1, \ldots, 4$, and $k'$ is the largest positive integer such that $k_{k'} \neq 0$. Note that $k' = \lfloor \log_5 k \rfloor$.

Now

$$kh_n = \sum_{j=0}^{k'} k_j 5^j h_n.$$

Using that $h_{n+1} = 5h_n + 1$ we obtain

$$5^j h_n = h_{n+j} - \frac{5}{4}(5^j - 1).$$

Thus

$$kh_n = \sum_{j=0}^{k'} (k_j h_{n+j} - \frac{5}{4} k_j (5^j - 1)).$$

Now

$$\sum_{j=0}^{k'} \frac{5}{4} k_j (5^j - 1) \leq \frac{5}{4} \sum_{j=0}^{k'} 4(5^j)$$
$$= 5\left(\frac{5^{k'+1} - 1}{4}\right)$$
$$< 5 \cdot 5^{k'+1}$$
$$\leq 25k.$$

Now let

$$a_0 = k_{k'} h_{n+k'},$$
$$a_1 = \sum_{j=0}^{k'-1} k_j h_{n+j},$$
$$a_2 = \sum_{j=0}^{k'} \frac{5}{4} k_j (5^j - 1).$$

Then $T^{kh_n} I = T^{(a_0 + a_1 - a_2)} I$. We first consider $T^{a_1} I$. For each $j = 0, \ldots, k' - 1$, apply the first part of Lemma 3.2 to finally obtain a full level in column $\mathcal{C}_{n+k'}$ that is contained in $T^{a_1} I$ and at distance $d$ above a copy of $J$ in $\mathcal{C}_{n+k'}$. By the second

part of Lemma 3.2, $T^{a_0}(T^{a_1}I)$ contains a subcrescent of $I$ that intersects the copy of $J$. Thus

$$\mu(T^{a_0}(T^{a_1}I) \cap J) > \frac{1}{8^{k+d}}\mu(J).$$

Finally $T^{-a_2}(T^{a_0}(T^{a_1}I))$ moves the subcrescent down at most $25k$ levels. From the choice of $k$ we know that $25k < d$, and therefore the subcrescent of $I$ that has been translated down at most $25k$ levels still intersects the copy of $J$, with a lower bound for the intersection still valid as calculated above.  $\square$

**Theorem 3.1.** *For any sequence of nonzero integers $\{k_1, \ldots, k_r\}$, the transformation $T^{k_1} \times \cdots \times T^{k_r}$ is ergodic.*

**Proof.** Let $K = \max\{|k_i|\}$. Let $A$ and $B$ be in $\Pi_{i=1}^r X$ with $\nu(A) > 0$ and $\nu(B) > 0$. Find rectangles $I = I_1 \times \cdots \times I_r$ and $J = J_1 \times \cdots \times J_r$ such that

$$\nu(A \cap I) > \frac{3}{4}\nu(I),$$

$$\nu(B \cap J) > \frac{3}{4}\nu(J),$$

and $I_m, J_m, m = 1, \ldots, r$ are all in the same column $\mathcal{C}_\ell$, and if $k_m$ is positive choose $I_m$ and $J_m$ in the fifth and second sections of $C_\ell$ respectively, and if $k_m$ is negative choose $I_m$ and $J_m$ in the second and fifth sections of $C_\ell$ respectively. We may further assume that $125K < h_\ell$. Let $d_i$ be the distance between $I_i$ and $J_i$ for all $i$, and put $d = \max\{d_i\}$. Since $d > h_\ell/5$ it follows that $25K < d$.

Choose $\delta$ so that

$$0 < \delta < \left(\frac{1}{8^{K+d}}\right)^r.$$

Apply the Double Approximation Lemma twice (with $1 - \epsilon = \frac{3}{4}$) to find $I' = I_1' \times \cdots \times I_r'$ and $J' = J_1' \times \cdots \times J_r'$ such that $I'$ and $J'$ are $(1 - \frac{\delta}{2})$-full of $A$ and $B$ respectively, $I_1', \ldots, I_r', J_1', \ldots, J_r'$ are all in some column $\mathcal{C}_n$, $n > \ell$, and for each $i$, $I_i'$ and $J_i'$ are in the same $\mathcal{C}_\ell$-copy in $\mathcal{C}_n$ (this follows from the fact that for each application of the lemma there are at least $\tau = 74 (< 100 \cdot \frac{3}{4})$-percent of the copies satisfying the conditions and thus one can choose a common copy for $A$ and $B$). Thus the distance between $I_i'$ and $J_i'$ is still $d_i$.

Let $H = h_n$. Then for all positive $k_i$, by Lemma 3.3,

$$\mu(T^{k_i H}I_i' \cap J_i') \geq \frac{1}{8^{k_i+d_i}}\mu(I_i') \geq \frac{1}{8^{K+d}}\mu(I_i').$$

By an analogous argument to Lemma 3.3, for all negative $k_i$,

$$\mu(T^{k_i H}I_i' \cap J_i') = \mu(I_i' \cap T^{|k_i|H}J_i') \geq \frac{1}{8^{k_i+d_i}}\mu(J_i') \geq \frac{1}{8^{K+d}}\mu(I_i').$$

Therefore,

$$\nu[(T^{k_1} \times \cdots \times T^{k_r})^H I' \cap J'] \geq \left(\frac{1}{8^{K+d}}\right)^r \nu(I').$$

Thus,

$$\nu((T^{k_1} \times \cdots \times T^{k_r})^H A \cap B)$$

$$\geq \nu([(T^{k_1} \times \cdots \times T^{k_r})^H I' \cap J'] \setminus [(T^{k_1} \times \cdots \times T^{k_r})^H A \cap B])$$

$$\geq \nu((T^{k_1} \times \cdots \times T^{k_r})^H I' \cap J') - \nu((T^{k_1} \times \cdots \times T^{k_r})^H (I \setminus A)) - \nu(J \setminus B)$$

$$\geq (\frac{1}{8^{K+d}})^r \nu(J') - \frac{\delta}{2}\nu(I') - \frac{\delta}{2}\nu(J') > 0.$$

Therefore $T^{k_1} \times \cdots \times T^{k_r}$ is ergodic.    $\square$

**Remark.** 1. The same proof will apply to a transformation where at the $\ell^{\text{th}}$ stage column $C_\ell$ is cut into $c > 1$ equally-spaced subcolumns $\mathcal{C}_{\ell,1}, \ldots, \mathcal{C}_{\ell,c}$, a single (staircase) spacer is put on top of column $\mathcal{C}_{\ell,c}$ and a stack of $h_\ell$ spacers is put on top of any of the middle subcolumns.

2. There exists a rank one infinite measure preserving transformation $S$ such that $S$ has infinite ergodic index but $S \times S^2$ is not conservative, hence $S$ is not power weakly mixing [AFS2].

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