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ON SOME FUNCTION SPACES THAT APPEAR IN APPLIED MATHEMATICS

Vladimir Jovanović

Abstract. The linear spaces generated by the eigenfunctions of a differential operator are well-known in applied mathematics. In this paper we examine their interpolation properties, connection with Sobolev spaces and apply these results to the solving of hyperbolic equation in Sobolev spaces of fractional order.

1. Interpolation and function spaces

Let A_1 and A_2 be two Banach spaces, linearly and continuously embedded in a topological linear space $\mathcal{A}$. Such two spaces are called *interpolation pair* $\{A_1, A_2\}$. The space $A_1 + A_2$ we define as

$$A_1 + A_2 = \{\, a \in \mathcal{A} : a = a_1 + a_2, a_j \in A_j, j = 1, 2 \,\},$$

with the norm $\|a\|_{A_1+A_2} = \inf_{a=a_1+a_2,\, a_j \in A_j} (\|a_1\|_{A_1} + \|a_2\|_{A_2})$. Introduce the function

$$K(t, a, A_1, A_2) = \inf_{\substack{a \in A_1 + A_2 \\ a = a_1 + a_2,\, a_j \in A_j}} (\|a_1\|_{A_1} + t\|a_2\|_{A_2}).$$

This function is a norm in $A_1 + A_2$ equivalent to the standard norm $\|a\|_{A_1+A_2}$.

For $0 < \theta < 1$, $1 \leqslant q < \infty$, *the interpolation space* $(A_1, A_2)_{\theta,q}$ obtained by *K-method of real interpolation* is defined as the set of all elements $a \in A_1 + A_2$ with the finite norm

$$\|a\|_{(A_1,A_2)_{\theta,q}} \left(\int_0^\infty [t^{-\theta} K(t, a, A_1, A_2)]^q \, \frac{dt}{t} \right)^{1/q}$$

(see [1]).

Proposition 1. *Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be two interpolation pairs such that $B_i \subset A_i$, $i = 1, 2$ (with continuous injections). Then for $0 < \theta < 1$, $1 \leqslant q < \infty$, $(B_1, B_2)_{\theta,q} \subset (A_1, A_2)_{\theta,q}$ (with continuous injection).*

Proof. Using continuity of injections $B_i \subset A_i$, $i = 1, 2$, one obtains that $K(t, a, A_1, A_2) \leqslant K(t, a, B_1, B_2)$ $(a \in B_1 + B_2)$. The conclusion is obvious. ■

Next, we shall present one more method of interpolation.

Let X and Y be two separable Hilbert spaces such that

$$X \subset Y, \quad X \text{ dense in } Y \text{ with continuos injection.} \tag{1}$$

For $0 \leqslant \theta \leqslant 1$ define the interpolation space as $[X,Y]_\theta = D(\Lambda^{1-\theta})$ (domain of $\Lambda^{1-\theta}$), where Λ is a self-adjoint, lower-bounded operator in Y, with domain X and which satisfies the relation $(u,v)_X = (\Lambda u, \Lambda v)_Y$ $(u,v \in X)$. We shall take $\|v\|_{[X,Y]_\theta} = \|\Lambda^{1-\theta} v\|_Y$ as the norm of this space. The space $[X,Y]_\theta$ does not depend on the choice of the operator Λ, although it is not unique (see [5]).

The next proposition improves the connection between two methods of interpolation we mentioned before (see [5]).

Proposition 2. *Suppose X, Y be two separable Hilbert spaces which satisfy (1). Then $[X,Y]_\theta = (X,Y)_{\theta,2}$ (with equivalent norms).* ∎

This equivalence will be often used, but we shall not mention it explicitly.

Let $L_q = L_q(0,1)$ $(1 \leqslant q \leqslant \infty)$ be Lebesgue spaces of integrable functions, $H^s = H^s(0,1)$ standard Sobolev spaces, $\mathcal{D}$ the space of infinitely differentiable functions with compact support in $(0,1)$ and H_0^s the closure of $\mathcal{D}$ in H^s. $(\ ,\ )$ and $\|\ \|$ will denote inner product and norm in L_2, respectively.

For Sobolev spaces the following interpolation theorems are valid (see [5]):

Proposition 3. *Suppose* $0 < \theta < 1$.

(i) *If* $0 \leqslant s_1, s_2 < \infty$, $s_1 \neq s_2$ *then* $(H^{s_1}, H^{s_2})_{\theta,2} = H^{(1-\theta)s_1+\theta s_2}$.

(ii) *If* $0 \leqslant s_1, s_2 < \infty$, $s_1 \neq s_2$ *and* s_1, s_2, $(1-\theta)s_1 + \theta s_2 \neq \text{integer} + 1/2$ *then*

$$(H_0^{s_1}, H_0^{s_2})_{\theta,2} = H_0^{(1-\theta)s_1+\theta s_2}.$$

2. The spaces V^α

Let

$$a \in L_\infty, \quad a \geqslant a_0 > 0 \text{ in } (0,1) \text{ a.e.} \tag{2}$$

Let us define a bounded linear operator $L\colon H_0^1 \to H^{-1}$ by $Lv = -(av')'$. Then, there exist $0 < \lambda_1 < \lambda_2 < \cdots$, $\lim_k \lambda_k = \infty$, such that $L\varphi_k = \lambda_k \varphi_k$ $(k \in \mathbf{N})$; the sequence of eigenfunctions $(\varphi_k)_{k\in\mathbf{N}} \subset H_0^1$ is an orthonormed topological basis of L_2 (see [3]). We introduce the spaces V^α $(\alpha \geqslant 0)$ by

$$V^\alpha = \left\{ v \in L_2 \mid \|v\|_{V^\alpha}^2 = \sum_{k=1}^{\infty} \lambda_k^\alpha \tilde{v}_k^2 < \infty \right\},$$

where $\tilde{v}_k = (v, \varphi_k)$ are Fourier coefficients of v in the basis $(\varphi_k)_{k\in\mathbf{N}}$. Obviously, $V^0 = L_2$. It is not hard to verify the following assertions:

1° V^α is a separable Hilbert space;

2° If $\alpha > \beta \geqslant 0$, then $V^\alpha \subset V^\beta$ (with dense and continuous injection).

PROPOSITION 4. *If* $\alpha > \beta \geqslant 0$ *then* $[V^\alpha, V^\beta]_\theta = V^{(1-\theta)\alpha+\theta\beta}$.

Proof. Let us define an operator in V^β by $\Lambda v = \sum_{k=1}^{\infty} \lambda_k^{\frac{\alpha-\beta}{2}} \tilde{v}_k \varphi_k$. It is easy to verify that $D(\Lambda) = V^\alpha$, $\mathrm{Im}(\Lambda) = V^\beta$ and $(u,v)_{V^\alpha} = (\Lambda u, \Lambda v)_{V^\beta}$ $(u, v \in V^\alpha)$. The family of projectors in V^β defined by

$$E(\lambda)v = \sum_{\mu_k \leqslant \lambda} \tilde{v}_k \varphi_k, \tag{3}$$

where $\mu_k = \lambda_k^{\frac{\alpha-\beta}{2}}$ $(k \in \mathbf{N})$, is a resolution of identity in V^β (see [6]). From equalities

$$\int_{\mu_1}^{\infty} \mu \, dE(\mu)v = \sum_{k=1}^{\infty} \mu_k \tilde{v}_k \varphi_k = \sum_{k=1}^{\infty} \lambda_k^{\frac{\alpha-\beta}{2}} \tilde{v}_k \varphi_k = \Lambda v \quad (v \in V^a),$$

one concludes that (3) is the spectral decomposition of Λ (see also [6]). Using the definition of the power of operator, we have

$$\Lambda^{1-\theta} v = \int_{\mu_1}^{\infty} \mu^{1-\theta} \, dE(\mu)v = \sum_{k=1}^{\infty} \mu_k^{1-\theta} \tilde{v}_k \varphi_k = \sum_{k=1}^{\infty} \lambda_k^{(1-\theta)\frac{\alpha-\beta}{2}} \tilde{v}_k \varphi_k.$$

Hence,

$$\begin{aligned} [V^\alpha, V^\beta]_\theta &= D(\Lambda^{1-\theta}) = \{ v \in V^\beta \mid \|\Lambda^{1-\theta} v\|_{V^\beta}^2 < \infty \} \\ &= \{ v \in V^\beta \mid \sum_{k=1}^{\infty} \lambda_k^{(1-\theta)\alpha+\theta\beta} \tilde{v}_k^2 < \infty \} = V^{(1-\theta)\alpha+\theta\beta}. \quad \blacksquare \end{aligned}$$

Further, we want to improve the connection between V^α and Sobolev spaces.

LEMMA 1. *If* $v \in L_2$ *such that* $(\forall \varphi \in \mathcal{D}) \, |\int_0^1 v\varphi' \, dx| \leqslant C\|\varphi\|$, *then* $v \in H^1$.

LEMMA 2. *If* $v \in H_0^1$ *is the solution of the variational problem*

$$\int_0^1 a v' \varphi' \, dx = \int_0^1 f \varphi \, dx \quad (\varphi \in H_0^1), \tag{4}$$

then

$$a \in C^1, f \in L_2 \implies v \in H^2 \quad \textit{and} \quad \|v\|_{H^2} \leqslant C\|f\|, \tag{5}$$

$$a \in C^2, f \in H^1 \implies v \in H^3 \quad \textit{and} \quad \|v\|_{H^3} \leqslant C\|f\|_{H^1}, \tag{6}$$

$$a \in C^3, f \in H^2 \implies v \in H^4 \quad \textit{and} \quad \|v\|_{H^4} \leqslant C\|f\|_{H^2}, \tag{7}$$

The proofs of these lemmas one can find in [2].

Let $L^{(\alpha)}v = \sum\limits_{k=1}^{\infty} \lambda_k^{\alpha}\tilde{v}_k\varphi_k$. Obviously, $D(L^{(\alpha/2)}) = V^{\alpha}$ and $\|v\|_{V^{\alpha}} = \|L^{(\alpha/2)}v\|$.

THEOREM 1. *Suppose $a \in C^3$ and (2) is satisfied. Then:*

$$V^1 \subset H_0^1, \tag{8}$$

$$V^2 = H^2 \cap H_0^1 \quad \textit{and} \quad L^{(1)}v = Lv \quad (v \in V^2), \tag{9}$$

$$V^3 \subset H^3, \tag{10}$$

$$V^4 \subset H^4 \quad \textit{and} \quad L^{(2)}v = L^2v \quad (v \in V^4). \tag{11}$$

Proof. We start by showing (9). Let $S\colon L_2 \to H_0^1$ be the mapping which to every $f \in L_2$ assigns the solution of (4). Obviously, $S\varphi_k = \frac{1}{\lambda_k}\varphi_k$ ($k \in \mathbf{N}$). Using (5), one obtains that $Sf \in H^2 \cap H_0^1$ and

$$\|Sf\|_{H^2} \leqslant C\|f\|. \tag{12}$$

Suppose $v \in V^2$. Then the series $\sum\limits_{k=1}^{\infty} \lambda_k\tilde{v}_k\varphi_k$ converges in L_2. From this, using the relation $\sum\limits_{k=1}^{\infty} \tilde{v}_k\varphi_k = S\Big(\sum\limits_{k=1}^{\infty} \lambda_k\tilde{v}_k\varphi_k\Big)$ and (12), we conclude that $\sum\limits_{k=1}^{\infty} \tilde{v}_k\varphi_k$ converges in H^2. Hence, $v \in H^2$ and, therefore, $V^2 \subset H^2 \cap H_0^1$. Conversely, if $v \in H^2 \cap H_0^1$ then $Lv \in L_2$, and from self-evident equalities

$$Lv = \sum_{k=1}^{\infty}(Lv, \varphi_k)\varphi_k = \sum_{k=1}^{\infty}(v, L\varphi_k)\varphi_k = \sum_{k=1}^{\infty} \lambda_k\tilde{v}_k\varphi_k, \tag{13}$$

one obtains that $v \in V^2$ and, therefore, $H^2 \cap H_0^1 \subset V^2$. From (13), it follows that $Lv = L^{(1)}v$ ($v \in V^2$).

Let us prove (11). Using (9) one obtains

$$v \in V^4 \implies Lv = L^{(1)}v \in V^2 \implies L^2v = L^{(2)}v.$$

Then from $v \in V^4 \implies Lv \in V^2 = H^2 \cap H_0^1$ and (7), we have $v \in H^4$, i.e. $V^4 \subset H^4$.

Now, we are going to verify (8). Suppose $v \in V^1$. Then, there is a sequence $(v_n)_{n\in\mathbf{N}} \subset V^2$ tending to v, i.e. $v_n \to v$ in V^1. Obviously

$$(Lv_n, v_n) = -\int_0^1 (av_n')'v_n\,dx = \int_0^1 a(v_n')^2\,dx \implies c\|v_n\|_{H^1}^2 \leqslant (Lv_n, v_n) \leqslant C\|v_n\|_{H^1}^2. \tag{14}$$

Further, one has $(L^{(1/2)}v_n, L^{(1/2)}v_n) = (L^{(1)}v_n, v_n) = (Lv_n, v_n)$. From this, using (14) one obtains

$$c\|v_n\|_{H^1} \leqslant \|v_n\|_{V^1} \leqslant C\|v_n\|_{H^1}. \tag{15}$$

This relation implies that $(v_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in H^1. Hence, $v_n \to \bar{v} \in H^1$ in H^1. The relations $v_n \to v$ in V^1 and $v_n \to \bar{v}$ in H^1 yield $v_n \to v$ and $v_n \to \bar{v}$ in L_2. Therefore,

$$v = \bar{v} \text{ in } L_2. \tag{16}$$

For every $\varphi \in \mathcal{D}$, the implication

$$\left|\int_0^1 v_n\varphi'\,dx\right| = \left|\int_0^1 v_n'\varphi\,dx\right| \leqslant \|v_n'\|\,\|\varphi\| \implies \left|\int_0^1 v\varphi'\,dx\right| \leqslant \|\bar{v}'\|\,\|\varphi\|$$

holds. According to Lemma 1 and the last inequality, we obtain that $v \in H^1$. The equalities

$$\int_0^1 v'\varphi\,dx = \int_0^1 v\varphi'\,dx = \lim_n \int_0^1 v_n\varphi'\,dx = \lim_n \int_0^1 v_n'\varphi\,dx = \int_0^1 \bar{v}'\varphi\,dx,$$

show that $v' = \bar{v}'$ in L_2. This fact, together with (16) implies that $v = \bar{v}$ in H^1. Since $v_n \in H_0^1$ and $v_n \to \bar{v}$ in H^1, it follows that $v \in H_0^1$. Hence, $V^1 \subset H_0^1$.

At last, we shall prove (10). Suppose $v \in V^3$. Then $Lv = L^{(1)}v \in V^1 \subset H^1$ and using (6) one obtains (10). ■

LEMMA 3. *Suppose $a \in C^3$ and (2) is satisfied.*

(i) *If $v \in H^2 \cap H_0^1$, then $c\|v\|_{H^2} \leqslant \|Lv\| \leqslant C\|v\|_{H^2}$;*

(ii) *If $v \in H^4 \cap H_0^1$ satisfying $Lv \in H_0^1$, then $c\|v\|_{H^4} \leqslant \|L^2v\| \leqslant C\|v\|_{H^4}$.*

Proof. The upper estimates in (i) and (ii) are evident. Let us prove the lowers.

(i) If $v \in H^2 \cap H_0^1$ then $Lv = f \in L_2$. (5) and fact that (4) has the unique solution $v \in H_0^1$ imply

$$\|Lv\| \geqslant c\|v\|_{H^2}. \tag{17}$$

(ii) Relation $Lv \in H_0^1$, (7) and (17) yield $\|L^2v\| = \|L(Lv)\| \geqslant c\|Lv\|_{H^2} \geqslant c\|v\|_{H^4}$. ■

PROPOSITION 5. *Suppose $a \in C^3$ and (2) is satisfied. Then*

$$c\|v\|_{H^i} \leqslant \|v\|_{V^i} \leqslant C\|v\|_{H^i} \quad (v \in V^i), \tag{18}$$

for $i = 1, 2, 3, 4$.

Proof. The cases $i = 2, 4$ are immediate consequences of Lemma 3. Since V^2 is dense in V^1 then, according to (15), we conclude that for $i = 1$ Proposition 5 holds (see the proof of Theorem 1). Applying (6) and (17) when $i = 1$, one can easily obtain (18) for $i = 3$. ■

THEOREM 2. *Suppose $a \in C^3$ and (2) is satisfied. Then:*

$H_0^1 \subset V^1$ (19), $\quad H_0^2 \subset V^2$ (20), $\quad H_0^2 \subset V^3$ (21), $\quad H_0^4 \subset V^4$ (22).

Proof. (20) is obvious. Firstly, let us prove (22). Since $V^4 = \{v \in V^2 \mid L^{(1)}v \in V^2\}$ and $L^{(1)}v = Lv$ ($v \in V^2 = H^2 \cap H_0^1$), then, obviously, (22) holds. From $H_0^4 \subset V^4 \subset V^3 \subset V^2 \subset V^1$, it follows that

$$\mathcal{D} \subset V^1. \tag{23}$$

To verify (19), we shall prove that V^1 is complete in the norm of the space H^1. Indeed, let $(v_n)_{n\in\mathbf{N}} \subset V^1$ be a Cauchy sequence in H^1. Then according to (15) we conclude that it is also a Cauchy sequence in V^1. Hence, $v_n \to v \in V^1$ in V^1. From this, thanks to (8), we obtain $v_n \to v$ in H^1, i.e. V^1 is complete in the norm of the space H^1. Combining this fact with (23), one obtainss (19). (21) may be easily verified in the same manner. ∎

Remark 1. The relations (8) and (19) show that $V^1 = H_0^1$ under hypothesis cited in the last proposition.

Remark 2. Applying technics used in the proving of (19), one may verify that V^i ($i = 1, 2, 3, 4$) are closed subspaces of Sobolev spaces H^i ($i = 1, 2, 3, 4$), respectively.

Proposition 6. *Suppose $a \in C^3$ and (2) is satisfied. Then:*

(i) $V^\alpha \subset H^\alpha$ *for* $1 \leqslant \alpha \leqslant 4$, *with continuous injection.*

(ii) $H_0^\alpha \subset V^\alpha$ *for* $1 \leqslant \alpha \leqslant 4$, $\alpha \neq$ integer $+1/2$, *with continuous injection.*

Proof. (i) follows from Propositions 1, 3(i) and injections (8), (11). Similarly, (ii) follows from Propositions 1, 3(ii) and injections (19), (22). ∎

3. Solving hyperbolic equation

For a Banach space B, let $C(B)$ denote the space of continuous functions defined on $[0,T]$ with values in B, furnished with the norm $\|v\|_{C(B)} = \max\limits_{t\in[0,T]} \|v(t)\|_B$. Similarly, $L_1(B)$ denotes the space of strong integrable functions with the norm $\|v\|_{L_1(B)} = \int_0^T \|v(t)\|_B \, dt$.

Consider initial boundary value problem (IBVP) for homogeneous hyperbolic equation in the domain $Q = (0,1) \times (0,T]$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(a(x), \frac{\partial u}{\partial x}\right), \quad (x,t) \in Q$$
$$u(0,t) = u(1,t) = 0, \quad t \in [0,T]$$
$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial x}(x,0) = u_1(x), \quad x \in (0,1).$$

The weak solution of this problem (see [4]) is a function $u \in C(V^1)$ satisfying conditions $u(x,0) = u_0(x)$, $\partial u/\partial t \in C(V^0)$ and the integral equality

$$\int_0^T \left(\frac{\partial u}{\partial t}, \frac{\partial \eta}{\partial t}\right) dt + \int_0^T \left(a\frac{\partial u}{\partial x}, \frac{\partial \eta}{\partial x}\right) dt = (u_1, \eta_0),$$

for every $\eta \in L_1(V^1)$ such that $\partial\eta/\partial t \in L_1(V^1)$ and $\eta|_{t=T} = 0$, where $\eta_0 = \eta|_{t=0}$. For $u_0 \in V^1$, $u_1 \in V^0$ there is the unique weak soluton of (IBVP). Furthermore, if $u_0 \in V^{k+l+1}$, $u_1 \in V^{k+l}$ then the following inequality holds (see [7]):

$$\left\|\frac{\partial^{l-1}u}{\partial t^{l-1}}\right\|_{C(V^{k+2})} + \left\|\frac{\partial^{l}u}{\partial t^{l}}\right\|_{C(V^{k+1})} + \left\|\frac{\partial^{l+1}u}{\partial t^{l+1}}\right\|_{C(V^{k})} \leqslant C(\|u_0\|_{V^{k+l+1}} + \|u_1\|_{V^{k+l}})$$

(here $k, l \in \mathbf{Z}$, $k \geqslant 0$, $l \geqslant 1$).

Similarly to the last inequality, one can easily obtain (see the proof of Propositon 1.3 in [7]) that if $u_0 \in V^{\alpha}$, $u_1 \in V^{\alpha-1}$ then

$$\left\|\frac{\partial^{l}u}{\partial t^{l}}\right\|_{C(V^{\alpha-l})} \leqslant C(\|u_0\|_{V^{\alpha}} + \|u_1\|_{V^{\alpha-1}}), \tag{24}$$

for every real $\alpha \geqslant 1$, where $l \in \mathbf{Z}$, $0 \leqslant l \leqslant \alpha$. According to (24) and Proposition 6 we finally have

THEOREM 3. *Suppose $a \in C^3$ and (2) is satisfied. Let u be the unique weak solution of (IBVP).*

(i) *If $u_0 \in V^{\alpha}$, $u_1 \in V^{\alpha-1}$ then $\partial^l u/\partial t^l \in C(H^{\alpha-l})$ and*

$$\left\|\frac{\partial^{l}u}{\partial t^{l}}\right\|_{C(H^{\alpha-l})} \leqslant C(\|u_0\|_{V^{\alpha}} + \|u_1\|_{V^{\alpha-1}}),$$

where $1 \leqslant \alpha \leqslant 4$, $l \in \mathbf{Z}$, $0 \leqslant l \leqslant \alpha$.

(ii) *If $u_0 \in H_0^{\alpha}$, $u_1 \in H_0^{\alpha-1}$ then $\partial^l u/\partial t^l \in C(H^{\alpha-l})$ and*

$$\left\|\frac{\partial^{l}u}{\partial t^{l}}\right\|_{C(H^{\alpha-l})} \leqslant C(\|u_0\|_{H^{\alpha}} + \|u_1\|_{H^{\alpha-1}}),$$

where $1 \leqslant \alpha \leqslant 4$, $\alpha \neq \text{integer} + 1/2$, $l \in \mathbf{Z}$, $0 \leqslant l \leqslant \alpha$. ■

REFERENCES

[**1**] J. Bergh, J. Löfström, *Interpolation Spaces*, Springer, Berlin 1976.

[**2**] N. Brezis, *Analyse fonctionélle*, Masson, Paris 1983.

[**3**] B.S. Jovanović, *Parcijalne jednačine*, BS Procesor – Matematički fakultet, Beograd 1993.

[**4**] O.A. Ladizhenskaya, *Boundary Value Problems of Mathematical Physics*, Nauka, Moscow 1973. (in Russian)

[**5**] J.L. Lions, E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin 1972.

[**6**] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin 1965.

[**7**] A.A. Zlotnik, *Convergence rate estimates of projection difference method for second-order hyperbolic equations*, Vychisl. protsessy sist. **8** (1991), 116–167 (in Russian).

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Faculty of mathematics, Studentski trg 16, 11000 Beograd, Yugoslavia