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SUBHARMONIC BEHAVIOUR OF SMOOTH FUNCTIONS

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Abstract. We prove that $|f|^p$, $p>0$, behaves like a subharmonic function if f is a C^2-function such that, for some constants K and K_0,

$$|\Delta f(x)| \leqslant K r^{-1} \sup|\nabla f| + K_0 r^{-2} \sup|f|,$$

where the supremum is taken over $B_r(x) = \{\, z : |z-x| < r \,\}$. If in addition $K_0 = 0$, then $|\nabla f|^p$ has a similar property.

Throughout the paper we fix a positive integer n and denote by $\mathbf{R}^n$ the euclidean n-space. The euclidean ball of radius r, centered at $x \in \mathbf{R}^n$, is denoted by $B_r(x)$, and the unit ball is $B = B_1(0)$. By dm we denote the Lebesgue measure in $\mathbf{R}^n$ normalized so that $m(B) = 1$, hence $m(B_r(x)) = r^n$. Throughout, G will denote a *proper* subdomain of $\mathbf{R}^n$.

If f is a function harmonic in G, then the function $|f|^p$ for $p \geqslant 1$ is subharmonic in G and therefore has the sub-mean-value property over balls. If $p<1$, then $|f|^p$ need not be subharmonic but, by a result of Hardy and Littlewood [4], Fefferman and Stein [3] and Kuran [6], there exists a constant $K = K(n,p) < \infty$ such that

$$|f(x)|^p \leqslant K r^{-n} \int_{B_r(x)} |f|^p \, dm$$

whenever $B_r(x) \subset G$. As observed in [1] and [7] a slight modification of Fefferman and Stein's proof yields a more general result. To state it we define a class of functions with subharmonic behaviour.

The class $\mathrm{sh}(G)$. Let $\mathrm{sh}(G)$ denote teh class of non-negative, continuous functions u on G such that, for some constant $K \geqslant 1$,

$$u(x) \leqslant K r^{-n} \int_{B_r(x)} u \, dm$$

whenever $B_r(x) \subset G$.

Theorem A. *If* $u \in \mathrm{sh}(G)$, *then* $u^p \in \mathrm{sh}(G)$ *for every* $p > 0$.

A very short proof is in [8].

In [8] we discussed some simple sufficient conditions for a C^1-function f in order that $|f| \in \mathrm{sh}(G)$ or $|\nabla f| \in \mathrm{sh}(G)$. More precisely, $|f| \in \mathrm{sh}(G)$ if

$$|\nabla f(x)| \leqslant K r^{-1} \sup\{\, |f(z)| \, : z \in B_r(x) \,\} \tag{1}$$

for some constant $K \geqslant 0$. And $|\nabla f| \in \mathrm{sh}(G)$ if

$$|\nabla f(x)| \leqslant K r^{-1} \omega f(x,r), \tag{2}$$

where $\omega f(x,r)$ is the oscillation of f over $B_r(x)$. (In Section 2 we will state a vector variant of these results.)

In this paper we consider sufficient conditions for a C^2-function f to satisfy (1) or (2). The main result, Theorem 2, asserts that (1) is implied by

$$|\Delta f(x)| \leqslant K r^{-1} \sup_{B_r(x)} |\nabla f| + K_0 r^{-2} \sup_{B_r(x)} |f|, \tag{3}$$

while (2) is implied by $(3, K_0 = 0)$. (Here and elsewhere K, K_0, K_1, $\ldots$ denote constants independent of $B_r(x) \subset G$.)

Condition $(3, K_0 = 0)$ is satisfied if, for instance,

(i) f is harmonic,

(ii) G is bounded and f is an eigenfunction of the ordinary Laplacian,

(iii) $G = B$ and f is an eigenfunction of the hyperbolic Laplacian.

Using Theorem 2 we prove that condition $(3, K_0 = 0)$ is satisfied if f is a polyharmonic function (Corollary 5).

1. Classes of smooth functions

For a sufficiently smooth function $f \colon G \to \mathbf{R}^k$ let $\partial_j f = \partial f / \partial x_j$ $(j = 1, \ldots, n)$, whence $\partial_j f(x) \in \mathbf{R}^k$ for $x \in G$. The Laplacian Δ is defined by $\Delta f = \sum_{j=1}^n \partial_j(\partial_j f)$ and so $\Delta f(x) \in \mathbf{R}^k$ for $x \in G$. We use the symbol ∇f only in the case $k = 1$: $\nabla f = (\partial_1 f, \ldots, \partial_n f)$. Thus if $k = 1$, then ∇f maps G into $\mathbf{R}^n$. In the general case we denote by $Df(x)$ the derivative of f at $x \in G$ treated as a linear operator from $\mathbf{R}^n$ to $\mathbf{R}^k$. If f is real valued, then $\|Df(x)\| = |\nabla f(x)|$, $\nabla(\Delta f) = \Delta(\nabla f)$ and

$$\|D(\nabla f)(x)\| \asymp \{\, \sum_{i,j} (\partial_i \partial_j f(x))^2 \}^{1/2}, \quad x \in G.$$

(We write $A(x) \asymp B(x)$, $x \in G$, to denote that $A(x)/B(x)$ is between two positive constants independent of x.)

The class $HC^1(G)$. Let $C^1(G)$ be the class of all C^1-functions defined in G and with values in some $\mathbf{R}^k$ (k is constant when f is fixed). The class $HC^1(G)$ is the subclass of $C^1(G)$ consisting of those f for which there is a constant $K \geqslant 0$ such that

$$\|Df(x)\| \leqslant K \sup\{\, |f(z)| \, : z \in B_r(x) \,\}, \quad B_r(x) \subset G. \tag{4}$$

Note that (4) is implied by

$$\|Df(x)\| \leqslant K|f(x)|/\delta_G(x), \qquad (\delta_G(x) = \operatorname{dist}(x, \partial G)) \tag{5}$$

which is a restriction on the growth of f and therefore is much stronger than (4). For example, if $f > 0$ is a real function on $G = (0, +\infty) \subset \mathbf{R}^1$, then (5) reduces to $f'(x) \leqslant Kf(x)/x$, which implies that $f(x)/x^K$ is decreasing and $x^K f(x)$ is increasing for $x > 0$.

Note also that $HC^1(0, +\infty)$ does not contain the function $f(x) = \sin x$, which is seen by choosing $x = 2s\pi$, $r = 2s\pi$ $(s = 1, 2, \dots)$.

The class $OC^1(G)$. This is the subclass of $C^1(G)$ consisting of those f such that, for some constant K,

$$\|Df(x)\| \leqslant Kr^{-1}\omega f(x, r), \quad B_r(x) \subset G, \tag{6}$$

where $\omega f(x, r) = \sup\{\, |f(z) - f(x)| : z \in B_r(x)\,\}$. Clearly $OC^1(G) \subset HC^1(G)$.

It was observed in [8] that every convex (or concave) function from $C^1(G)$ belongs to $OC^1(G)$. In particular the function $f(x) = e^x$ is in $HC^1(0, +\infty)$ but f does not satisfy (5).

The class $HC^2(G)$. This class consists of those $f \in C^2(G)$ for which

$$|\Delta f(x)| \leqslant Kr^{-1}\sup\{\, \|Df(z)\| : z \in B_r(x)\,\} + K_0 r^{-2}\sup\{\, |f(z)| : z \in B_r(x)\,\} \tag{7}$$

for some constants K, K_0.

The condition $|\Delta f(x)| \leqslant K\|Df(x)\|\delta_G(x)^{-1} + K_0|f(x)|\delta_G(x)^{-2}$ implies (7) with the same values of K and K_0.

The class $OC^2(G)$. It consists of those $f \in C^2(G)$ such that

$$|\Delta f(x)| \leqslant Kr^{-1}\sup\{\, \|Df(z)\| : z \in B_r(x)\,\} \tag{8}$$

for some constant K. In particular, $OC^2(G)$ contains every function f for which

$$|\Delta f(x)| \leqslant K\|Df(x)\|/\delta_G(x), \quad x \in G. \tag{9}$$

EXAMPLE 1. Condition (8) is satisfied if f is a harmonic function on an arbitrary domain G. Let f be an eigenfunction of Δ, i.e. $\Delta f \equiv \lambda f$ for some constant λ. Assuming that $\operatorname{cl}(rB) \subset G$, where $rB = B_r(0)$, we have

$$n\int_{\partial B} \frac{d}{dr} f(ry)\, d\sigma(y) = r^{1-n}\int_{rB} \Delta f\, dm,$$

where $d\sigma$ is the normalized surface measure on ∂B, which is a special case of Green's formula. Hence

$$r\Delta f(0) = n\int_{\partial B} \frac{d}{dr} f(ry)\, d\sigma(y) - r^{1-n}\int_{rB} (f - f(0))\, dm,$$

and hence $|\Delta f(0)| \leqslant nr^{-1}\sup_{rB}|\nabla f| + r|\lambda|\sup_{rB}|\nabla f|$. Applying this to the function $z \mapsto f(x+z)$ we conclude that $f \in OC^1(G)$ provided G is bounded. On the other hand, if $f(x) = \sin x_1$, $G = \{\, x \in \mathbf{R}^n \,:\, x_1 > 0 \,\}$, then $\Delta f = -f$ but f is not in $OC^2(G)$.

EXAMPLE 2. A function $f \in C^2(B)$ is said to be hyperharmonic if $\Delta_h f \equiv 0$, where

$$\Delta_h f(x) = (1 - |x|^2)^2 [\Delta f(x) + 2(n-2)(1-|x|^2)^{-1} x \cdot \nabla f(x)].$$

($x \cdot y$ denotes the inner product in $\mathbf{R}^n$.) It is clear that a hyperharmonic function satisfies (9) with $K = 2(n-2)$ and therefore belongs to $OC^2(B)$. More generally, every eigenfunction of Δ_h belongs to $OC^2(B)$, which can be proved by using the hyperbolic variant of Green's formula. (See [5], where a complex hyperbolic analog of (8) was considered.)

2. Results

The following theorem was proved in [8] in the case of scalar functions. The proof of the vector variant is similar and we omit it.

THEOREM 1. (a) *If $f \in HC^1(G)$, then $|f| \in \mathrm{sh}(G)$.*

(b) *If $f \in OC^1(G)$, then the function $x \mapsto \|Df(x)|$ belongs to $\mathrm{sh}(G)$.*

COROLLARY 1. *Let $p > 0$. A function f from $C^1(G)$ belongs to $HC^1(G)$ if and only if there is a constant K such that*

$$|\nabla F(x)| \leqslant K r^{-n-p} \int_{B_r(x)} |f|^p \, dm, \quad 0 < r < \delta_G(x).$$

Let $\omega_p f(x,r) = \{r^{-n} \int_{B_r(x)} |f(z) - f(x)|^p \, dm(z)\}^{1/p}$.

COROLLARY 2. *Let $p > 0$. A function f belongs to $OC^1(G)$ if and only if $|\nabla f(x)| \leqslant K r^{-1} \omega_p f(x,r)$, $0 < r < \delta_G(x)$, for some constant K.*

Proof. "If" part is trivial. Let $f \in OC^1(G)$. Then $f - c$ is in $OC^1(G) \subset HC^1(G)$ for an arbitrary vector c. Hence, by Corollary 1,

$$\|Df(x)\|^p \leqslant K_1 r^{-n-p} \int_{B_r(x)} |f - c|^p \, dm.$$

As follows from [8], he constant K_1 depends only on K from (6), p and n (not on c). The desired result now follows by taking $c = f(x)$. ∎

The main result of this paper is the following

THEOREM 2. *The following relations hold:*

(a) $HC^2(G) \subset HC^1(G)$ (b) $OC^2(G) \subset OC^1(G)$.

Before proving the theorem we deduce some consequences.

COROLLARY 3. *A function $f \in C^2(G)$ belongs to $HC^2(G)$ if and only if there is a constant K such that $|\Delta f(x)| \leqslant K r^{-2} \sup\{\, |f(z)| \,:\, z \in B_r(x) \,\}$.*

Proof. "If" part is trivial. "Only if" part is a consequence of Theorem 2(a). ∎

COROLLARY 4. *Let $p > 0$. For a function $f \in C^2(G)$ the following assertions are equivalent:*

(i) *There is a constant K such that $|\Delta f(x)| \leqslant K r^{-2} \omega f(x,r)$.*

(ii) *There is a constant K such that $|\Delta f(x)| \leqslant K r^{-2} \omega_p f(x,r)$.*

(iii) *There is a constant K such that $|\Delta f(x)|^p \leqslant K r^{-n-p} \int_{B_r(x)} \|Df(z)\|^p \, dm(z)$.*

(iv) *$f \in OC^2(G)$.*

Proof. The implications (ii) ⇒ (i) ⇒ (iv) and (iii) ⇒ (iv) are trivial. The validity of implication (iv) ⇒ (iii) is easily derived from Theorem 2(b), Theorem 1(b) and Theorem A. That (iv) implies (ii) is deduced from Theorem 2(b) and Corollary 2. ∎

As a further application of Theorem 2 we note a sufficient condition for a C^3-function to be in $OC^2(G)$.

THEOREM 3. *A real valued C^3-function f belongs to $OC^2(G)$ if there are constants K_1 and K_2 such that*

$$|\nabla(\Delta f)(x)| \leqslant K_1 r^{-1} \sup_{B_r(x)} \|D(\nabla f)\| + K_2 r^{-2} \sup_{B_r(x)} |\nabla f|. \tag{10}$$

Proof. Since $\nabla(\Delta f) = \Delta(\nabla f)$ condition (10) means that $\nabla f \in HC^2(G)$. Thus (10) implies $\nabla f \in HC^1(G)$, by Theorem 2(b), which means $\|D(\nabla f)(x)\| \leqslant K r^{-1} \sup_{B_r(x)} |\nabla f|$ for some constant K. Since obviously $|\Delta f| \leqslant \mathrm{const} \cdot \|D(\nabla f)\|$, it follows that $f \in OC^2(G)$. ∎

COROLLARY 5. *A C^4-function $f \colon G \to \mathbf{R}$ belongs to $OC^2(G)$ if so does Δf. Consequently a C^∞-function f belongs to $OC^2(G)$ if so does $\Delta^k f$ for some integer k. In particular every polyharmonic function of finite order belongs to OC^2.* (A function f is polyharmonic if $\Delta^k f \equiv 0$ for some integer k. For information see [2].)

Proof. Let $\Delta f \in OC^2$. Then $\Delta f \in HC^1$, by Theorem 2, i.e. $|\nabla(\Delta f)(x)| \leqslant K r^{-1} \sup_{B_r(x)} |\nabla f|$. Now the desired conclusion follows from Theorem 3. ∎

REMARK 1. As noted in the proof of Theorem 3 condition (10) means $\nabla f \in HC^2$. By Corollary 3 condition (10) implies the existence of a constant K such that $|\nabla(\Delta f)(x)| \leqslant K r^{-2} \sup_{B_r(x)} |\nabla f|$, which is aparently stronger than (10).

REMARK 2. It follows from the proofs of Theorem 3 and Corollary 5 that there hold the following implications:

$$\Delta f \in HC^1 \implies \nabla f \in OC^2 \implies \nabla f \in HC^1 \implies f \in OC^2.$$

3. Proof of theorem 2.

The proof is based on the following consequence of Green's formula.

LEMMA 1. *If $f\colon B_r(x) \to \mathbf{R}^k$ is a C^2-function, then*

$$\|Df(x)\| \leqslant nr^{-1} \sup_{B_r(x)} |f| + \frac{n}{n+1} r \sup_{B_r(x)} |\Delta f|. \tag{11}$$

Proof. In the case $k = 1$ a proof is in [2] (Proposition 3.1). If $k > 1$, we consider the functions $u(z) = f(z) \cdot \xi$, $\xi \in \mathbf{R}^k$, then use the formula $\nabla u(x) = Df(x)^* \xi$ and choose ξ so that $|\xi| = 1$ and $|Df(x)^*\xi| = \|Df(x)^*\| = \|Df(x)\|$. Then the result follows from the inequalities $|u(z)| \leqslant |f(z)|$ and $|\Delta u(z)| = |(\Delta f)(z) \cdot \xi| \leqslant |\Delta f(z)|$. ∎

LEMMA 2. *Let F_1, F_2 and F_3 be nonnegative, continuous functions on G such that, for some constant K,*

$$F_1(x)/K \leqslant r^{-1} \sup_{B_r(x)} F_2 + r \sup_{B_r(x)} F_3 \tag{12}$$

and

$$F_3(x)/K \leqslant r^{-1} \sup_{B_r(x)} F_1 + r^{-2} \sup_{B_r(x)} F_2 \tag{13}$$

whenever $B_r(x) \subset G$. Then there is a constant $C = C(K)$ such that

$$F_1(x) \leqslant Cr^{-1} \sup_{B_r(x)} F_2. \tag{14}$$

Proof. By translations the proof of (14) reduces to the case $x = 0$. Let $\mathrm{cl}(B_\varepsilon(0)) \subset G$ and $F_2 \leqslant 1$ on $B_\varepsilon(0)$. (In the general case we consider the functions F_i/A, where A is chosen so that $F_2(z) \leqslant A$ for all $z \in B_\varepsilon(0)$.) Choose $x \in B_\varepsilon(0)$ so that $F_1(y)(\varepsilon - |y|) \leqslant F_1(x)(\varepsilon - |x|)$ for all $y \in B_\varepsilon(0)$. This implies that $F_1(y) \leqslant 2F_1(x)$ for $y \in B_\delta(x)$, where $\delta = (\varepsilon - |x|)/2$. Now we use the hypotheses to find $y \in \mathrm{cl}(B_\delta(x))$ so that

$$F_1(x)/K \leqslant r^{-1} + (Kr/t)F_1(y) + Krt^{-2}$$

for all r, $t > 0$ such that $r + t = \delta$, which implies $F_1(x)/K \leqslant r^{-1} + (2Kr/t)F_1(x) + Krt^{-2}$. Now choose r, t so that $r + t = \delta$ and $2Kr/t = 1/2K$, which implies that $r = c_1(\varepsilon - |x|)$, $t = c_2(\varepsilon - |x|)$ for some $c_i = c_i(K)$, to obtain

$$F_1(x)/K \leqslant F_1(x)/2K + K_1(\varepsilon - |x|)^{-1},$$

where $K_1 = c_1^{-1} + c_1 c_2^{-2}$. Hence $F_1(0)\varepsilon \leqslant F_1(x)(\varepsilon - |x|) \leqslant 2KK_1$, and this concludes the proof. ∎

Proof of Theorem 2. Let f satisfy (7). We may assume that $K \geqslant n$ and $K_0 \geqslant n$. Define functions

$$F_1(x) = \|Df(x)\|, \quad F_2(x) = |f(x)|, \quad F_3(x) = |\Delta f(x)|.$$

Then (12) is sastisfied because of (11), and (13) is satisfied because of (7). Hence $f \in HC^1(G)$, by Lemma 2. This proves assertion (a).

To prove (b) let $f \in OC^2(G)$. Applying (a), together with its proof, to the functions $f - c$ we find a constant K_1 independent of x, r, c so that $\|Df(x)\| \leqslant K_1 r^{-1} \sup_{B_r(x)} |f - c|$. Finally we take $c = f(x)$ to finish the proof. ■

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