МАТЕМАТИЧКИ ВЕСНИК
47 (1995), 1–6

UDC 517.926
оригинални научни рад
research paper

FINDING THE BOUNDARY CURVES OF THE THIRD ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Zdravko F. Starc

Abstract. In this paper the boundary curves of the third order linear ordinary differential equations are given.

In this paper, using the boundary curves of the third order linear differential equations, we shall determine the boundary curves of the initial value problem

$$y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = 0, \tag{1}$$

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad y''(x_0) = y_0'', \tag{2}$$

where $a_i(x) \in C(I)$ $(i = 1, 2, 3)$.

The similar results for the second order linear equation have been obtained in [1], [6].

1. Suppose that $p = p(x)$ is a particular solution of the equation (1) and $p(x_0) \neq 0$. By the substitution

$$y = pz \qquad (z \in C^3(I)) \tag{3}$$

the initial value problem (1), (2) becomes

$$pz''' + (3p' + a_1(x)p)z'' + (3p'' + 2a_1(x)p' + a_2(x)p)z' = 0, \tag{4}$$

$$z(x_0) = z_0 = \frac{y_0}{p(x_0)}, \quad z'(x_0) = z_0' = \frac{p(x_0)y_0' - p'(x_0)y_0}{p(x_0)^2}, \tag{5.1}$$

$$z''(x_0) = z_0'' = \frac{p(x_0)^2y_0'' - 2p'(x_0)p(x_0)y_0' + (2p'(x_0)^2 - p''(x_0)p(x_0))y_0}{p(x_0)^3}. \tag{5.2}$$

Substituting the function

$$z(x) = z_0 + \int_{x_0}^{x} w(t)\,dt \qquad (w \in C^1(I)) \tag{6}$$

into (4) we arrive at the initial value problem

$$pw'' + (3p' + a_1(x)p)w' + (3p'' + 2a_1(x)p' + a_2(x)p)w = 0, \tag{7}$$
$$w(x_0) = z_0', \quad w'(x_0) = z_0''. \tag{8}$$

Now, we can formulate the following result.

PROPOSITION 1. *Suppose that h, $H \in C$. Let $p = p(x)$ be a particular solution of the equation (1) and $p(x) > 0$ on the interval (x_0, x_1). Let $w = w(x)$ be the solution of the initial value problem (7), (8) such that*

$$h(x) < w(x) < H(x) \qquad (x_0 < x < x_2, \quad x_2 > x_1)$$

and $h(x_0) = H(x_0) = w(x_0)$. Then for the solution $y = y(x)$ of the initial value problem (1), (2) the following inequalities

$$p(x)\left(\frac{y_0}{p(x_0)} + \int_{x_0}^{x} h(t)\,dt\right) < y(x) < p(x)\left(\frac{y_0}{p(x_0)} + \int_{x_0}^{x} H(t)\,dt\right) \tag{9}$$

$(x_0 < x < x_1)$ are valid.

The proof immediately follows from (6) and (3).

For the equation [4] $(f(x),\ g(x) \in C(I))$

$$y''' + x^2 f(x)y'' - x(2f(x) - g(x))y' + (2f(x) - g(x))y = 0,$$

with the particular solution $p(x) = x$, the equation (7) has the form

$$xw'' + (x^3 f(x) + 3)w' + x^2 g(x)w = 0.$$

EXAMPLE 1. The solution $y = y(x)$ of the initial value problem

$$x^3(x+2)y''' - x^2(x+2)(9x+10)y'' + 23x(x+1)y' - 23(x+1)y = 0,$$
$$y(1) = 0, \quad y'(1) = 1, \quad y''(1) = 3$$

satisfies the inequalities

$$x\left(\frac{x^2}{2} - \frac{1}{2}\right) < y(x) < x(\exp(x-1) - 1) \qquad (x > 1)$$

because the solution $w = w(x)$ of the initial value problem [6]

$$(x^3 + 2x^2)w'' - (6x^2 + 4x)w' + (5x + 3)w = 0, \quad w(1) = w'(1) = 1,$$

satisfies the inequalities $x < w(x) < \exp(x-1)$ $(x > 1)$.

If $a_1(x) \in C^2(I)$, $a_i(x) \in C(I)$ $(i = 2, 3)$ and [5]

$$a_3(x) + \frac{2}{27}a_1(x)^3 - \frac{1}{3}a_1(x)a_2(x) - \frac{1}{3}a_1''(x) = 0 \tag{10}$$

then a particular solution of the equation (1) is

$$p(x) = \exp\left(-\frac{1}{3}\int_{x_0}^{x} a_1(t)\,dt\right).$$

In this case the equation (7) has the form

$$w'' + R(x)w = 0, \tag{11}$$

where $R(x) = a_2(x) - a_1'(x) - \frac{1}{3}a_1(x)^2$.

PROPOSITION 2. *Suppose that (10) holds and $R(x) < 0$ on the interval (x_0, x_1). Then for the solution $y = y(x)$ of the initial value problem (1), (2) the following inequalities*

$$\begin{aligned}\exp\left(-\frac{1}{3}\int_{x_0}^{x} a_1(t)\,dt\right)&(y_0 + \ln|Y_0(x-x_0)+1|) < y(x)\\ &< \exp\left(-\frac{1}{3}\int_{x_0}^{x} a_1(t)\,dt\right)\left(y_0 + Y_0(x-x_0) - \int_{x_0}^{x}\left(\int_{x_0}^{s} R(t)\,dt\right)ds\right)\end{aligned} \tag{12}$$

$(x_0 < x < x_1)$ are valid, where $Y_0 = z_0''/z_0'$ $(z_0',\ z_0'' \neq 0)$.

Proof. By the substitution $w = z_0 \exp \int_{x_0}^{x} Y(t)\,dt$ $(Y \in C^1(x_0, x_1))$ the equation (11) becomes $Y' + Y^2 + R(x) = 0$, $Y(x_0) = Y_0$. According to the Chapligin's theorem for a first order differential equation, from $-Y^2 < -Y^2 - R(x) < -R(x)$ we obtain

$$h(x) = Y_0/(Y_0(x-x_0)+1), \quad H(x) = Y_0 - \int_{x_0}^{x} R(t)\,dt$$

and by means of (9) we have the inequalities (12). ∎

EXAMPLE 2. The solution $y = y(x)$ of the initial value problem

$$y''' + 3xy'' + (3x^2 - x + 2)y' + (x^3 - x^2 + 2x)y = 0, \quad y(0) = y'(0) = y''(0) = 1,$$

satisfies the inequalities

$$(1 + \ln(2x+1))\exp\left(-\frac{x^2}{2}\right) < y(x) < \left(1 + 2x + \frac{x^2}{2} + \frac{x^3}{6}\right)\exp\left(-\frac{x^2}{2}\right)$$

$(x > 0)$. Here $R(x) = -x - 1$, $z_0' = 1$, $z_0'' = 2$.

2. We shall now assume that

$$D^3 + a_1(x)D^2 + a_2(x)D + a_3(x) = (D + b_0(x))(D^2 + b_1(x)D + b_2(x)), \tag{13}$$

where $b_0(x) \in C(I)$, $b_i(x) \in C^1(I)$ $(i = 1, 2)$. Some conditions under which this factorization is possible are considered in [3]. It means that we have

$$z' + b_0(z) = 0, \tag{14}$$

$$y'' + b_1(x)y' + b_2(x)y = z. \tag{15}$$

The solution of the initial value problem (10) and

$$z(x_0) = z_0 = y_0'' + b_1(x_0)y_0' + b_2(x_0)y_0$$

is given by $z(x) = z_0 \exp(-\int_{x_0}^{x} b_0(t)\,dt)$. By virtue of (14), (15) the initial value problem (1), (2) reduces to the initial value problem

$$y'' + b_1(x)y' + b_2(x)y = z_0 \exp\left(-\int_{x_0}^{x} b_0(t)\,dt\right) \tag{16}$$

$$y(x_0) = y_0, \quad y'(x_0) = y_0'. \tag{17}$$

Let us consider the following initial value problem

$$y'' + b_1(x)y' + b_2(x)y = 0, \tag{18}$$

$$y(x_0) = \alpha_1, \quad y'(x_0) = \alpha_2 \tag{19}$$

and (16) with the initial conditions

$$y(x_0) = \beta_1, \quad y'(x_0) = \beta_2, \tag{20}$$

where $\alpha_1 + \beta_1 = y_0$, $\alpha_2 + \beta_2 = y_0'$.

Now, we can formulate the following result.

PROPOSITION 3. *Suppose that (13) holds. If $h(x)$, $H(x)$ are boundary curves of the initial value problem (18), (19) on the interval (x_0, x_1) and if $y_p(x)$ is a particular solution of (16), (20), then for the solution $y = y(x)$ of the initial value problem (1), (2) the following inequalities*

$$h(x) + y_p(x) < y(x) < H(x) + y_p(x) \qquad (x_0 < x < x_1)$$

are valid.

The proof follows from the fact that the general solution of the equation (16) has the form $y(x) = y_h(x) + y_p(x)$.

EXAMPLE 3. The solution $y = y(x)$ of the initial value problem

$$(x+2)y''' - (x+2)^2 y'' - (2x+6)y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}, \quad y''(0) = 3$$

satisfies the inequalities

$$\exp\left(\frac{x^2}{2}\right) + \frac{x}{3} + 1 < y(x) < \exp\left(\frac{x^2}{2}\left(2 + \frac{x^2}{24}\right)\right) + \frac{x}{3} + 1 \quad (x > 0).$$

Here $b_0(x) = -1/(x+2)$, $b_1(x) = -x$, $b_2(x) = -2$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 1/3$ and $h(x) = \exp(x^2/2)$, $H(x) = \exp(\frac{x^2}{2}(2 + \frac{x^2}{24}))$, $y_p(x) = \frac{x}{3} + 1$.

3. Suppose that $f(x)$, $g(x) \in C^1(I)$. The equation [2; eq. (3.26)]

$$y''' + 3f(x)y'' + (f'(x) + 2f(x)^2 + 4g(x))y' + (4f(x)g(x) + 2g'(x))y = 0 \tag{21}$$

has the general solution $y = C_1u^2+C_2uv+C_3v^2$, where u, v are linearly independent solutions of the equation

$$w'' + f(x)w' + g(x)w = 0. \tag{22}$$

We now assume that $u(x_0) = v'(x_0) = 1$, $u'(x_0) = v(x_0) = 0$. The solution of the initial value problem (22) and

$$w(x_0) = \sqrt{y_0} \quad (y_0 > 0), \qquad w'(x_0) = \frac{y_0'}{2\sqrt{y_0}} \tag{23}$$

is given by $w(x) = \sqrt{y_0}\, u + \dfrac{y_0'}{2\sqrt{y_0}} v$. If

$$0 < h(x) < w(x) < H(x) \qquad (x_0 < x < x_1), \tag{24}$$

then for the solution

$$y(x) = y_0u^2 + y_0'uv + \frac{y_0'}{4y_0}v^2 \tag{25}$$

we have

$$h(x)^2 < y(x) < H(x)^2 \qquad (x_0 < x < x_1). \tag{26}$$

From (25) we obtain

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad y''(x_0) = \frac{y_0'^2}{2y_0} - 2y_0g(x_0) - y_0'f(x_0). \tag{27}$$

Finally, we arrive at the following result.

PROPOSITION 4. *Let $w = w(x)$ be the solution of the initial value problem (22), (23) such that (24) holds. Then for the solution $y = y(x)$ of the initial value problem (21), (27) the inequalities (26) are valid.*

EXAMPLE 4. The solution $y = y(x)$ of the initial value problem

$$y''' - 3xy'' + (2x^2 - 4a - 1)y' + 4axy = 0, \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = 2a \quad (a > 1)$$

satisfies the inequalities $\exp x^2 < y(x) < \exp x^2(a + x^2/24)$ $(x > 0)$ because the solution $w = w(x)$ of the initial value problem [1]

$$w'' - xw' - aw = 0, \quad w(0) = 1, \quad w'(0) = 0$$

satisfies the inequalities $\exp\dfrac{x^2}{2} < w(x) < \exp\dfrac{x^2}{2}\left(a + \dfrac{x^2}{24}\right)$ $(x > 0)$.

4. Now we consider the case when $p = p(x)$ is not a particular solution of the equation (1). By the substitution (3) the initial value problem (1), (2) becomes

$$pz''' + (3p' + a_1(x)p)z'' + (3p'' + 2a_1(x)p' + a_2(x)p)z' + (p''' + a_1(x)p'' + a_2(x)p' + a_3(x)p)y = 0 \tag{28}$$

with the initial conditions (5.1), (5.2).

We have the following result.

Proposition 5. *If the solution $z = z(x)$ of the initial value problem (28), (5.1), (5.2) satisfies $h(x) < z(x) < H(x)$ $(x_0 < x < x_2,\ x_2 > x_1)$, then for the solution $y = y(x)$ of the initial value problem (1), (2) the following is valid*

$$p(x)h(x) < y(x) < p(x)H(x) \qquad (x_0 < x < x_1).$$

The proof is direct.

This proposition shows that the boundary curves of (1) can be obtained by those of (28).

Example 5. The solution $y = y(x)$ of the initial value problem

$$y''' - (3x+3)y'' + (2x^2+6x-4a+2)y' - (2x^2+(3-4a)x-4a)y = 0 \quad (a > 1)$$
$$y(0) = y'(0) = 1, \quad y''(0) = 1 + 2a$$

satisfies the inequalities

$$\exp(x+x^2) < y(x) < \exp\left(x + x^2\left(a + \frac{x^2}{24}\right)\right) \qquad (x > 0).$$

Here $p(x) = e^x$ and the initial value problem (28), (5.1), (5.2) is given by Example 4.

REFERENCES

[**1**] M. Bertolino, *Inéqualités différentielles et l'analyse qualitative des équations différentielles ordinaries*, Ed. spéciales Inst. Math. Beograd **7** (1969), 59–152

[**2**] Э. Камке, *Справочник по обыкновенным дифференциальным уравнениям*, «Наука», Москва 1976

[**3**] П. Р. Лазов, Д. С. Димитровски, Н. Рајовић, *Редукция линейних дифференциальных уравнений третего порядка*, Bull. Soc. Math. Phys. Macédoine **26** (1975/76), 25–28

[**4**] D. S. Mitrinović, *Compléments au traité de Kamke V*, Univ. Beograd Publ. Elektroteh. Fak. Ser. Mat. Fiz. **11** (1957), 1–10

[**5**] B. S. Popov, *Über die Integration der linearen Differentialgleichungen drite Ordnung in geschlossener Form*, Bull. Soc. Math. Phys. Macédoine **7** (1956), 17–19

[**6**] Z. Radašin, *Nalaženje okvirnih krivih za linearne diferencijalne jednačine drugog reda*, Mat. Vesnik **31** (1979), 497–509

(received 23.10.1986.)

Žarka Zrenjanina 93

YU-26300 Vršac