

ON LOCALLY SUBADDITIVE FUNCTIONS

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Abstract. We define locally (on $C \subset \mathbf{R}^2$) subadditive functions $f, f: C \rightarrow \mathbf{R}$, by

$$f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2), \quad (x_1, y_1), (x_2, y_2) \in C,$$

where C is some cone in \mathbf{R}^2 . The purpose of the paper is to find explicit form of such functions.

A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is said to be additive if it satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x, y \in \mathbf{R}.$$

Under some smoothing restrictions (measurability or Baire property) the only form of additive functions, as is well known, is that of cx .

Two-dimensional case of Cauchy equation, i.e.

$$f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2), \quad (x_1, y_1), (x_2, y_2) \in \mathbf{R}^2,$$

in a similar way has the solution $f(x, y) = c_1x + c_2y$.

We define *locally* (on $C \subset \mathbf{R}^2$) *subadditive functions* $f, f: C \rightarrow \mathbf{R}$, by

$$f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2), \quad (x_1, y_1), (x_2, y_2) \in C, \quad (1)$$

where C is some cone in \mathbf{R}^2 . A non-empty convex subset C of \mathbf{R}^n is called a cone if $\lambda C \subset C$ for all $\lambda \geq 0$. For the definition of a cone in an arbitrary vector space see [2]. We shall denote the class of all such functions on C by LS_C . In these considerations we also admit sets C which are cones without the point 0.

Our task in this paper is to "solve" functional inequality (1), i.e. to give an explicit form of $f \in LS_C$.

We begin with the following results:

PROPOSITION 1. *If $f_k \in LS_{C_k}$, $k = 1, 2, \dots, n$ then:*

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = f \in LS_C, \quad (2)$$

where $C = \bigcap_{k=1}^n C_k$ and c_1, c_2, \dots, c_n are arbitrary positive constants.

Proof follows immediately from definition (1) of locally subadditive functions and the fact that intersection of any family of cones is a cone. ■

PROPOSITION 2. *If $g(t)$ is a convex function defined for $t \in (a, b)$, then*

$$x \cdot g(y/x) = f(x, y) \in LS_C,$$

where $C = \{(x, y) \mid a < y/x < b, x > 0\}$ is a subset of \mathbf{R}^2 .

Proof. It is clear that C is a non-empty convex subset of \mathbf{R}^2 with $\lambda C \subset C$ for all $\lambda > 0$. From this it follows that $C + C \subset C$. Now, according to definition of a convex function $g(t)$, $t \in (a, b)$

$$g(pr + qs) \leq pg(r) + qg(s) \quad (3)$$

for each $r, s \in (a, b)$ and each $p, q \geq 0$, $p + q = 1$, and since $(x_1, y_1), (x_2, y_2) \in C$ implies that $(x_1 + x_2, y_1 + y_2) = (x_1, y_1) + (x_2, y_2) \in C + C \subset C$, we have

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) &= (x_1 + x_2)g\left(\frac{y_1 + y_2}{x_1 + x_2}\right) \\ &= (x_1 + x_2)g\left(\frac{x_1}{x_1 + x_2} \cdot \frac{y_1}{x_1} + \frac{x_2}{x_1 + x_2} \cdot \frac{y_2}{x_2}\right) \\ &\leq (x_1 + x_2)\left(\frac{x_1}{x_1 + x_2}g\left(\frac{y_1}{x_1}\right) + \frac{x_2}{x_1 + x_2}g\left(\frac{y_2}{x_2}\right)\right) \\ &= x_1g\left(\frac{y_1}{x_1}\right) + x_2g\left(\frac{y_2}{x_2}\right) = f(x_1, y_1) + f(x_2, y_2), \end{aligned}$$

i.e. $f \in LS_C$. ■

REMARK 1. Since $0 \notin C$, the subset C from the proposition 2 is not a cone, but this is permitted, by our previous convention.

REMARK 2. We can conclude that a system of functions $g_i(t)$, convex for $t \in (a, b)$, produces a system of subadditive functions $f_i(x, y)$ over $C \subset \mathbf{R}^2$ (denoted as $g(t) \Rightarrow f(x, y)$), so, according to proposition 1, we obtain a solution of (1) in the form

$$f(x, y) = \sum_{i=1}^n c_i f_i(x, y), \quad c_i > 0, \quad (x, y) \in C.$$

Conversely to proposition 2, we have the following

PROPOSITION 2'. *If the function $f \in LS_C$, where C is the same subset of \mathbf{R}^2 as in the proposition 2 and $f(\alpha x, \alpha y) = \alpha f(x, y)$ for every $\alpha \in \mathbf{R}^+$, then $f(x, y) = x \cdot g(y/x)$, where $g(t)$ is a convex function.*

Proof. The function $g(y) = f(1, y)$ is convex. Indeed, for $p \geq 0$, $q \geq 0$, $p + q = 1$:

$$\begin{aligned} g(py_1 + qy_2) &= f(1, py_1 + qy_2) = f(p + q, py_1 + qy_2) \leq f(p, py_1) + f(q, qy_2) \\ &= pf(1, py_1) + qf(1, y_2) = pg(y_1) + qg(y_2). \end{aligned}$$

Now, for $\alpha = 1/x$ we obtain

$$\frac{1}{x} \cdot f(x, y) = f\left(\frac{1}{x} \cdot x, \frac{1}{x} \cdot y\right) = f\left(1, \frac{y}{x}\right) = g\left(\frac{y}{x}\right),$$

i.e. $f(x, y) = xg(y/x)$. This prove the proposition. ■

REMARK 3. A method of obtaining the function from LS_C is following: If $\sup_{(x,y)}(f(x+a, y+b) - f(x, y)) = g(a, b)$, then $g \in LS_C$, where $f: C \rightarrow \mathbf{R}$.

Proof. Since

$$\begin{aligned} g(a_1 + b_1, a_2 + b_2) &= \sup_{(x,y)}(f(x + a_1 + b_1, y + a_2 + b_2) - f(x, y)) \\ &= \sup_{(x,y)}(f(x + a_1 + b_1, y + a_2 + b_2) - f(x + b_1, y + b_2) + f(x + b_1, y + b_2) - f(x, y)) \\ &\leq \sup_{(x,y)}(f(x + a_1 + b_1, y + a_2 + b_2) - f(x + b_1, y + b_2)) + \sup_{(x,y)}(f(x + b_1, y + b_2) - f(x, y)) \\ &= g(a_1, a_2) + g(b_1, b_2), \end{aligned}$$

then $g \in LS_C$. ■

Another property of subadditive functions is the following

PROPOSITION 3. If $f \in LS_C$ then

$$f\left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) \leq \sum_{i=1}^n f(x_i, y_i) \quad \text{for } (x_i, y_i) \in C, i = 1, 2, \dots, n.$$

Proof. This is easy to prove by induction on n , since from $(x_i, y_i) \in C, i = 1, 2, \dots, n$ it follows that

$$\begin{aligned} \left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) &= \sum_{i=1}^n (x_i, y_i) \in \underbrace{C + C + \dots + C}_n \\ &\subset \underbrace{C + C + \dots + C}_{n-1} \subset \dots \subset C + C \subset C. \quad \blacksquare \end{aligned}$$

Propositions 2 and 3 are the source for obtaining several kinds of two-parameter inequalities. We illustrate this with some examples.

EXAMPLE 1. Since $\ln t \rightleftharpoons -x \ln(y/x), x, y > 0$, using proposition 3 and putting $x_i = b_i, y_i = a_i b_i, i = 1, 2, \dots, n$, we obtain generalized arithmetic-geometric inequality

$$\prod_{i=1}^n a_i^{b_i} \leq \left(\sum_{i=1}^n a_i b_i / \sum_{i=1}^n b_i\right)^{\sum_{i=1}^n b_i}, \quad a_i, b_i > 0,$$

i.e. putting $b_i / \sum_{i=1}^n b_i = p_i, i = 1, 2, \dots, n$:

$$\prod_{i=1}^n a_i^{p_i} \leq \sum_{i=1}^n a_i p_i, \quad p_i, a_i > 0, \sum_{i=1}^n p_i = 1.$$

EXAMPLE 2. Since $t^r \Rightarrow \begin{cases} -x(y/x)^r, & \text{for } r \in (0, 1), \\ x(y/x)^r, & \text{for } r \in \mathbf{R} \setminus [0, 1], \end{cases}$ $x, y > 0$, putting $x_i = b_i^q$, $y_i = a_i^p$ and $r = 1/p$, $1 - r = 1/q$ in proposition 3, we obtain Hölder's inequality

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p \right)^{1/p} \left(\sum_{i=1}^n b_i^q \right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1; \quad p, q > 1, \text{ and}$$

$$\sum_{i=1}^n a_i b_i \geq \left(\sum_{i=1}^n a_i^p \right)^{1/p} \left(\sum_{i=1}^n b_i^q \right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1; \quad p < 1, \text{ or } q < 1.$$

EXAMPLE 3. Since $\ln \sin t \Rightarrow -x \ln \sin y/x$, using proposition 3 with $x_i = 1$, $i = 1, 2, \dots, n$ we have

$$\prod_{i=1}^n \sin y_i \leq \sin^n \left(\frac{1}{n} \sum_{i=1}^n y_i \right), \quad y_i \in (0, \pi).$$

For the n -dimensional case of locally subadditive functions we give the following definition: A function $f \in LS_C$ if

$$f(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \leq f(x_1, x_2, \dots, x_n) + f(y_1, y_2, \dots, y_n) \quad (4)$$

for each $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in C$, where C is a cone in \mathbf{R}^n .

Now we have the following

PROPOSITION 4. A function $g(t)$, convex for $t \in (a, b)$, produces a locally subadditive function $f(\cdot)$ on $C \subset \mathbf{R}^n$ by

$$f(x_1, x_2, \dots, x_n) = \left(\sum_{i=1}^n A_i x_i \right) g \left(\frac{\sum_{i=1}^n B_i x_i}{\sum_{i=1}^n A_i x_i} \right),$$

where

$$C = \left\{ (x_1, x_2, \dots, x_n) \mid \sum_{i=1}^n A_i x_i > 0, a < \frac{\sum_{i=1}^n B_i x_i}{\sum_{i=1}^n A_i x_i} < b \right\},$$

and A_i, B_i are arbitrary constants, not all equal to zero.

Proof is similar to that of proposition 2. Since $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in C$ imply that $(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in C$, putting

$$p = \frac{\sum_{i=1}^n A_i x_i}{\sum_{i=1}^n A_i (x_i + y_i)}, \quad q = \frac{\sum_{i=1}^n A_i y_i}{\sum_{i=1}^n A_i (x_i + y_i)}, \quad r = \frac{\sum_{i=1}^n B_i x_i}{\sum_{i=1}^n A_i x_i}, \quad s = \frac{\sum_{i=1}^n B_i y_i}{\sum_{i=1}^n A_i y_i}$$

in (3), we obtain (4), i.e. that $f \in LS_C$. ■

It is obvious that propositions 1 and 3 could be easily transformed to \mathbf{R}^n .

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REFERENCES

- [1] D. S. Mitrinović, P. M. Vasić, *Analytic Inequalities*, Beograd 1970
- [2] Y. Wong and Ng, K.F., *Partially Ordered Topological Vector Spaces*, Oxford 1973

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