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EXISTENCE OF THREE WEAK SOLUTIONS FOR A QUASILINEAR DIRICHLET PROBLEM

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Abstract. The aim of this note is to establish the existence of three solutions for a two-point boundary value problem. The approach is based on variational methods. Some particular cases and two concrete examples are then presented.

1. Introduction

In this paper, we consider the following Dirichlet boundary value problem

$$\begin{cases} -u'' = [\lambda f(x,u) + g(u)]h(x,u') & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2}$$

where λ is a positive parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, with $g(0) = 0$, and $h : [0,1] \times \mathbb{R} \to [0, +\infty)$ is a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$.

Motivated by the fact that such problems are used to describe a large class of physical phenomena, many authors looked for existence of solutions for second order ordinary differential non-linear equations. The existence of solutions for problem (2) or, more generally, for nonlinear differential problems has been widely investigated (see, for instance, [1–3, 5, 7, 10, 12] and the references therein).

In [2], using variational methods, the authors established the existence of at least one non-trivial solution for problem (2).

Using Ricceri's variational principle, in [1] the existence of infinitely many weak solutions has been proved for the problem

$$\begin{cases} -u'' + uh(x,u') = [\lambda f(x,u) + \mu g(x,u) + p(u)]h(x,u') & \text{in } (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

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where λ is a positive parameter, μ is a non-negative parameter, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, $p(0) = 0$, and $h : [0,1] \times \mathbb{R} \to [0, +\infty)$ is a bounded and continuous function with $m := \inf_{(x,t) \in [0,1] \times \mathbb{R}} h(x,t) > 0$.

In the present paper, employing two three critical points theorems which we recall in the next section (Theorems 2.1 and 2.2), we establish the existence of three solutions for the problem (2).

The following theorem represents a special case of Theorem 3.1.

THEOREM 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$36 \int_0^2 f(x)\, dx < \int_0^3 f(x)\, dx$$

and

$$\limsup_{|\xi| \to +\infty} \frac{\int_0^\xi f(x)\, dx}{\xi^2} \le 0.$$

Then, for each

$$\lambda \in \Big(\frac{90}{\int_0^3 f(x)\, dx}, \frac{5}{2 \int_0^2 f(x)\, dx} \Big),$$

the problem

$$\begin{cases} -u'' + u = \lambda f(u) & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

admits at least three classical solutions.

Moreover, the following result is a consequence of Theorem 3.6.

THEOREM 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$75 \int_0^4 f(x)\, dx < 2 \int_0^5 f(x)\, dx$$

and

$$3 \int_0^{40} f(x)\, dx < 4 \int_0^5 f(x)\, dx.$$

Then, for each

$$\lambda \in \Big(\frac{375}{\int_0^5 f(x)\, dx}, \min\Big\{ \frac{10}{\int_0^4 f(x)\, dx}, \frac{500}{\int_0^{40} f(x)\, dx} \Big\} \Big),$$

the problem

$$\begin{cases} -u'' + u = \lambda f(u) & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

admits at least three classical solutions.

2. Preliminaries

We now state two critical point theorems from Bonanno and coauthors [4,6] which are the main tools for proving our results. The first result has been obtained in [6] and it is a more precise version of Theorem 3.2 in [4]. The second one has been established in [4]. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, while in the second a suitable sign hypothesis is assumed.

THEOREM 2.1 ([6, Theorem 2.6]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that*

(i) $\sup_{\Phi(x)\leq r} \Psi(x) < r\Psi(\bar{x})/\Phi(\bar{x})$,

(ii) for each λ in

$$\Lambda_r := \Big(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\Big),$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

THEOREM 2.2 ([4, Theorem 3.2]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist two positive constants $r_1, r_2 > 0$ and $\bar{x} \in X$, with $2r_1 < \Phi(\bar{x}) < \frac{r_2}{2}$, such that*

(i) $$\frac{\sup_{\Phi(x)\leq r_1} \Psi(x)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})},$$

(ii) $$\frac{\sup_{\Phi(x)\leq r_2} \Psi(x)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})},$$

(iii) for each λ in $\Lambda^*_{r_1,r_2} := \Big(\frac{3}{2}\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \min\big\{\frac{r_1}{\sup_{\Phi(x)\leq r_1} \Psi(x)}, \frac{r_2}{2\sup_{\Phi(x)\leq r_2} \Psi(x)}\big\}\Big)$

and for every $x_1, x_2 \in X$, which are local minima for the functional $\Phi - \lambda\Psi$, and such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, one has $\inf_{t\in[0,1]} \Psi(tx_1 + (1-t)x_2) \geq 0$.

Then, for each $\lambda \in \Lambda^*_{r_1,r_2}$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Let us introduce some notation which will be used later. Define

$$H_0^1([0,1]) := \Big\{ u \in L^2([0,1]) : u' \in L^2([0,1]),\ u(0) = u(1) = 0 \Big\}.$$

Take $X = H_0^1([0,1])$ endowed with the usual norm defined as follows:

$$\|u\| := \left(\int_0^1 |u'(x)|^2 \, dx \right)^{1/2}.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e. $|g(t_1) - g(t_2)| \le L|t_1 - t_2|$, for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$, $h : [0,1] \times \mathbb{R} \to [0, +\infty)$ be a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function.

We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(i) the mapping $x \longmapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;

(ii) the mapping $\xi \longmapsto f(x,\xi)$ is continuous for almost every $x \in [0,1]$;

(iii) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that for almost every $x \in [0,1] :\ \sup_{|\xi|\le\rho} |f(x,\xi)| \le l_\rho(x)$.

Corresponding to f, g and h we introduce the functions $F : [0,1] \times \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ and $H : [0,1] \times \mathbb{R} \to [0, +\infty)$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)\, d\xi, \qquad G(t) := -\int_0^t g(\xi)\, d\xi,$$

$$H(x,t) := \int_0^t \left(\int_0^\tau \frac{1}{h(x,\delta)} d\delta \right) d\tau$$

for all $x \in [0,1]$ and $t \in \mathbb{R}$.

In the following, let $M := \sup_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t)$ and suppose that the Lipschitz constant $L > 0$ of the function g satisfies the condition $LM < 4$.

We say that a function $u \in X$ is a *weak solution* of problem (2) if

$$\int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)} d\tau \right) v'(x)\, dx - \lambda \int_0^1 f(x,u(x)) v(x)\, dx - \int_0^1 g(u(x)) v(x)\, dx = 0$$

holds for all $v \in X$.

By standard regularity results, if f is continuous in $[0,1] \times \mathbb{R}$, then weak solutions of problem (2) belong to $C^2([0,1])$, thus they are classical solutions.

For other basic notations and definitions, we refer the reader to [8, 11, 13, 15].

3. Main results

Put

$$A := \frac{4 - LM}{8M}, \qquad B := \frac{4 + Lm}{8m},$$

and suppose that $B \le 4A$. We formulate our main results as follows.

Theorem 3.1. *Assume that there exist two positive constants c and d with $c < \sqrt{2}d$, such that*

$$(A1) \qquad F(x,t) \geq 0 \text{ for all } (x,t) \in \left([0,\frac{1}{4}] \cup [\frac{3}{4},1]\right) \times [0,d];$$

$$(A2) \qquad \frac{\int_0^1 \max_{|t|\leq c} F(x,t)\,dx}{c^2} < \frac{1}{8}\frac{\int_{1/4}^{3/4} F(x,d)\,dx}{d^2};$$

$$(A3) \qquad \limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,1]} F(x,\xi)}{\xi^2} \leq \frac{4A}{B}\frac{\int_0^1 \max_{|t|\leq c} F(x,t)\,dx}{c^2}.$$

Then, for every λ in

$$\Lambda := \Big(\frac{8Bd^2}{\int_{1/4}^{3/4} F(x,d)\,dx}, \frac{Bc^2}{\int_0^1 \max_{|t|\leq c} F(x,t)\,dx}\Big),$$

problem (2) *has at least three distinct weak solutions.*

Proof. Fix λ as in the conclusion. Our aim is to apply Theorem 2.1 to our problem. To this end, for every $u \in X$, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ by setting

$$\Phi(u) := \int_0^1 H(x,u'(x))\,dx + \int_0^1 G(u(x))\,dx,$$

$$\Psi(u) := \int_0^1 F(x,u(x))\,dx$$

and put
$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \quad \forall\, u \in X.$$

Note that the weak solutions of (2) are exactly the critical points of I_λ. It is well known that the functionals Φ, Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}d\tau\right) v'(x)\,dx - \int_0^1 g(u(x))v(x)\,dx,$$

$$\Psi'(u)(v) = \int_0^1 f(x,u(x))v(x)\,dx$$

for any $v \in X$. Also, the functionals Φ and Ψ satisfy all regularity assumptions imposed in Theorem 2.1 (for more details, see the proof of [9, Theorem 2.1]).

Since g is Lipschitz continuous and satisfies $g(0) = 0$, while h is bounded away from zero, the inequality

$$\max_{x\in[0,1]} |u(x)| \leq \frac{1}{2}\|u\| \quad \text{for all } u \in X \tag{3}$$

(see, e.g. [14]) yields for any $u \in X$ the estimate

$$A\|u\|^2 \leq \Phi(u) \leq B\|u\|^2. \tag{4}$$

We will verify (i) and (ii) of Theorem 2.1. Put $r = Bc^2$. Taking (3) into account, for every $u \in X$ such that $\Phi(u) \le r$, one has $\max_{x\in[0,1]} |u(x)| \le c$. Consequently,

$$\sup_{\Phi(u)\le r} \Psi(u) \le \int_0^1 \max_{|t|\le c} F(x,t)\, dx,$$

that is,

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} \le \frac{\int_0^1 \max_{|t|\le c} F(x,t)\, dx}{Bc^2}.$$

Hence,

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{1}{\lambda}. \tag{5}$$

Put

$$w(x) = \begin{cases} 4dx, & x \in [0, \frac{1}{4}[, \\ d, & x \in [\frac{1}{4}, \frac{3}{4}], \\ 4d(1-x), & x \in]\frac{3}{4}, 1]. \end{cases}$$

It is easy to verify that $w \in X$ and, in particular, $\|w\|^2 = 8d^2$. So, taking (4) into account, we deduce $8Ad^2 \le \Phi(w) \le 8Bd^2$. Hence, from $c < \sqrt{2}d$ and $B \le 4A$, we obtain $r < \Phi(w)$.

Since $0 \le w(x) \le d$ for each $x \in [0,1]$, assumption $(A1)$ ensures that

$$\int_0^{1/4} F(x, w(x))\, dx + \int_{3/4}^1 F(x, w(x))\, dx \ge 0,$$

and so

$$\Psi(w) \ge \int_{1/4}^{3/4} F(x, d)\, dx.$$

Therefore, we obtain

$$\frac{\Psi(w)}{\Phi(w)} \ge \frac{1}{8} \frac{\int_{1/4}^{3/4} F(x,d)\, dx}{Bd^2} > \frac{1}{\lambda}. \tag{6}$$

Therefore, from (5) and (6), condition (i) of Theorem 2.1 is fulfilled. Now, to prove the coercivity of the functional I_λ, due to $(A3)$, we have

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,1]} F(x,\xi)}{\xi^2} < \frac{4A}{\lambda}.$$

So, we can fix $\varepsilon > 0$ satisfying

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,1]} F(x,\xi)}{\xi^2} < \varepsilon < \frac{4A}{\lambda}.$$

Then, there exists a positive constant θ such that

$$F(x,t) \le \varepsilon |t|^2 + \theta \quad \forall x \in [0,1],\ \forall t \in \mathbb{R}.$$

Taking into account (3) and (4), it follows that

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \ge A\|u\|^2 - \lambda\varepsilon \|u\|^2_{L^2[0,1]} - \lambda\theta \ge \left(A - \frac{\lambda\varepsilon}{4}\right)\|u\|^2 - \lambda\theta$$

for all $u \in X$. So, the functional I_λ is coercive. Now, the conclusion of Theorem 2.1

can be used. It follows that, for every

$$\lambda \in \Lambda \subseteq \Big(\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\Big),$$

the functional I_λ has at least three distinct critical points in X, which are the weak solutions of the problem (2). This completes the proof. $\square$

Now, we point out the following consequence of Theorem 3.1.

Corollary 3.2. *Let $\alpha \in L^1([0,1])$ be a non-negative and non-zero function and let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $\alpha_0 := \int_{1/4}^{3/4} \alpha(x)dx$, $\|\alpha\|_1 := \int_0^1 \alpha(x)\,dx$ and $\Gamma(t) = \int_0^t \gamma(\xi)d\xi$ for all $t \in \mathbb{R}$, and assume that there exist two positive constants c and d with $c < \sqrt{2}d$, such that*

$(A1')$ $$\Gamma(t) \geq 0 \text{ for all } t \in [0,d]$$

$(A2')$ $$\frac{\max_{|t|\leq c}\Gamma(t)}{c^2} < \frac{1}{8}\frac{\alpha_0}{\|\alpha\|_1}\frac{\Gamma(d)}{d^2};$$

$(A3')$ $$\limsup_{|\xi|\to+\infty}\Gamma(\xi)/\xi^2 \leq 0.$$

Then, for every

$$\lambda \in \Big(\frac{8Bd^2}{\alpha_0\Gamma(d)}, \frac{Bc^2}{\|\alpha\|_1 \max_{|t|\leq c}\Gamma(t)}\Big),$$

the problem

$$\begin{cases} -u'' = [\lambda\alpha(x)\gamma(u) + g(u)]h(x,u') & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{7}$$

has at least three weak solutions.

Proof. The proof follows from Theorem 3.1 by choosing $f(x,t) := \alpha(x)\gamma(t)$ for all $(x,t) \in [0,1]\times\mathbb{R}$. $\square$

Remark 3.3. Clearly, if γ is non-negative then assumption $(A1')$ is verified and $(A2')$ becomes

$$\frac{\Gamma(c)}{c^2} < \frac{1}{8}\frac{\alpha_0}{\|\alpha\|_1}\frac{\Gamma(d)}{d^2}.$$

Remark 3.4. Theorem 1.1 from the introduction is an immediate consequence of Corollary 3.2, by choosing $g(u) = -u$, $h \equiv 1$, $c = 2$ and $d = 3$.

Lemma 3.5. *Assume that $f(x,t) \geq 0$ for all $(x,t) \in [0,1]\times\mathbb{R}$. If u is a weak solution of (2), then $u(x) \geq 0$ for all $x \in [0,1]$.*

Proof. Arguing by contradiction, if we assume that u is negative at a point of $[0,1]$, the set $\Omega := \{x \in [0,1] : u(x) < 0\}$ is non-empty and open. Moreover, let us consider $\bar{v} := \min\{u, 0\}$, one has, $\bar{v} \in X$. So, taking into account that g is a Lipschitz continuous function, $g(0) = 0$, u is a weak solution and by choosing $v = \bar{v}$, one has

$$
\begin{aligned}
0 &\geq \lambda \int_\Omega f(x, u(x))u(x)\, dx \\
&= \int_\Omega \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)} d\tau \right) u'(x)\, dx - \int_\Omega g(u(x))u(x)\, dx \\
&\geq \frac{4 - LM|\Omega|}{4M} \|u\|^2_{H^1_0(\Omega)},
\end{aligned}
$$

where $|\Omega|$ is the Lebesgue measure of the set Ω. Therefore, $\|u\|_{H^1_0(\Omega)} = 0$ which is irrational. Hence, the conclusion is achieved. □

Our other main result is as follows.

THEOREM 3.6. *Assume that there exist three positive constants c_1, c_2 and d with $c_1 < d < c_2/4$, such that*

$(B1)$ $f(x,t) \geq 0$ *for all* $(x,t) \in [0,\, 1] \times [0, c_2]$;

$(B2)$ $$\frac{\int_0^1 F(x, c_1)\, dx}{c_1^2} < \frac{1}{12} \frac{\int_{1/4}^{3/4} F(x,d)\, dx}{d^2};$$

$(B3)$ $$\frac{\int_0^1 F(x, c_2)\, dx}{c_2^2} < \frac{1}{24} \frac{\int_{1/4}^{3/4} F(x,d)\, dx}{d^2}.$$

Let $$\Lambda' := \Big(\frac{12Bd^2}{\int_{1/4}^{3/4} F(x,d)\, dx}, B \min\Big\{ \frac{c_1^2}{\int_0^1 F(x,c_1)\, dx}, \frac{c_2^2}{2\int_0^1 F(x,c_2)\, dx} \Big\} \Big).$$

Then, for every $\lambda \in \Lambda'$ the problem (2) has at least three weak solutions u_i, $i = 1, 2, 3$, such that $0 < \|u_i\|_\infty \leq c_2$.

Proof. Without loss of generality, we can assume $f(x,t) \geq 0$ for all $(x,t) \in [0,1] \times \mathbb{R}$. Fix λ as in the conclusion and take X, Φ and Ψ as in the proof of Theorem 3.1. Put w as in Theorem 3.1, $r_1 = Bc_1^2$ and $r_2 = Bc_2^2$. Therefore, one has $2r_1 < \Phi(w) < \frac{r_2}{2}$ and we have

$$\frac{1}{r_1} \sup_{\Phi(u) < r_1} \Psi(u) \leq \frac{1}{Bc_1^2} \int_0^1 F(x,c_1)\, dx < \frac{1}{\lambda} < \frac{1}{12} \frac{\int_{1/4}^{3/4} F(x,d)\, dx}{Bd^2} \leq \frac{2}{3} \frac{\Psi(w)}{\Phi(w)},$$

and $$\frac{2}{r_2} \sup_{\Phi(u) < r_2} \Psi(u) \leq \frac{2}{Bc_2^2} \int_0^1 F(x,c_2)\, dx < \frac{1}{\lambda} < \frac{1}{12} \frac{\int_{1/4}^{3/4} F(x,d)\, dx}{Bd^2} \leq \frac{2}{3} \frac{\Psi(w)}{\Phi(w)}.$$

So, conditions (i) and (ii) of Theorem 2.2 are satisfied. Finally, let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then, u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (2). Hence, owing to Lemma 3.5, we obtain $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for all $x \in [0,1]$. So, one has $\Psi(su_1 + (1-s)u_2) \geq 0$ for all $s \in [0,1]$. From Theorem 2.2 the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are weak solutions of (2). This complete the proof. □

Now, we point out the following consequence of Theorem 3.6.

Corollary 3.7. *Let $\alpha \in L^1([0,1])$ be such that $\alpha(x) \geq 0$ a.e. $x \in [0,1]$, $\alpha \not\equiv 0$, and let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $\alpha_0 := \int_{1/4}^{3/4} \alpha(x)dx$, $\|\alpha\|_1 := \int_0^1 \alpha(x)\,dx$ and $\Gamma(t) = \int_0^t \gamma(\xi)d\xi$ for all $t \in \mathbb{R}$, and assume that there exist three positive constants c_1, c_2 and d with $c_1 < d < c_2/4$, such that*

$(B1')$ $$\gamma(t) \geq 0 \text{ for all } t \in [0, c_2];$$

$(B2')$ $$\frac{\Gamma(c_1)}{c_1^2} < \frac{1}{12}\frac{\alpha_0}{\|\alpha\|_1}\frac{\Gamma(d)}{d^2};$$

$(B3')$ $$\frac{\Gamma(c_2)}{c_2^2} < \frac{1}{24}\frac{\alpha_0}{\|\alpha\|_1}\frac{\Gamma(d)}{d^2}.$$

Then, for every

$$\lambda \in \Big(\frac{12Bd^2}{\alpha_0\Gamma(d)}, B\min\Big\{\frac{c_1^2}{\|\alpha\|_1\Gamma(c_1)}, \frac{c_2^2}{2\|\alpha\|_1\Gamma(c_2)}\Big\}\Big),$$

the problem (7) has at least three weak solutions u_i, $i = 1, 2, 3$, such that $0 < \|u_i\|_\infty \leq c_2$.

Proof. The proof follows from Theorem 3.6 by choosing $f(x,t) := \alpha(x)\gamma(t)$ for all $(x,t) \in [0,1] \times \mathbb{R}$. □

Remark 3.8. Theorem 1.2 from the introduction is an immediate consequence of Corollary 3.7, by choosing $g(u) = u$, $h \equiv 1$, $c_1 = 4$, $c_2 = 40$ and $d = 5$.

Finally, we present the following examples to illustrate the results.

Example 3.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows

$$f(t) := \begin{cases} 1 & \text{if } |t| \leq 1, \\ t^{10} & \text{if } 1 < |t| \leq 2, \\ 2^{12}\, t^{-2} & \text{if } |t| > 2. \end{cases}$$

Put $g(t) = -t$ and $h(x,t) = (2 + x + \cos t)^{-1}$ for all $x \in [0,1]$ and $t \in \mathbb{R}$ and choose $c = 1$ and $d = 2$. Therefore, according to Corollary 3.2, for each $\lambda \in [0.8, 2]$, the problem

$$\begin{cases} -u''(2 + x + \cos u') + u = \lambda f(u) & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

admits at least three weak solutions.

Example 3.10. Consider the problem

$$\begin{cases} -u''(3 + \sin u') - u = \lambda f(u) & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{8}$$

where f be the function defined as in the Example 3.9. Setting $g(t) = t$ and $h(x,t) = (3 + \sin t)^{-1}$ for all $x \in [0,1]$ and $t \in \mathbb{R}$, we see that (B2') and (B3') are satisfied with $c_1 = 1$, $d = 2$ and $c_2 = 64$. According to Corollary 3.7, for each $\lambda \in [1.09, 2]$, the problem (8) admits at least three classical solutions u_i, $i = 1, 2, 3$, such that $0 < \|u_i\|_\infty \leq 64$.

References

[**1**] G. A. Afrouzi, A. Hadjian, *Infinitely many solutions for a Dirichlet boundary value problem depending on two parameters*, Glas. Mat. **48** (2013), 357–371.

[**2**] G. A. Afrouzi, A. Hadjian, S. Heidarkhani, *Non-trivial solutions for a two-point boundary value problem*, Ann. Polon. Math. **108** (2013), 75–84.

[**3**] G. A. Afrouzi, S. Shokooh, *Three solutions for a fourth-order boundary-value problem*, Electron. J. Diff. Eqs. **45** (2015), 1–11.

[**4**] G. Bonanno, A. Chinni, *Existence of three solutions for a perturbed two-point boundary value problem*, Appl. Math. Lett. **23** (2010), 807–811.

[**5**] G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations, **244** (2008), 3031–3059.

[**6**] G. Bonanno, S. A. Marano, *On the structure of the critical set of non-differentiable functionals with a weak compactness condition*, Appl. Anal. **89** (2010), 1–10.

[**7**] L. H. Erbe, H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 743–748.

[**8**] M. Ghergu, V. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analaysis*, Oxford Lecture Ser. Math. Appl. 37, Oxford Univ. Press, New York, 2008.

[**9**] S. Heidarkhani, D. Motreanu, *Multiplicity results for a two-point boundary value problem*, Panamer. Math. J. **19** (2009), 69–78.

[**10**] N. Hirano, *Existence of infinitely many solutions for sublinear elliptic problems*, J. Math. Anal. Appl. **278** (2003), 83–92.

[**11**] A. Kristály, V. Rădulescu, C. Varga, *Variational Principles in Mthematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia Math. Appl. 136, Cambridge Univ. Press, Cambridge, 2010.

[**12**] Y. Naito, S. Tanaka, *On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations*, Nonlinear Anal. **56** (2004), 919–935.

[**13**] B. Ricceri, *Nonlinear eigenvalue problems*, in: D. Y. Gao and D. Motreanu (eds.), Handbook of Nonconvex Analysis and Applications, Int. Press, Someerville, MA, 2010, 543–595.

[**14**] G. Talenti, *Some inequalities of Sobolev type on two-dimensional spheres*, in: W. Walter(ed.), General Inequalities, Vol. 5, Int. Ser. Numer. Math. 80, Birkhäuser, Basel, 1987, 401–408.

[**15**] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, vol. II/B and III, Berlin-Heidelberg-New York, 1990 and 1985.

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