COUPLED FIXED POINTS FOR MAPPINGS ON A b-METRIC SPACE WITH A GRAPH

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Abstract. In this paper, we will develop a new method to study coupled fixed points of a mapping $T : X \times X \to X$, where (X, d) is a special class of b-metric spaces endowed with a graph. We will prove some general fixed point theorems which enable us to extend some old results in fixed point theory. Moreover, we will extend Edelstein's fixed point theorem for two variable mappings in ε-chainable b-metric spaces.

1. Introduction

In 1922, Banach [2] established a fixed point theorem known as the Banach Contraction Principle which is one of the most important results in analysis. There are many generalizations of Banach Contraction Principle. In particular, Ran and Reurings [22] extended Banach's fixed point theorem in complete metric spaces endowed with a partial ordering as follows.

THEOREM 1.1. *[22] Let (X, d) be a complete metric space endowed with a partial ordering $\precsim$ such that every pair of elements of X has an upper and lower bound. Let $T : X \to X$ be continuous and monotone and such that for some $\alpha \in (0, 1)$,*

$$x \precsim y \Rightarrow d(Tx, Ty) \leq \alpha d(x, y) \quad (x, y \in X).$$

If there is $x_0 \in X$ with $x_0 \precsim Tx_0$ or $Tx_0 \precsim x_0$, then T has a unique fixed point x^* and $x^* = \lim_{n \to \infty} T^n x$ for all $x \in X$.

Subsequently, a few generalizations of Theorem 1.1 were obtained in [18–21].

In 2007, Jachymski [12] investigated the class of generalized Banach contractions on a metric space endowed with a directed graph (see also [1] and [4]). Therefore many corresponding results on partially ordered metric spaces were extended.

Let X be a topological space. Following Opoitsev [23,24] a pair $(x^*, y^*) \in X \times X$ is called a *coupled fixed point* for a mapping $T : X \times X \to X$ provided that $x^* = T(x^*, y^*)$ and $y^* = T(y^*, x^*)$. We inductively define $\{T^n\}$ as follows: $T^1(x, y) = T(x, y)$ and $T^{n+1}(x, y) = T\big(T^n(x, y), T^n(y, x)\big)$ for each $n \in \mathbb{N}$ and $x, y \in X$.

The mapping T is called a Picard operator if the sequence $\big\{\big(T^n(x, y), T^n(y, x)\big)\big\}$ converges to a unique coupled fixed point of T for each $(x, y) \in X^2$. T is a weakly Picard operator if for each $(x, y) \in X^2$, the sequence $\big\{\big(T^n(x, y), T^n(y, x)\big)\big\}$ converges to a coupled fixed point of T.

The theory of coupled fixed points was expanded by Guo and Lakshmikantham [10], who used the monotone iteration method for two variable contractive type mappings. In 2006, Bhaskar and Lakshmikantham proved the following.

THEOREM 1.2. *[3] Let $(X, \precsim)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \times X \to X$ be a continuous mapping having the mixed monotone property on X i.e. $T(\cdot, y)$ is monotone increasing and $T(x, \cdot)$ is monotone decreasing. Assume that there exists a $\lambda \in [0, 1)$ with $d(T(x, y), T(u, v)) \leq \frac{\lambda}{2}(d(x, u) + d(y, v))$ for all $x, y, u, v \in X$ for which $x \succsim u$ and $y \precsim v$. If there exist $x_0, y_0 \in X$ such that $x_0 \precsim T(x_0, y_0)$ and $y_0 \succsim T(y_0, x_0)$, then T has a coupled fixed point.*

They also proved that Theorem 1.2 is still valid when continuity of T is replaced by the following properties.

(i) If a nondecreasing sequence $\{x_n\}$ converges to x, then $x_n \precsim x$, for all $n \in \mathbb{N}$.

(ii) If a nonincreasing sequence $\{x_n\}$ converges to x, then $x \precsim x_n$, for all $n \in \mathbb{N}$.

In 2015, Bota et al. obtained the following result in b-metric spaces.

THEOREM 1.3. *[5] Let (X, d) be a complete b-metric space with parameter $k \geq 1$ and $\precsim$ be partial order on X. If $T : X \times X \to X$ is a continuous mapping with the mixed monotone property and the following conditions are satisfied:*

(i) there exists $\lambda \in [0, 1/k)$ such that $d\big(T(x, y), T(u, v)\big) \leq \frac{\lambda}{2}[d(x, u) + d(y, v)]$ for all $x \succsim u, y \precsim v$,

(ii) there exist $x_0, y_0 \in X$ such that $x_0 \precsim T(x_0, y_0)$ and $y_0 \succsim T(x_0, y_0)$.

Then T has a coupled fixed point.

In 1995, Czerwik [6] introduced a new class of b-metric spaces. In this paper, we use Frink's lemma on this class of b-metric spaces endowed with a graph to obtain some general results for the existence of coupled fixed points of generalized contractions. Our results enable us to extend the above results in b-metric spaces endowed with a graph. Moreover, we will prove Edelstein's fixed point theorem in ε-chainable b-metric spaces for two variable mappings.

2. Results

Let us start by recalling the following definition.

DEFINITION 2.1. [6] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ satisfies the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq k[d(x, y) + d(y, z)]$

for all $x, y, z \in Z$ and for some $k > 1$. Then (X, d) is called a b-metric space.

In 1995, Czerwik [6] defined a special class of b-metrics by replacing (iii) with the following:

(iv) For each $\varepsilon > 0$ and $x, y, z \in X$ if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$.

It is easy to verify that "$<$" can be replaced by "$\leq$" in (iv). Moreover, if d satisfies (iv), then (iii) holds for $k = 2$.

EXAMPLE 2.2. Let $X = \ell^{\frac{1}{2}}(\mathbb{R})$. For $x, y \in X$ define $d(x, y) = ||x - y||_{\frac{1}{2}}$. Then (X, d) is not a metric space. But it satisfies (iv), hence it is a b-metric space in the sense of Czerwik.

However, for the b-metric $(\mathbb{R}, d')$ defined by $d'(x, y) = |x - y|^2$, $x, y \in \mathbb{R}$, we have $d'(1, 0) = d'(0, -1) = 1$ but $d'(1, -1) = 4 \not< 2$. It is easy to verify that $(\mathbb{R}, d')$ satisfies (iii) for $k = 2$. Hence the class of those b-metrics with the property (iv) is strictly smaller than the class of all b-metric spaces for $k = 2$.

We refer the interested reader to $[11, 14\text{--}17]$ for some recent results on b-metric spaces.

The following result plays an important role in the sequel.

LEMMA 2.3. ($[8, 9]$) Suppose $d : X \times X \to [0, \infty)$ satisfies the following condition. For any $\varepsilon > 0$ and $x, y, z \in X$, if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$. Define $\rho : X \times X \to [0, \infty)$ by

$$\rho(x, y) = \inf \left\{ \sum_{i=1}^{n} d(x_{i-1}, x_i); \ \text{where } n \in \mathbb{N}, x_0 = x \text{ and } x_n = y \right\} \ (x, y \in X).$$

Then ρ has the following properties:

(i) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, for all $x, y, z \in X$.

(ii) $\frac{d(x,y)}{4} \leq \rho(x, y) \leq d(x, y)$ for all $x, y \in X$. Further, ρ is symmetric (i.e. $\rho(x, y) = \rho(y, x)$) if d is.

Hereafter, we will assume that (X, d) is a b-metric space in the sense of Czerwik [6], that is a b-metric with the property (iv). Moreover, we will assume that Δ is the diagonal of the Cartesian product $X \times X$ and that G is a graph on X such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains Δ.

The metric space (X, d) endowed with the graph G is called a G-b-metric space. We also denote by G^{-1} the graph obtained from G by reversing the direction of edges. Define the graph G^2 on X^2 by

$$E(G^2) = \{\big((x, y), (u, v)\big) \in X^2 \times X^2 : (x, u) \in E(G), \quad (y, v) \in E(G^{-1})\}.$$

EXAMPLE 2.4. Let $\precsim$ be a partial order in X. We define the graph G_1 by $E(G_1) = \{(x, y) \in X^2 : x \precsim y\}$. Then a mapping $T : X \times X \to X$ has the mixed monotone property with respect to $\precsim$ if and only if it is G_1^2-edge preserving.

Now, we recall some basic notions concerning connectivity of graphs [13]. Let G be a graph on X and x and y be vertices in G. Then a path in G from x to y of length N ($N = 0, 1, 2, \dots$) is a finite sequence $x_0 = x, x_1, \dots, x_N = y$, where $(x_{i-1}, x_i) \in E(G)$ for $1 \le i \le N$.

A graph G is called connected if there is a path between any two vertices. The graph G is called weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ is the undirected graph obtained from G by ignoring the direction of edges.

Let $E(G)$ be symmetric and $x \in V(G)$. The component of G which contains x, denoted by G_x, is the subgraph of G consisting of all edges and vertices which are contained in some path beginning at x. In this case, the relation "yRz if and only if there is a path in G from y to z" defines an equivalent relation on X and $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of x.

DEFINITION 2.5. A function $T : X \times X \to X$ is called

(i) G^2-edge preserving if $\big((x, y), (u, v)\big) \in E(G^2)$ implies that
$$\big((T(x, y), T(y, x)), (T(u, v), T(v, u))\big) \in E(G^2).$$

(ii) G^2-continuous if $x_n \to x$, $y_n \to y$ and $\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \in E(G^2)$ implies that $T(x_n, y_n) \to T(x, y)$.

(iii) G-contraction if it is G^2 edge preserving and there is $\lambda \in [0, 1)$ such that $\big((x, y), (u, v)\big) \in E(G^2)$ implies that $d(T(x, y), T(u, v)) \le \frac{\lambda}{2}\big[d(x, u) + d(y, v)\big]$.

In order to state the main results of this paper, we need the following auxiliary results.

LEMMA 2.6. *Let $T : X \times X \to X$ be a G-contraction with a constant $0 \le \lambda < 1$ and $(x, y) \in [(x_0, y_0)]_{\tilde{G}_2}$, where $x_0, y_0, x, y \in X$. Then there is some $\varphi(x, y)$ such that*

$$d\big(T^n(x, y), T^n(x_0, y_0)\big) \le \lambda^n \varphi(x, y) \quad (n \in \mathbb{N}). \tag{2}$$

Proof. By induction on n, we will show that

$$d\big(T^n(a, b), T^n(c, d)\big) \le \frac{\lambda^n}{2}[d(a, c) + d(b, d)], \quad (n \in \mathbb{N}), \tag{3}$$

whenever $\big((a,b),(c,d)\big) \in E(\tilde{G}^2)$. Since T is a G-contraction, (3) holds for $n = 1$.

Let (3) hold for some $n \in \mathbb{N}$ and $\big((a,b),(c,d)\big) \in E(\tilde{G}^2)$. Since T is G^2-edge preserving, $\big((T(a,b),\, T(b,a)),(T(c,d),\, T(d,c))\big) \in E(\tilde{G}^2)$. Thus by our hypothesis,

$$
\begin{aligned}
d\big(T^{n+1}(a,b),\, T^{n+1}(c,d)\big) &= d\big(T^n(T(a,b),\, T(b,a)),\, T^n(T(c,d),\, T(d,c))\big) \\
&\leq \frac{\lambda^n}{2}[d\big(T(a,b),\, T(c,d)\big) + d\big(T(b,a),\, T(d,c)\big)] \\
&\leq \frac{\lambda^{n+1}}{2}[d(a,c) + d(b,d)].
\end{aligned}
$$

This proves (3) for each $n \in \mathbb{N}$ and $\big((a,b),(c,d)\big) \in E(\tilde{G}^2)$.

Suppose that $(t_i, w_i)_{i=0}^N$ is a path from (x,y) to (x_0, y_0) in $\tilde{G}^2$. Then $(x,y) = (t_0, w_0), (x_0, y_0) = (t_N, w_N)$ and $\big((t_{i-1}, w_{i-1}),(t_i, w_i)\big) \in E(\tilde{G}^2)$ for $i = 1, \ldots, N$. Therefore, by (3), for all $n \in \mathbb{N}$ and $1 \leq i \leq N$,

$$
d\big(T^n(t_{i-1}, w_{i-1}),\, T^n(t_i, w_i)\big) \leq \frac{\lambda^n}{2}[d(t_{i-1}, t_i) + d(w_{i-1}, w_i)].
$$

Let ρ be as in Lemma 2.3. Then

$$
\begin{aligned}
d\big(T^n(x,y),\, T^n(x_0, y_0))\big) &\leq 4\rho\big(T^n(x,y),\, T^n(x_0, y_0)\big) \\
&\leq 4\sum_{i=1}^N \rho\big(T^n(t_{i-1}, w_{i-1}),\, T^n(t_i, w_i)\big) \\
&\leq 4\sum_{i=1}^N d\big(T^n(t_{i-1}, w_{i-1}),\, T^n(t_i, w_i)\big) \\
&\leq 2\lambda^n \sum_{i=1}^N [d(t_{i-1}, t_i) + d(w_{i-1}, w_i)].
\end{aligned}
$$

This proves (2) with $\varphi(x,y) = 2\sum_{i=1}^N [d(t_{i-1}, t_i) + d(w_{i-1}, w_i)]$. $\hfill\square$

LEMMA 2.7. *Let $T : X \times X \to X$ be G^2-edge preserving such that for some $(x_0, y_0) \in X^2$, $\big(T(x_0, y_0),\, T(y_0, x_0)\big) \in [(x_0, y_0)]_{\tilde{G}^2}$. Then for each $(x,y) \in [(x_0, y_0)]_{\tilde{G}^2}$,*

$$
\big(T^n(x,y),\, T^n(y,x)\big) \in [(x_0, y_0)]_{\tilde{G}^2}, \quad (n \in \mathbb{N}). \tag{4}
$$

Proof. We will prove (4) by induction. Let $(x,y) \in [(x_0, y_0)]_{\tilde{G}^2}$. Then there is a path $(x,y) = (t_0, w_0), \ldots, (t_N, w_N) = (x_0, y_0)$, from (x,y) to (x_0, y_0) in $\tilde{G}^2$. Since T is G^2-edge preserving,

$$
\Big(\big(T(t_{i-1}, w_{i-1}),\, T(w_{i-1}, t_{i-1})\big),\big(T(t_i, w_i),\, T(w_i, t_i)\big)\Big) \in E(\tilde{G}^2);\ i = 1, \ldots, N.
$$

It follows that $\big(T(x,y),\, T(y,x)\big)$ and $\big(T(x_0, y_0),\, T(y_0, x_0)\big)$ are in the same class. Hence $\big(T(x,y),\, T(y,x)\big) \in [(x_0, y_0)]_{\tilde{G}^2}$. This proves (4) for $n = 1$.

Let (4) hold for some n. Put $u = T^n(x,y)$ and $v = T^n(y,x)$. Then $(u,v) \in [(x_0, y_0)]_{\tilde{G}^2}$. Repeating the argument from the first part of the proof, we see that

$$
\big(T^{n+1}(x,y),\, T^{n+1}(y,x)\big) = \big(T(u,v),\, T(v,u)\big) \in [(x_0, y_0)]_{\tilde{G}^2}.
$$

This proves (4). $\hfill\square$

In order to state our next auxiliary result, we need the following definition.

DEFINITION 2.8. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X. We say that these sequences are Cauchy equivalent if each of them is Cauchy and $d(x_n, y_n) \to 0$ as $n \to \infty$.

LEMMA 2.9. *Let $T : X \times X \to X$ be a G-contraction with a constant $\lambda \in [0,1)$ and let $(T(x_0, y_0), T(y_0, x_0)) \in [(x_0, y_0)]_{\tilde{G}^2}$ for some $(x_0, y_0) \in X^2$. Then for each $(x, y) \in [(x_0, y_0)]_{\tilde{G}^2}$ the following statements hold.*

(i) The sequences $\{T^n(x, y)\}$ and $\{T^n(x_0, y_0)\}$ are Cauchy equivalent.

(ii) The sequences $\{T^n(y, x)\}$ and $\{T^n(y_0, x_0)\}$ are Cauchy equivalent.

Proof. Since $\big(T(x_0, y_0), T(y_0, x_0)\big) \in [(x_0, y_0)]_{\tilde{G}^2}$, by Lemma 2.6, there is some $\varphi(x_0, y_0)$ such that for each $n \in \mathbb{N}$,

$$d\big(T^n(x_0, y_0), T^{n+1}(x_0, y_0)\big) = d\big(T^n(x_0, y_0), T^n(T(x_0, y_0), T(y_0, x_0))\big) \le \lambda^n \varphi(x_0, y_0).$$

Let ρ be as in Lemma 2.3. Then for each $m > n$,

$$\begin{aligned}
d\big(T^n(x_0, y_0), T^m(x_0, y_0)\big) &\le 4\rho\big(T^n(x_0, y_0), T^m(x_0, y_0)\big) \\
&\le 4 \sum_{i=n}^{m-1} \rho\big(T^i(x_0, y_0), T^{i+1}(x_0, y_0)\big) \\
&\le 4 \sum_{i=n}^{m-1} d\big(T^i(x_0, y_0), T^{i+1}(x_0, y_0)\big) \\
&\le 4\varphi(x_0, y_0) \sum_{i=n}^{m-1} \lambda^i \le \frac{4\varphi(x_0, y_0)\lambda^n}{1 - \lambda}.
\end{aligned}$$

Since the right-hand side of the above inequality tends to zero as n tends to infinity, $\{T^n(x_0, y_0)\}$ is a Cauchy sequence. Let $(x, y) \in [(x_0, y_0)]_{\tilde{G}^2}$. By Lemma 2.7, $(T(x, y), T(y, x)) \in [(x, y)]_{\tilde{G}^2}$. By a similar argument as used above, one can show that $\{T^n(x, y)\}$ is also a Cauchy sequence. This, together with Lemma 2.6, implies that the first statement holds. The proof of the second statement is similar. $\square$

We also need the following definition.

DEFINITION 2.10. A mapping $T : X \times X \to X$ is called orbitally G^2-continuous if for each $x, y, x^*, y^* \in X$ and any sequence k_n of positive integers,

(i) $T^{k_n}(x, y) \to x^*$, $T^{k_n}(y, x) \to y^*$ as $n \to \infty$, and

(ii) $\big(\big(T^{k_n}(x, y), T^{k_n}(y, x)\big), \big(T^{k_{n+1}}(x, y), T^{k_{n+1}}(y, x)\big)\big) \in E(G^2)$ for each $n \in \mathbb{N}$ imply that $T(T^{k_n}(x, y), T^{k_n}(y, x)) \to x^*$ and $T(T^{k_n}(y, x), T^{k_n}(x, y)) \to y^*$ as $n \to \infty$.

Now, we are ready to state one of the main results of this section.

THEOREM 2.11. *Let (X, d) be a complete b-metric space and $T : X \times X \to X$ a G-contraction with a constant $0 \le \lambda < 1$. Suppose that T is orbitally G^2-continuous*

and the set $X_T^2 = \left\{(x,y) : \left((x,y), \left(T(x,y), T(y,x)\right)\right) \in E(G^2)\right\}$, is not empty. Then the following statements hold.

(i) For all $(x_0, y_0) \in X_T^2$, $(x,y) \in [(x_0, y_0)]_{\tilde{G}^2}$, the sequence $\left\{\left(T^n(x,y), T^n(y,x)\right)\right\}$ converges to a fixed point of T and the limit does not dependent on (x,y).

(ii) If G^2 is weakly connected, then T is a Picard operator.

(iii) If $X_T^2 = X^2$, then T is a weakly Picard operator.

Proof. Let $(x_0, y_0) \in X_T^2$. By Lemma 2.9, the sequences $\{T^n(x_0, y_0)\}$ and $\{T^n(y_0, x_0)\}$ are Cauchy. Thanks to completeness of X, there are $x^*, y^* \in X$ such that $x^* = \lim_{n \to \infty} T^n(x_0, y_0)$, $\quad y^* = \lim_{n \to \infty} T^n(y_0, x_0)$. By using the fact that T is G^2-edge preserving, one can inductively prove that

$$\left(\left(T^n(x_0, y_0), T^n(y_0, x_0)\right), \left(T^{n+1}(x_0, y_0), T^{n+1}(y_0, x_0)\right)\right) \in E(G^2) \quad (n \in \mathbb{N}). \quad (5)$$

Since T is orbitally G^2-continuous, by (5),

$$T(x^*, y^*) = T\left(\lim_{n \to \infty} T^n(x_0, y_0), \lim_{n \to \infty} T^n(y_0, x_0)\right)$$
$$= \lim_{n \to \infty} T\left(T^n(x_0, y_0), T^n(y_0, x_0)\right) = \lim_{n \to \infty} T^{n+1}(x_0, y_0) = x^*.$$

Similarly, one can show that $T(y^*, x^*) = y^*$. Hence (x^*, y^*) is a fixed point of T. If $(x,y) \in [(x_0, y_0)]_{\tilde{G}^2}$, by Lemma 2.9,

1) $\{T^n(x,y)\}$ and $\{T^n(x_0, y_0)\}$ are Cauchy equivalent, also

2) $\{T^n(y,x)\}$ and $\{T^n(y_0, x_0)\}$ are Cauchy equivalent.

So that $x^* = \lim_{n \to \infty} T^n(x, y)$ and $y^* = \lim_{n \to \infty} T^n(y, x)$. This proves *(i)*.

Statement *(ii)* follows from *(i)*.

If $X_T^2 = X^2$, by the argument used in the beginning of the proof one can easily see that for each $(x, y) \in X^2$, the sequence $\left\{\left(T^n(x,y), T^n(y,x)\right)\right\}$ converges to a fixed point of T. This proves *(iii)*. $\qquad\square$

The following example shows that our results are genuine generalization of some old results.

EXAMPLE 2.12. Let $X = [0,1]$ equipped with usual topology. Define $\precsim$ on X by
$$x \precsim y \Leftrightarrow (x, y \in (0,1) \& \ x \leq y) \text{ or } (x = y = 0) \text{ or } (x = y = 1).$$
Let G_1 be the graph defined in Example 2.4. Consider the function $T : X \times X \to X$ defined by

$$T(x, y) = \begin{cases} \frac{x+y}{4} & \text{if } x, y \in (0,1), \\ 1 & \text{otherwise}. \end{cases}$$

It is easy to verify that T is G_1-contraction. Since

$$|T(1,1) - T(0.9, 0.9)| = 1 - \frac{0.9 + 0.9}{4} = 5.5\left(\frac{|1 - 0.9| + |1 - 0.9|}{2}\right),$$

T is not a contraction. But T is $\tilde{G_1}^2$-orbitally continuous and $(1,1) \in X_T^2$, so that Theorem 2.11 implies that T has a coupled fixed point. However, T is not continuous, since $\lim_{n \to \infty} T(\frac{1}{n}, \frac{1}{n}) = 0 \neq T(0,0) = 1$. Hence Theorem 1.3 cannot be applied.

The next result, which is an extension of Theorem 1.3, follows immediately from Theorem 2.11 by considering the graph G_1 from Example 2.4.

COROLLARY 2.13. *Let (X, d) be a complete b-metric space and $\precsim$ be partial order on X. Suppose that $T : X \times X \to X$ is a continuous mapping with the mixed monotone property and for some $0 \leq \lambda < 1$*

$$d\big(T(x,y), T(u,v)\big) \leq \frac{\lambda}{2}[d(x,u) + d(y,v)] \text{ for all } x \succsim u, y \precsim v.$$

Let $X_T^2 = \{(x,y) : x \succsim T(x,y), \ y \precsim T(y,x)\}$. Then the following statements hold.

(i) If X_T^2 is nonempty, then T has a coupled fixed point.

(ii) If $X_T^2 = X^2$, then T is a weakly Picard operator.

The next result states that when T is not G^2-orbitally continuous, the results of Theorem 2.11 are still valid provided that the underlying b-metric space has some additional properties.

THEOREM 2.14. *Let (X, d) be a complete b-metric space and $T : X \times X \to X$ a G-contraction with a constant $\lambda \in [0, 1)$. Assume that the following properties hold.*

(i) $X_T^2 = \Big\{(x,y) : \Big((x,y), \big(T(x,y), T(y,x)\big)\Big) \in E(G^2)\Big\}$ is not empty.

(ii) If $(x_n, x_{n+1}) \in E(G)$ and $\{x_n\}$ converges to x, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

(iii) If $(x_{n+1}, x_n) \in E(G)$ and $\{x_n\}$ converges to x, then $(x, x_n) \in E(G)$ for all $n \in \mathbb{N}$.

Then the following statements hold.

(1) For all $(x_0, y_0) \in X_T^2$, $(x,y) \in [(x_0, y_0)]_{\tilde{G}^2}$, the sequence $\Big\{\big(T^n(x,y), T^n(y,x)\big)\Big\}$ converges to a fixed point of T and the limit does not dependent on (x,y).

(2) If G^2 is weakly connected, then T is a Picard operator.

(3) If $X_T^2 = X^2$, then T is a weakly Picard operator.

Proof. Let $(x_0, y_0) \in X_T^2$. By the proof of Theorem 2.11, we need only to show that (x^*, y^*) is a coupled fixed point of T, where

$$x^* = \lim_{n \to \infty} T^n(x_0, y_0), \quad y^* = \lim_{n \to \infty} T^n(y_0, x_0). \tag{6}$$

Let $\varepsilon > 0$. By (6), there is some $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$,

$$\max\{d(x^*, T^n(x_0, y_0)), d(y^*, T^n(y_0, x_0))\} < \frac{\varepsilon}{6}.$$

Then for each $n \geq n_0$, we have

$$
\begin{aligned}
d(x^*, T(x^*, y^*)) &\leq 2d(x^*, T^{n+1}(x_0, y_0)) + 2d(T^{n+1}(x_0, y_0), T(x^*, y^*)) \\
&\leq \frac{\varepsilon}{3} + 2d\big(T(T^n(x_0, y_0), T^n(y_0, x_0)), T(x^*, y^*)\big) \\
&\leq \frac{\varepsilon}{3} + \lambda[d(T^n(x_0, y_0), x^*) + d(T^n(y_0, x_0), y^*)] \\
&\leq \frac{\varepsilon}{3} + \lambda[\frac{\varepsilon}{6} + \frac{\varepsilon}{6}] < \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, $x^* = T(x^*, y^*)$. A similar argument shows that $y^* = T(y^*, x^*)$. This completes our proof. $\qquad\square$

The following example shows that conditions *(ii)* and *(iii)* are necessary in Theorem 2.14.

EXAMPLE 2.15. Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$. Define $E(G) = \big\{(x, y) : x, y \in (0, 1) \text{ or } x, y \in \{0, 1\}\big\}$ and $T : X \times X \to X$ by

$$
T(x, y) = \begin{cases} \frac{x+y}{4} & \text{if } x, y \in (0, 1), \\ \frac{1}{2} & \text{otherwise}. \end{cases}
$$

It is easy to verify that T is a G-contraction with constant $\lambda = \frac{1}{2}$ and $(x, y) \in X_T^2$ for each $x, y \in (0, 1)$. However, T has no fixed points. Therefore the conditions *(ii)* and *(iii)* are necessary in Theorem 2.14.

By considering the graph G_1 from Example 2.4, we get to the following result which represents an extension of Theorem 2.2 from [3].

COROLLARY 2.16. *Let (X, d) be a complete b-metric space and $\precsim$ be partial order on X. Suppose that $T : X \times X \to X$ has the following properties.*

(i) T has the mixed monotone property.

(ii) There is $0 \leq \lambda < 1$ such that $d\big(T(x, y), T(u, v)\big) \leq \frac{\lambda}{2}[d(x, u) + d(y, v)]$ whenever $x \succsim u$ and $y \precsim v$.

(iii) If a nondecreasing sequence $\{x_n\}$ converges to x, then $x_n \precsim x$, for all $n \in \mathbb{N}$.

(iv) If a nonincreasing sequence $\{x_n\}$ converges to x, then $x \precsim x_n$, for all $n \in \mathbb{N}$.

Let $X_T^2 = \{(x, y) : x \succsim T(x, y),\ y \precsim T(y, x)\}$. Then the following statements hold.

(1) If X_T^2 is nonempty, then T has a coupled fixed point.

(2) If $X_T^2 = X^2$, then T is a weakly Picard operator.

In order to state the next main result, we need the following lemma.

LEMMA 2.17. *Let $T : X \times X \to X$ be a mapping and $A \subset X^2$ be such that for some $(x^*, y^*) \in X^2$, $T^n(x, y) \to x^*$ and $T^n(y, x) \to y^*$ for each $(x, y) \in A$. If $\{T^n\}$ is equicontinuous on $\bar{A}$, then $T^n(x, y) \to x^*$ and $T^n(y, x) \to y^*$ for each $(x, y) \in \bar{A}$.*

Proof. Let $(x, y) \in \bar{A}$ and $\varepsilon > 0$. Thanks to equicontinuity of $\{T^n\}$, there is some $\delta > 0$ such that for all $n \in \mathbb{N}$, $(x', y') \in \bar{A}$, $(d(x, x') < \delta$ and $d(y, y') < \delta)$ imply $(d(T^n(x, y), T^n(x', y') < \varepsilon/4$ and $d(T^n(y, x), T^n(y', x')) < \varepsilon/4)$. Let $(x', y') \in A$ be such that $d(x, x') < \delta$ and $d(y, y') < \delta$. Take some $n_0 > 0$ such that $d(T^n(x', y'), x^*)$ for all $n > n_0$, then

$$d\big(T^n(x, y), x^*\big) \leq 2\big(d(T^n(x, y), T^n(x', y')) + d\big(T^n(x', y'), x^*\big)$$
$$< 2(\varepsilon/4 + \varepsilon/4) = \varepsilon \quad (n > n_0).$$

Hence $\lim_{n\to\infty} T^n(x, y) = x^*$.

By an obvious change, one can prove that $\lim_{n\to\infty} T^n(y, x) = y^*$. $\qquad \square$

The next result states that under some circumstances equicontinuity of $\{T^n\}$ implies that T is Picard operator on a closed subset of $X \times X$.

THEOREM 2.18. *Let (X, d) be a complete b-metric and $T : X \times X \to X$ be a G-contraction with a constant $\lambda \in [0, 1)$. Suppose that for some $(x, y) \in X$, the sequence $\{T^n\}$ is equicontinuous on $[(x, y)]_{\tilde{G}}$ and $\big(T(x, y), T(y, x)\big) \in [(x, y)]_{\tilde{G}}$. Then $T|_{\overline{[(x,y)]_{\tilde{G}}}}$ is a Picard operator.*

Proof. By Lemma 2.7, $\big\{\big(T^n(x, y), T^n(y, x)\big)\big\} \in [x, y)]_{\tilde{G}^2}$ for each $n \in \mathbb{N}$. According to Lemma 2.9, for each $(u, v) \in [x, y)]_{\tilde{G}^2}$, the sequences $\big\{\big(T^n(x, y), T^n(y, x)\big)\big\}$ and $\big\{\big(T^n(u, v), T^n(v, u)\big)\big\}$ are Cauchy equivalent. By the completeness of $\overline{[(x, y)]}_{\tilde{G}^2}$, there is some $(x^*, y^*) \in \bar{[}x, y)]_{\tilde{G}^2}$ such that

$$\lim_{n\to\infty} T^n(u, v) = x^* \text{ and } \lim_{n\to\infty} T^n(v, u) = y^* \tag{7}$$

for all $(u, v) \in [x, y)]_{\tilde{G}^2}$. Lemma 2.17 guarantees that (7) is true for all $(u, v) \in \overline{[(x, y)]}_{\tilde{G}^2}$. Finally, continuity of T implies that (x^*, y^*) is the unique fixed point of T in $\overline{[(x, y)]}_{\tilde{G}^2}$. $\qquad \square$

In order to state an application of the above result, we need the following definition.

DEFINITION 2.19. *Let (X, d) be a b-metric space and $\varepsilon > 0$. Then X is called ε-chainable if for each $x, y \in X$, there is some $n_0 \in \mathbb{N}$ and $x_0 = x, x_1, \ldots, x_{n_0} \in X$ such that $d(x_{i-1}, x_i) < \varepsilon$ for $i = 1, \ldots, n_0$.*

Edelstein [7] proved that if a complete metric space (X, d) for some $\varepsilon > 0$ is ε-chainable and $T : X \to X$ for some $0 \leq \lambda < 1$ satisfies

$$d(x, y) < \varepsilon \Rightarrow d(Tx, Ty) < \lambda d(x, y),$$

then T has a unique fixed point. In the following result, we extend Edelstein's theorem for two variable mappings $T : X \times X \to X$, where X is a complete b-metric space.

COROLLARY 2.20. *Let (X, d) be a complete and ε-chainable b-metric space for some $\varepsilon > 0$. Let $T : X \times X \to X$ be such that for some $0 \leq \lambda < 1$,*

$$d(x, x') < \varepsilon \text{ and } d(y, y') < \varepsilon \Rightarrow d\big(T(x, y), T(x', y')\big) \leq \frac{\lambda}{2}\big[d(x, x') + d(y, y')\big]. \tag{8}$$

Then T is a Picard operator.

Proof. We inductively prove that for each $n \in \mathbb{N}$,

$$d(x, x') < \varepsilon, \ d(y, y') < \varepsilon \Rightarrow d\big(T^n(x, y), T^n(x', y')\big) \leq \frac{\lambda^n}{2}[d(x, x') + d(y, y')]. \quad (9)$$

For $n = 1$, (9) becomes (8). Assume that for some $n \in \mathbb{N}$ (9) holds. If $d(x, x') < \varepsilon$ and $d(y, y') < \varepsilon$, then $d\big(T^n(x, y), T^n(x', y')\big) \leq \frac{\lambda^n}{2}[d(x, x') + d(y, y')] < \lambda^n \varepsilon < \varepsilon$. Similarly, $d\big(T^n(y, x), T^n(y', x')\big) < \varepsilon$. Therefore we have

$$d\big(T^{n+1}(x, y), T^{n+1}(x', y')\big) = d\left(T\big(T^n(x, y), T^n(y, x)\big), T\big(T^n(x', y'), T^n(y', x')\big)\right)$$

$$\leq \frac{\lambda}{2}\left[d\big(T^n(x, y), T^n(x', y')\big) + d\big(T^n(y, x), T^n(y', x')\big)\right]$$

$$\leq \frac{\lambda}{2}\left[\lambda^n\big(d(x, x') + d(y, y')\big)\right] = \frac{\lambda^{n+1}}{2}[d(x, x') + d(y, y')].$$

Hence (9) holds for all $n \in \mathbb{N}$. Therefore, $\{T^n\}$ is equicontinuous.

Let G be a graph with $V(G) = X$ and $E(G) = \{(x, x') : d(x, x') < \varepsilon\}$. Then ε-chainability of (X, d) implies that G^2 is connected. So that $X^2 = [(x, y)]_{G^2}$ for each $(x, y) \in X^2$. By Theorem 2.18, T is a Picard operator. $\qquad \square$

REFERENCES

[1] M. N. Aslam Khan, A. Azam, L. D. R. Kočinac, *Coincidence of multivalued mappings on metric spaces with a graph*, to appear in Filomat.

[2] S. Banach, *Sur les opérations dans les ensemble abstraits et leur application aux équations intégrales,* Fund. Mat. 3 (1922), 133–181.

[3] T. G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (2006), 1379–1393. 10.1016/j.na.2005.10.017

[4] F. Bojor, *Fixed point of φ-contraction in metric spaces endowed with a graph,* Ann. Univ. Cariova, Math. Comp. Sci. Series, 37(4), (2010) 85–92.

[5] M. F. Bota, A. Petruçel, B. Samet, *Coupled fixed point theorems for single-valued operators in b-metric spaces,* Fixed Point Theory Appl. (2015), 2015:231.

[6] S. Czerwik, *Contraction mappings in b-metric spaces,* Acta Math. Inform. Univ. Ostraviensis 1, (1995), 5–11.

[7] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. 12, (1961), 7–10.

[8] A. H. Frink, *Generalization of the G_δ-property of complete metric spaces,* Bull. Amer. Math. Soc. 43, (1937), 133–142.

[9] G. Gruenhage, *Generalized metric spaces*, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, (1984), 423–501.

[10] D. Guo, V. Lakshmikantham, *Coupled fixed points of nonlinear operations with applications,* Nonlinear Anal. **11**, 52, (1987), 525–562.

[11] H. Huang, S. Hu, B. Z. Popović, S. Radenović, *Common fixed point theorems for four mappings on cone b-metric space over Banach algebras*, J. Nonlinear Sci. Appl. **9** (2016), 3655–3671.

[12] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Am. Math. Soc. 1(136), (2008) 1359–1373.

[**13**] R. Johnsonbaugh, *Discrete Mathematics,* Prentice-Hall, Inc., New Jersey, 1997.

[**14**] M. Jovanović, Z. Kadelburg, S. Radenović, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl. (2010), Article ID 978121, 15 pages.

[**15**] Z. Kadelburg, A. Nastasi, S. Radenović, P. Verto, *Remarks on the paper "Fixed point theorems for cyclic contractions in C^*-algebra-valued b-metric spaces",* Adv. Oper. Theory **1**(3), (2016), 93–104.

[**16**] Z. Kadelburg, S. Radenović, M. Rajović, *A note on fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces,* Krag. J. Math. **39**(2), (2015), 187–195.

[**17**] A. K. Mirmostafaee, *Fixed point theorems for set-valued mappings in b-metric spaces,* to apperar in Fixed Point Theory.

[**18**] J. J. Nieto, R. L. Pouso, R. Rodríguez-López, *Fixed point theorems in ordered abstract spaces,* Proc. Amer. Math. Soc. 135 (2007), 2505–2517.

[**19**] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,* Order 22 (2005), 223–239.

[**20**] J. J. Nieto, R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations,* Acta Math. Sinica, English Ser. (2007), 2205–2212.

[**21**] A. Petruşel, I. A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc. 134 (2006), 411–418.

[**22**] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Am. Math. Soc. 132, (2004) 1435-1443.

[**23**] Vl. Opoitsev, *Hetrogenic and combined-concave operations,* Sib. Math. J. **16**, (1975), 781–792.

[**24**] Vl. Opoitsev, *Dynamics of collective behavior III*, Avtom Telemech **36**, (1975), 124–138.

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