МАТЕМАТИЧКИ ВЕСНИК
66, 4 (2014), 343–350
December 2014

оригинални научни рад
research paper

β-GREEDOIDS

Talal Al-Hawary

Abstract. In this paper, we introduce the notion of β-greedoids and discuss four basic constructions of β-greedoids namely, deletion, contraction, direct sum and ordered sum. We show that the operations of deletion and contraction commute and the direct sum and ordered sum of β-greedoids G_1 and G_2 are interval β-greedoids if and only if G_1 and G_2 are both interval β-greedoids. We also give a necessary and sufficient condition for the direct sum and ordered sum of balanced β-greedoids to be balanced.

1. Background

We begin with some background material, which follows the terminology and notation in [7]. A greedoid G is a pair $(E, \mathfrak{F})$ where E is a nonempty set and $\mathfrak{F} \subseteq 2^E$ is a set system satisfying the following conditions.

(**G1**) For every non-empty $X \in \mathfrak{F}$, there is an $x \in X$ such that $X - x \in \mathfrak{F}$.

(**G2**) For $X, Y \in \mathfrak{F}$ such that the cardinality $|X|$ of X is greater than the cardinality $|Y|$ of Y, there is an $x \in X - Y$ such that $Y \cup x \in \mathfrak{F}$.

Thus every matroid is a greedoid and a greedoid is a matroid if and only if the axiom

(**M1**) If $X \in \mathfrak{F}$ and $Y \subseteq X$, then $Y \in \mathfrak{F}$

is satisfied. For an introduction on matroids the reader is referred to [7] and [8]. Observe that axioms **M1** and **G2** together define a matroid and axiom **G1** and

(**G2'**) For $X, Y \in \mathfrak{F}$ such that $|X| = |Y| + 1$, there is an $x \in X - Y$ such that $Y \cup x \in \mathfrak{F}$

define a greedoid. The set E is called the ground set of G and the sets in $\mathfrak{F}$ are called feasible. For $A \subseteq E$, the rank of A is $r(A) = \max\{|X| : X \subseteq A,\ X \in \mathfrak{F}\}$. We remark that several structural properties of greedoids related to the rank function was discussed in [5] and we point out that the usual (feasible) rank function for a greedoid is not monotone. Thus A is feasible if and only if $r(A) = |A|$ and it is called a basis if $r(A) = |A| = r(G)$. The collection of all basis of G is

2010 Mathematics Subject Classification: 52C99
Keywords and phrases: β-greedoid; basis; feasible; flat; density; direct sum; ordered sum.

denoted by $\mathcal{B}(G)$. Axiom **G2** implies that bases elements have the same size r (or $r(G)$). Unfortunately, the rank function of a general greedoid need not satisfy the semimodularity:

$$r(A \cap B) + r(A \cup B) \leq r(A) + r(B) \text{ for every } A, B \subseteq E.$$

On the other hand, the basis rank of A is defined to be $\beta(A) = \max\{|X \cap A| : X \in \mathfrak{F}\}$, which is the maximal size of the intersection of A with a basis, does satisfy the semimodularity:

$$\beta(A \cap B) + \beta(A \cup B) \leq \beta(A) + \beta(B) \text{ for every } A, B \subseteq E.$$

Moreover, $\beta(A) \geq r(A)$. For $A \subseteq E$, define

$$\mathfrak{F} \setminus A := \{X \subseteq E - A \,:\, X \in \mathfrak{F}\},$$

and, if A is feasible, define

$$\mathfrak{F}/A := \{X \subseteq E - A \,:\, X \cup A \in \mathfrak{F}\}.$$

Then it is easy to see that the set systems obtained in both cases are greedoids on the ground set $E - A$. The greedoid $G \setminus A = (E - A, \mathfrak{F} \setminus A)$ is called G delete A or the restriction of G to $E - A$ and $G/A = (E - A, \mathfrak{F}/A)$ is called G contract A. For all $X \subseteq E - A$, it is easy to see that

$$\beta_{G \setminus A}(X) = \beta(X) \text{ and } \beta_{G/A}(X) = \beta(X \cup A) - \beta(A).$$

A greedoid $G = (E, \mathfrak{F})$ is called an interval greedoid if it satisfies the interval property: if $A \subseteq B \subseteq C$, $A, B, C \in \mathfrak{F}$, $x \in E - C$, $A \cup x \in \mathfrak{F}$, and $C \cup x \in \mathfrak{F}$, imply that $B \cup x \in \mathfrak{F}$. Thus, matroids are interval greedoids.

Operations as basic as deletion and contraction are those of direct sum and ordered sum. Let $G_1 = (E_1, \mathfrak{F}_1)$ and $G_2 = (E_2, \mathfrak{F}_2)$ be two greedoids on disjoint ground sets. Then their direct sum is the greedoid $G_1 \oplus G_2 = (E_1 \cup E_2, \mathfrak{F}_1 \oplus \mathfrak{F}_2)$, where

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \{X_1 \cup X_2 \,:\, X_1 \in \mathfrak{F}_1 \text{ and } X_2 \in \mathfrak{F}_2\},$$

and the ordered sum of G_1 and G_2 is the greedoid $G_1 \otimes G_2 = (E_1 \cup E_2, \mathfrak{F}_1 \otimes \mathfrak{F}_2)$, where

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \mathfrak{F}_1 \cup \{B \cup X \,:\, B \in \mathcal{B}(G_1),\, X \in \mathfrak{F}_2\}.$$

Observe that $\varnothing \in \mathfrak{F}_1 \cap \mathfrak{F}_2$ and $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \subseteq \mathfrak{F}_1 \oplus \mathfrak{F}_2$, thus $G_1 \otimes G_2$ is a subgreedoid of $G_1 \oplus G_2$.

Remark 1. Although we will only consider greedoids on disjoint ground sets when talking about the operations of direct sum and ordered sum, these operations can easily be defined on the disjoint union of the ground sets of any greedoids.

The density of a loopless greedoid (i.e., has no elements of rank zero) $G = (E, \mathfrak{F})$ is given by $d(G) := \frac{|G|}{\beta(G)}$. A greedoid G is balanced if

$$d(K) \leq d(G) \text{ for all non-empty subgreedoids } K \text{ of } G.$$

Greedoids were invented in 1981 by Korte and Lovász [5]. Originally, the main motivation for proposing this generalization of the matroid concept came from combinatorial optimization. Korte and Lovász had observed that the optimality of a "greedy" algorithm could in several instances be traced back to an underlying combinatorial structure that was not a matroid—but (as they named it) a greedoid. In 1991, Korte, Lovász and Schrader [4] introduced greedoid as a special kind of antimatroids. In 1992, Björner and Ziegler [7] explained the basic ideas and gave a few glimpses of more specialized topics related to greedoids. Also, they studied the rank function and some connections to other rank functions (a closure based function and a kernel rank function). In 1992, Broesma and Li [2] extended the "connectivity" concept from matroids to greedoids and in 1997, Gordon [3] extended Crapo's β invariant from matroids to greedoids.

In this paper, we introduce the notion of β-greedoids and discuss four basic constructions of β-greedoids namely; deletion, contraction, direct sum and ordered sum. We define the notion of strong maps and the kernel for strong maps and then show that kernel notion coincides with the categorical notion of kernel. We show the operations of deletion and contraction commute and the direct sum and ordered sum of β-greedoids G_1 and G_2 are interval β-greedoids if and only if G_1 and G_2 are both interval β-greedoids. We also give a necessary and sufficient condition for the direct sum and ordered sum of balanced β-greedoids to be balanced. We extend the density concept from matroids and graphs to greedoids. Finally, we study some greedoid preserving operations.

2. β-greedoids

We begin this section by defining the closure of a given set in a greedoid and use that to define flats and our main notion of greedoid which we call β-greedoid.

Definition 1. Let $G = (E, \mathfrak{F})$ be a greedoid and $A \subseteq E$. The closure of A in G is defined to be $\overline{A} = \{x \in E : \beta(A \cup x) = \beta(A)\}$ and A is called a β-flat of G if $\beta(A \cup x) = \beta(A) + 1$ for every $x \in E \setminus A$.

Next, we show that the closure operation is monotone.

Lemma 2. *Let $G = (E, \mathfrak{F})$ be a greedoid and $A, B \subseteq E$ such that $A \subseteq B$. Then $\overline{A} \subseteq \overline{B}$.*

Proof. Let $A \subseteq B$ and $x \in \overline{A} - A$. Then $\beta(A \cup x) = \beta(A)$. Thus if B_A is a basis for A, then B_A is a basis for $A \cup x$. Now $B \cup x$ has a basis $B_{B\cup x}$ that contains B_A, but not x. Since $B_{B\cup x}$ must be a basis of B, it follows that $\beta(B \cup x) = |B_{B\cup x}| = \beta(B)$. Therefore $x \in \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$. ■

Definition 3. A greedoid $G = (E, \mathfrak{F})$ is a β-greedoid if $\overline{A} \cup \overline{B} = \overline{A \cup B}$ for all subsets A and B of E.

We first show that a β-greedoid can be defined in terms of flats. Then we use this definition to characterize all β-greedoids. For terminology and notation not explained here we refer the reader to [7] or [8].

Lemma 4. *A greedoid G is a β-greedoid if and only if unions of flats of G are again flats of G.*

Proof. The forward implication is obvious, we show the reverse. We first note that by Lemma 2, for all $A, B \subseteq E$, we have $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$; thus, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Hence, we need to show the reverse inclusion holds whenever the union of flats is a flat of G. So, assume $\overline{A} \cup \overline{B}$ is a flat of G. Now, $A \cup B \subseteq \overline{A} \cup \overline{B}$ implies $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$ by Lemma 2. By assumption, $\overline{\overline{A} \cup \overline{B}} \subseteq \overline{A} \cup \overline{B}$ and hence $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. ■

As for a full greedoid, $\beta(A) = |A|$, full greedoids are β-greedoids.

Definition 5. Let G be a greedoid. If x and y are not loops of G, then x and y are parallel elements if $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. By $\tilde{G}$ we mean the greedoid G after deleting all loops and identifying parallel elements.

One can easily check that adding or deleting loops and parallel elements preserves the β property, namely;

Lemma 6. *A greedoid G is a β-greedoid if and only if $\widetilde{G}$ is a β-greedoid.*

Next, we define modular greedoids and compare them with β-greedoids.

Definition 7. A greedoid G is modular if every flat A in G is modular, that is if for every other flat B, $\beta(A) + \beta(S - A) = \beta(G)$.

Lemma 8. *For a greedoid $G = (E, \mathfrak{F})$, the following are equivalent:*

(1) *G is modular.*

(2) *For all flats A and B such that $A \cap B = \overline{\varnothing}$, we have $\beta(A \cup B) = \beta(A) + \beta(B)$.*

(3) *For every flat A and every subset B such that $A \cap B = \overline{\varnothing}$ and $\overline{A \cup B} = E$, we have $\beta(A \cup B) = \beta(A) + \beta(B)$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. We shall show (3) $\Rightarrow$ (2) and omit the similar proof that (2) $\Rightarrow$ (1). Assume that (3) holds, but (2) does not. Let A and B be flats for which $A \cap B = \overline{\varnothing}$ and $\beta(A \cup B) \neq \beta(A) + \beta(B)$. By (3), $A \cap B \neq \overline{\varnothing}$ or $\overline{A \cup B} \neq E$ and so $\beta(X) + \beta(Y) < \beta(G)$. Let Z be a basis for $G/A \cup B$. We show $\overline{B \cup Z} \cap A = \overline{\varnothing}$ and $\overline{\overline{B \cup Z} \cup A} = E$. Clearly $\overline{\overline{B \cup Z} \cup A} = E$. Now suppose that x is a non-loop element of $\overline{B \cup Z} \cap A$. Then $x \notin B$ since $A \cap B = \overline{\varnothing}$. Thus if W is a basis of B, then $W \cup x$ is feasible in $G|_{A \cup B}$. But $Z \cup W \cup x$ is feasible in G, a contradiction to the fact that $x \in \overline{B \cup Z}$. Hence $\overline{B \cup Z} \cap A = \overline{\varnothing}$. ■

Theorem 9. *A β-greedoid is a modular greedoid.*

Proof. Let G be a β-greedoid, and let A and B be flats in G such that $A \cap B = \overline{\varnothing}$. By the semimodular property, we have $\beta(A \cup B) \leq \beta(A) + \beta(B)$.Without loss of generality, we may assume that $\beta(A) = n$ and $\beta(B) = m$ where $m \geq n$. Then there exist flats $A_1, A_2, \ldots, A_{n-1}$ and $B_1, B_2, \ldots, B_{m-1}$ such that

$$\overline{\varnothing} \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A \text{ and } \overline{\varnothing} \subset B_1 \subset B_2 \subset \cdots \subset B_{m-1} \subset B.$$

Moreover, these are maximal chains of flats in A and B, respectively. By Lemma 4, $A\cup B$,$A_i\cup B_i$ and $A\cup B_j$ are flats in G, for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, m-1$. Hence,

$$\begin{aligned}\overline{\varnothing} &\subset A_1 \subset A_1 \cup B_1 \subset A_2 \cup B_1 \subset \cdots \subset A_{n-1} \cup B_1 \\ &\subset A \cup B_1 \subset A \cup B_2 \cdots \subset A \cup B_{m-1} \subset A \cup B\end{aligned}$$

is a chain of flats in $A\cup B$ of size $n+m$. Therefore, $\beta(A\cup B) \geq n+m = \beta(A)+\beta(B)$. Thus by Lemma 8, G is modular. ■

3. Deletion and contraction greedoids

In this section, we study properties of greedoids deletion and contraction operations and show that these operations commute. We start by proving the following.

Proposition 10. *If B_A is a basis for the restriction $G|A$ of G to A, then*

$$\begin{aligned}\mathfrak{F}(G/A) &= \{X \subseteq E - A \,:\, G|A \textit{ has a basis } B \textit{ such that } X \cup B \in \mathfrak{F}(G)\} \\ &= \{X \subseteq E - A \,:\, X \cup B_A \in \mathfrak{F}(G)\}.\end{aligned}$$

Proof. Clearly $\{X \subseteq E - A \,:\, G|A$ has a basis B such that $X \cup B \in \mathfrak{F}(G)\}$ contains the set $\{X \subseteq E - A \,:\, X \cup B_A \in \mathfrak{F}(G)\}$. Suppose $X \cup B \in \mathfrak{F}(G)$ for some basis B of $G|A$. We shall show that $X \in \mathfrak{F}(G/A)$. Clearly $X \cup B$ is a basis of $X \cup A$, so $\beta(X \cup B) = \beta(X \cup A)$. Therefore,

$$\beta_{G/A}(X) = \beta(X \cup A) - \beta(B) = \beta(X \cup B) - \beta(B) = |X \cup B| - |B| = |X|,$$

that is, $X \in \mathfrak{F}(G/A)$. Hence,

$$\{X \subseteq E - A \,:\, G|A \text{ has a basis } B \text{ such that } X \cup B \in \mathfrak{F}(G)\} \subseteq \mathfrak{F}(G/A).$$

Finally we show $\{X \subseteq E - A \,:\, X \cup B_A \in \mathfrak{F}(G)\}$ contains $\mathfrak{F}(G/A)$. If $X \in \mathfrak{F}(G/A)$, then

$$\begin{aligned}|X| = \beta_{G/A}(X) &= \beta(X \cup A) - \beta(A) \\ &= \beta(X \cup B_A) - |B_A|.\end{aligned}$$

Hence $|X \cup B_A| = \beta(X \cup B_A)$, so $X \cup B_A \in \mathfrak{F}(G)$. ■

Corollary 11. *If B_A is a basis for $G|A$, then the bases of G/A are*

$$\begin{aligned}\mathcal{B}(G/A) &= \{B \subseteq E - A \,:\, G|A \textit{ has a basis } \acute{B} \textit{ such that } B \cup \acute{B} \in \mathcal{B}(G)\} \\ &= \{B \subseteq E - A \,:\, B \cup B_A \in \mathcal{B}(G)\}.\end{aligned}$$

Observe that $\mathcal{B}(G\backslash A)$ is the set of maximal members of $\{B-A \,:\, B \in \mathcal{B}(G)\}$ and $\mathfrak{F}(G/A) \subseteq \mathfrak{F}(G \setminus A)$ for every feasible set A in G. Next, we give a necessary and sufficient condition for the contraction of a feasible set to be the same as the deletion of that set.

Proposition 12. *If A is a feasible set in G, then*

$$G/A = G \setminus A \text{ if and only if } \beta(G \setminus A) = \beta(G) - \beta(A).$$

Proof. Suppose $G/A = G \setminus A$ and let B be a basis of $G \setminus A$. Then B is a basis of G/A and hence by Corollary 11, $B \cup B_A$ is a basis of G for some basis B_A of $G|A$. Thus

$$\begin{aligned}\beta(G) &= |B \cup B_A| = |B| + |B_A| \\ &= \beta(A) + \beta(G \setminus A).\end{aligned}$$

Suppose $\beta(G \setminus A) = \beta(G) - \beta(A)$. Since $\mathfrak{F}(G/A) \subseteq \mathfrak{F}(G \setminus A)$, to show $G/A = G \setminus A$, we need only show $\mathfrak{F}(G \setminus A) \subseteq \mathfrak{F}(G/A)$. But if $X \in \mathfrak{F}(G \setminus A)$, then X is a subset of a basis B of $G \setminus A$ and B is contained in a basis $B \cup \acute{B}$ of G. Evidently

$$\begin{aligned}\beta(G) &= |\acute{B} \cup B| = |B| + |\acute{B}| \\ &= \beta(G \setminus A) + |\acute{B}|.\end{aligned}$$

Since $\beta(G \setminus A) = \beta(G) - \beta(A)$, we have $\beta(A) = |\acute{B}|$, that is, $\acute{B}$ is a basis of $G|A$. Hence $B \in \mathcal{B}(G/A)$, so $X \in \mathfrak{F}(G/A)$ and $G/A = G \setminus A$. ∎

Corollary 13. *For all $A \in \mathfrak{F}$, $G/A = G \setminus A$ if and only if $\beta(G \setminus A) \leq \beta(G/A)$.*

Proof. If $G/A = G \setminus A$, then clearly $\beta(G \setminus A) \leq \beta(G/A)$. If $\beta(G \setminus A) \leq \beta(G/A)$, then as $\mathfrak{F}(G/A)$ is a subset of $\mathfrak{F}(G \setminus A)$ we must have $\beta(G \setminus A) \geq \beta(G/A)$. Thus $G/A = G \setminus A$. ∎

In the next proposition, we show that the operations of deletion and contraction commute.

Proposition 14. *Let $G = (E, \mathfrak{F})$ be a greedoid. Then $(G \setminus \acute{A})/A = (G/A) \setminus \acute{A} = (\{X \subseteq E - (\acute{A} \cup A) : X \cup A \in \mathfrak{F}\}, \mathfrak{F}_{(G \setminus \acute{A})/A})$.*

Proof. We need only to show $(G \setminus \acute{A})/A$ and $(G/A) \setminus \acute{A}$ have the same collections of feasible sets. If $X \in \mathfrak{F}_{(G \setminus \acute{A})/A}$, then $X \subseteq (E - \acute{A}) - A$ and $X \cup A \in \mathfrak{F}$. That is, $X \subseteq (E - A) - \acute{A}$ and $X \in \mathfrak{F}_{G/A}$ and hence $X \in \mathfrak{F}_{(G/A) \setminus \acute{A}}$. Conversely, if $X \in \mathfrak{F}_{(G/A) \setminus \acute{A}}$, then $X \subseteq (E - A) - \acute{A}$ and $X \in \mathfrak{F}_{G/A}$. That is, $X \subseteq (E - \acute{A}) - A$ and $X \cup A \in \mathfrak{F}$ and hence $X \in \mathfrak{F}_{(G \setminus \acute{A})/A}$. Therefore, $\mathfrak{F}_{(G \setminus \acute{A})/A} = \mathfrak{F}_{(G/A) \setminus \acute{A}}$. ∎

The straightforward proof of the following proposition is omitted.

Proposition 15. $\{B_1 \cup B_2 : B_1 \in \mathcal{B}(G_1)$ *and* $B_2 \in \mathcal{B}(G_2)\} = \mathcal{B}(G_1 \otimes G_2)$ *which is equal to* $\mathcal{B}(G_1 \oplus G_2)$.

Corollary 16. *Let $G_1 = (E_1, \mathfrak{F}_1)$ and $G_2 = (E_2, \mathfrak{F}_2)$ be greedoids on disjoint ground sets. If $X \subseteq E_1 \cup E_2$, then*

$$\beta_{G_1 \otimes G_2}(X) = \beta_{G_1 \oplus G_2}(X) = \beta_{G_1}(X \cap E_1) + \beta_{G_2}(X \cap E_2).$$

4. On greedoids preserving operations

In this section, we prove the operations of direct sum and ordered sum take interval greedoids (resp., β-greedoids) to interval greedoids (resp., β-greedoids). In fact, we show that the direct sum and ordered sum of greedoids G_1 and G_2 is an interval greedoid (resp., β-greedoids) if and only if G_1 and G_2 are both interval greedoids (resp., β-greedoids). We also give a condition for the direct sum and ordered sum of balanced greedoids to be balanced.

For $i = 1, 2$, we recall (see [7]), for all $X \subseteq E_1 \cup E_2$, $\overline{X}^{G_1 \oplus G_2} = \overline{X \cap E_1}^{G_1} \cup \overline{X \cap E_2}^{G_2}$.

Theorem 17. *Let G_1 and G_2 be greedoids on disjoint ground sets. Then $G_1 \oplus G_2$ ($G_1 \otimes G_2$) is a β-greedoid if and only if G_1 and G_2 are β-greedoids.*

Theorem 18. *Let $G_1 = (E_1, \mathfrak{F}_1)$ and $G_2 = (E_2, \mathfrak{F}_2)$ be greedoids on disjoint ground sets. Then G_1 and G_2 are interval β-greedoids if and only if $G_1 \oplus G_2$ is an interval β-greedoid.*

Proof. Suppose G_1 and G_2 are interval greedoids. If $A \subseteq B \subseteq C$, $A, B, C \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, $x \in E_1 \cup E_2 - C$, $A \cup x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, and $C \cup x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, then $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $C = C_1 \cup C_2$ where A_i, B_i, C_i are feasible sets in G_i for $i = 1, 2$, $A_i \cup x \in \mathfrak{F}_i$ (as $A_i \cup x = (A_1 \cup A \cup x) \cap E_i$). Similarly, $C_i \cup x \in \mathfrak{F}_i$. Moreover, $x \in (E_1 \cup E_2 - C_1) \cap (E_1 \cup E_2 - C_2)$. Hence suppose $x \in E_i - C_i$ for $i = 1$ or $i = 2$ and as $A_i \subseteq B_i \subseteq C_i$, $B_i \cup x \in \mathfrak{F}_i$. But

$$B \cup x = B_1 \cup B_2 \cup x = (B_1 \cup x) \cup B_2 \in \mathfrak{F}_1 \oplus \mathfrak{F}_2.$$

Therefore, $G_1 \oplus G_2$ is an interval greedoid.

Suppose $G_1 \oplus G_2$ is an interval greedoid. If $A \subseteq B \subseteq C$, $A, B, C \in \mathfrak{F}_1$, x an element in $E_1 - C$, $A \cup x \in \mathfrak{F}_1$, and $C \cup x \in \mathfrak{F}_1$, then as $\varnothing \in \mathfrak{F}_2$, $A \cup \varnothing \subseteq B \cup \varnothing \subseteq C \cup \varnothing$, $A \cup \varnothing, B \cup \varnothing, C \cup \varnothing \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, $x \in E_1 \cup E_2 - C$, $(A \cup x) \cup \varnothing, (B \cup x) \cup \varnothing \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$ and as $G_1 \oplus G_2$ is an interval greedoid, $B \cup x = (B \cup \varnothing) \cup x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$. But $B \cup x = (B \cup x) \cap E_1 \in \mathfrak{F}_1$ and hence G_1 is an interval greedoid. Similarly, G_2 is an interval greedoid. Now by Theorem 17, the result follows. ∎

Theorem 19. *Let $G_1 = (E_1, \mathfrak{F}_1)$ and $G_2 = (E_2, \mathfrak{F}_2)$ be greedoids on disjoint ground sets. Then G_1 and G_2 are interval β-greedoids if and only if $G_1 \otimes G_2$ is an interval β-greedoid.*

Proof. The proof is similar to that of the direct sum one in the preceding theorem and thus omitted. ∎

We end this section by giving a necessary and sufficient condition for the direct sum and ordered sum of balanced loopless greedoids to be balanced.

Theorem 20. *The direct sum (respectively, the ordered sum) of balanced loopless greedoids G_1 and G_2, on disjoint ground sets, is balanced if and only if*

$$d(G_1) = d(G_2) = d(G_1 \oplus G_2) \text{ (respectively, } d(G_1) = d(G_2) = d(G_1 \otimes G_2)).$$

Proof. We only prove the direct sum part since the order sum one is similar. Let $G_1 = (E_1, \mathfrak{F}_1)$ and $G_2 = (E_2, \mathfrak{F}_2)$ be balanced greedoids on disjoint ground sets. Suppose that $G_1 \oplus G_2$ is balanced. Evidently $d(G_i) \leq d(G_1 \oplus G_2)$ for $i = 1, 2$ and thus

$$|E_1|\beta(G_1) + |E_1|\beta(G_2) \leq |E_1|\beta(G_1) + |E_2|\beta(G_2) \text{ and}$$
$$|E_2|\beta(G_1) + |E_2|\beta(G_2) \leq |E_1|\beta(G_2) + |E_2|\beta(G_2).$$

Hence $|E_1|\beta(G_2) \leq |E_2|\beta(G_1) \leq |E_1|\beta(G_2)$ which implies $|E_2|\beta(G_1) = |E_1|\beta(G_2)$ or

$$d(G_2) = \frac{|E_2|}{\beta(G_2)} = \frac{|E_1|}{\beta(G_1)} = d(G_1) = d(G_1 \oplus G_2).$$

Conversely, suppose that $d(G_1) = d(G_2) = d(G_1 \oplus G_2)$. If N is a subgreedoid of $G_1 \oplus G_2$, then $N = N_1 \oplus N_2$ where each $N_i = N \cap E_i$. Thus $d(N_i) = \frac{|E(N_i)|}{\beta(N_i)} \leq \frac{|E_1|}{\beta(G_1)}$ and hence

$$|E(N_1)|\beta(G_1) + |E(N_2)|\beta(G_1) \leq |E_1|\beta(N_1) + |E_1|\beta(N_2),$$

and

$$d(N) = \frac{|E(N_1)| + |E(N_2)|}{\beta(N_1) + \beta(N_2)} \leq \frac{|E_1|}{\beta(G_1)} = d(G_1 \oplus G_2).$$

Therefore, $G_1 \oplus G_2$ is balanced. ■

REFERENCES

[**1**] A. Björner, M. Las Verganas, B. Sturmfields, W. White, G. Ziegler, *Oriented Matroids*, Encyclopedia of Mathematics and Its Applications, Vol. 46, Cambridge, Cambridge University Press, 1993.

[**2**] H. Broersma, X. Li, *The connectivity of basis graph of a branching greedoid*, J. Graph Theory **16** (1992), 233–237.

[**3**] G. Gordon, *A β invariant for greedoids and antigreedoids*, Elect. J. Comb. **4** (1997), 13 pages.

[**4**] B. Korte, L. Lovász, R. Schrader, *Greedoids*, Berlin, Springer-Verlang, 1991.

[**5**] B. Korte, L. Lovász, *Mathematical structures underlying greedy algorithms*, in: Fundamentals of Computation Theory (Szeged, 1981), Lecture Notes in Computer Science 117, New York, Springer-Verlang, pp. 205–209.

[**6**] B. Korte, L. Lovász, *Structural properties of greedoids*, Combinatorica **4** (1983), 359–374.

[**7**] N. White N., ed., *Matroid Applications*: A. Björner and G. Ziegler, *Introduction to greedoids*, Cambridge, Cambridge University Press, 1992.

[**8**] N. White, *Theory of Matroids*, Cambridge University Press, Cambridge, 1986.

(received 07.02.2013; in revised form 11.08.2013; available online 10.09.2013)

Department of Mathematics, Yarmouk University, Irbid-Jordan

E-mail: talalhawary@yahoo.com