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FIXED POINT THEOREMS ON S-METRIC SPACES

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Abstract. In this paper, we prove a general fixed point theorem in S-metric spaces which is a generalization of Theorem 3.1 from [S. Sedghi, N. Shobe, A. Aliouche, Mat. Vesnik 64 (2012), 258–266]. As applications, we get many analogues of fixed point theorems from metric spaces to S-metric spaces.

1. Introduction and preliminaries

In [13], S. Sedghi, N. Shobe and A. Aliouche have introduced the notion of an S-metric space as follows.

Definition 1.1. [13, Definition 2.1] Let X be a nonempty set. An *S-metric* on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

(S1) $S(x, y, z) = 0$ if and only if $x = y = z$.

(S2) $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an *S-metric space*.

This notion is a generalization of a G-metric space [11] and a D^*-metric space [14]. For the fixed point problem in generalized metric spaces, many results have been proved, see [1, 7, 9, 10], for example. In [13], the authors proved some properties of S-metric spaces. Also, they proved some fixed point theorems for a self-map on an S-metric space.

In this paper, we prove a general fixed point theorem in S-metric spaces which is a generalization of [13, Theorem 3.1]. As applications, we get many analogues of fixed point theorems in metric spaces for S-metric spaces.

Now we recall some notions and lemmas which will be useful later.

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Definition 1.2. [2] Let X be a nonempty set. A *B-metric* on X is a function $d : X^2 \to [0, \infty)$ if there exists a real number $b \geq 1$ such that the following conditions hold for all $x, y, z \in X$.

(B1) $d(x, y) = 0$ if and only if $x = y$.

(B2) $d(x, y) = d(y, x)$.

(B3) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

The pair (X, d) is called a *B-metric space*.

Definition 1.3. [13] Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, we define the *open ball* $B_S(x, r)$ and the *closed ball* $B_S[x, r]$ with center x and radius r as follows

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$
$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

The *topology induced by the S-metric* is the topology generated by the base of all open balls in X.

Definition 1.4. [13] Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\} \subset X$ *converges* to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.

(2) A sequence $\{x_n\} \subset X$ is a *Cauchy sequence* if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

(3) The S-metric space (X, S) is *complete* if every Cauchy sequence is a convergent sequence.

Lemma 1.5. [13, Lemma 2.5] *In an S-metric space, we have*

$$S(x, x, y) = S(y, y, x)$$

for all $x, y \in X$.

Lemma 1.6. [13, Lemma 2.12] *Let* (X, S) *be an S-metric space. If* $x_n \to x$ *and* $y_n \to y$ *then* $S(x_n, x_n, y_n) \to S(x, x, y)$.

As a special case of [13, Examples in page 260] we have the following

Example 1.7. Let $\mathbb{R}$ be the real line. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the *usual S-metric* on $\mathbb{R}$.

2. Main results

First, we prove some properties of S-metric spaces.

Proposition 2.1. *Let* (X, S) *be an S-metric space and let*

$$d(x, y) = S(x, x, y)$$

for all $x, y \in X$. *Then we have*

(1) *d is a B-metric on X;*

(2) *$x_n \to x$ in (X, S) if and only if $x_n \to x$ in (X, d);*

(3) *$\{x_n\}$ is a Cauchy sequence in (X, S) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d).*

Proof. For the statement (1), conditions (B1) and (B2) are easy to check. It follows from (S2) and Lemma 1.5 that

$$\begin{aligned} d(x,z) &= S(x,x,z) \le S(x,x,y) + S(x,x,y) + S(z,z,y) \\ &= 2S(x,x,y) + S(y,y,z) = 2d(x,y) + d(y,z) \\ d(x,z) &= S(z,z,x) \le S(z,z,y) + S(z,z,y) + S(x,x,y) \\ &= 2S(z,z,y) + S(x,x,y) = 2d(y,z) + d(x,y). \end{aligned}$$

It follows that $d(x,z) \le 3/2[d(x,y) + d(y,z)]$. Then d is a B-metric with $b = 3/2$.

Statements (2) and (3) are easy to check. ∎

The following property is trivial and we omit the proof.

Proposition 2.2. *Let (X, S) be an S-metric space. Then we have*

(1) *X is first-countable;*

(2) *X is regular.*

Remark 2.3. By Propositions 2.1 and 2.2 we have that every S-metric space is topologically equivalent to a B-metric space.

Corollary 2.4. *Let $f : X \to Y$ be a map from an S-metric space X to an S-metric space Y. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

Now, we introduce an implicit relation to investigate some fixed point theorems on S-metric spaces. Let $\mathcal{M}$ be the family of all continuous functions of five variables $M : \mathbb{R}_+^5 \to \mathbb{R}_+$. For some $k \in [0,1)$, we consider the following conditions.

(C1) For all $x, y, z \in \mathbb{R}_+$, if $y \le M(x, x, 0, z, y)$ with $z \le 2x + y$, then $y \le k\,x$.

(C2) For all $y \in \mathbb{R}_+$, if $y \le M(y, 0, y, y, 0)$, then $y = 0$.

(C3) If $x_i \le y_i + z_i$ for all $x_i, y_i, z_i \in \mathbb{R}_+$, $i \le 5$, then

$$M(x_1, \dots, x_5) \le M(y_1, \dots, y_5) + M(z_1, \dots, z_5).$$

Moreover, for all $y \in X$, $M(0, 0, 0, y, 2y) \le k\,y$.

Remark 2.5. Note that the coefficient k in conditions (C1) and (C3) may be different, for example, k_1 and k_3 respectively. But we may assume that they are equal by putting $k = \max\{k_1, k_3\}$.

A general fixed point theorem for S-metric spaces is as follows.

Theorem 2.6. *Let T be a self-map on a complete S-metric space (X, S) and*

$$S(Tx, Tx, Ty) \le M(S(x,x,y), S(Tx,Tx,x), S(Tx,Tx,y), S(Ty,Ty,x), S(Ty,Ty,y)) \quad (2.1)$$

for all $x, y, z \in X$ and some $M \in \mathcal{M}$. Then we have

(1) *If M satisfies the condition (C1), then T has a fixed point. Moreover, for any $x_0 \in X$ and the fixed point x, we have*

$$S(Tx_n, Tx_n, x) \leq \frac{2k^n}{1-k} S(x_0, x_0, Tx_0).$$

(2) *If M satisfies the condition (C2) and T has a fixed point, then the fixed point is unique.*

(3) *If M satisfies the condition (C3) and T has a fixed point x, then T is continuous at x.*

Proof. (1) For each $x_0 \in X$ and $n \in \mathbb{N}$, put $x_{n+1} = Tx_n$. It follows from (2.1) and Lemma 1.5 that

$$\begin{aligned}
S(x_{n+1}, x_{n+1}, x_{n+2}) &= S(Tx_n, Tx_n, Tx_{n+1}) \\
&\leq M\big(S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n+1}), \\
&\quad S(x_{n+2}, x_{n+2}, x_n), S(x_{n+2}, x_{n+2}, x_{n+1})\big) \\
&= M\big(S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), 0, \\
&\quad S(x_n, x_n, x_{n+2}), S(x_{n+1}, x_{n+1}, x_{n+2})\big).
\end{aligned}$$

By (S2) and Lemma 1.5 we have

$$\begin{aligned}
S(x_n, x_n, x_{n+2}) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1}) \\
&= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}).
\end{aligned}$$

Since M satisfies the condition (C1), there exists $k \in [0, 1)$ such that

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq kS(x_n, x_n, x_{n+1}) \leq k^{n+1} S(x_0, x_0, x_1). \tag{2.2}$$

Thus for all $n < m$, by using (S2), Lemma 1.5 and (2.2), we have

$$\begin{aligned}
S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\
&= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\
&\dots \\
&\leq 2\big[k^n + \cdots + k^{m-1}\big] S(x_0, x_0, x_1) \\
&\leq \frac{2k^n}{1-k} S(x_0, x_0, x_1).
\end{aligned}$$

Taking the limit as $n, m \to \infty$ we get $S(x_n, x_n, x_m) \to 0$. This proves that $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). Then $x_n \to x \in X$. Moreover, taking the limit as $m \to \infty$ we get

$$S(x_n, x_n, x) \leq \frac{2k^{n+1}}{1-k} S(x_0, x_0, x_1).$$

It implies that

$$S(Tx_n, Tx_n, x) \leq \frac{2k^n}{1-k} S(x_0, x_0, Tx_0).$$

Now we prove that x is a fixed point of T. By using (2.1) again we get

$$\begin{aligned} S(x_{n+1},x_{n+1},Tx) &= S(Tx_n,Tx_n,Tx) \\ &\le M\big(S(x_n,x_n,x),S(Tx_n,Tx_n,x),S(Tx_n,Tx_n,x_n), \\ &\qquad S(Tx,Tx,x_n),S(Tx,Tx,x)\big) \\ &= M\big(S(x_n,x_n,x),S(x_{n+1},x_{n+1},x),S(x_{n+1},x_{n+1},x_n), \\ &\qquad S(Tx,Tx,x_n),S(Tx,Tx,x)\big). \end{aligned}$$

Note that $M \in \mathcal{M}$, then using Lemma 1.6 and taking the limit as $n \to \infty$ we obtain

$$S(x,x,Tx) \le M\big(0,0,0,S(Tx,Tx,x),S(Tx,Tx,x)\big).$$

Then, from Lemma 1.5, we obtain

$$S(x,x,Tx) \le M\big(0,0,0,S(x,x,Tx),S(x,x,Tx,)\big).$$

Since M satisfies the condition (C1), then $S(x,x,Tx) \le k \cdot 0 = 0$. This proves that $x = Tx$.

(2) Let x, y be fixed points of T. We shall prove that $x = y$. It follows from (2.1) and Lemma 1.5 that

$$\begin{aligned} S(x,x,y) &= S(Tx,Tx,Ty) \\ &\le M\big(S(x,x,y),S(Tx,Tx,x),S(Tx,Tx,y),S(Ty,Ty,x),S(Ty,Ty,y)\big) \\ &= M\big(S(x,x,y),0,S(x,x,y),S(y,y,x),0\big) \\ &= M\big(S(x,x,y),0,S(x,x,y),S(x,x,y),0\big). \end{aligned}$$

Since M satisfies the condition ((C2), then $S(x,x,y) = 0$. This proves that $x = y$.

(3) Let x be the fixed point of T and $y_n \to x \in X$. By Corollary 2.4, we need to prove that $Ty_n \to Tx$. It follows from (2.1) that

$$\begin{aligned} S(x,x,Ty_n) &= S(Tx,Tx,Ty_n) \\ &\le M\big(S(x,x,y_n),S(Tx,Tx,x),S(Tx,Tx,y_n), \\ &\qquad S(Ty_n,Ty_n,x),S(Ty_n,Ty_n,y_n)\big) \\ &= M\big(S(x,x,y_n),0,S(x,x,y_n),S(Ty_n,Ty_n,x),S(Ty_n,Ty_n,y_n)\big). \end{aligned}$$

Since M satisfies the condition (C3) and by (S2)

$$S(Ty_n,Ty_n,y_n) \le 2S(Ty_n,Ty_n,x) + S(y_n,y_n,x)$$

then we have

$$\begin{aligned} S(x,x,Ty_n) &\le M\big(S(x,x,y_n),0,S(x,x,y_n),0,S(x,x,y_n)\big) \\ &\qquad + M\big(0,0,0,S(Ty_n,Ty_n,x),2.S(Ty_n,Ty_n,x)\big) \\ &\le M\big(S(x,x,y_n),0,S(x,x,y_n),0,S(x,x,y_n)\big) + k\,S(Ty_n,Ty_n,x). \end{aligned}$$

Therefore

$$S(x,x,Ty_n) \le \frac{1}{1-k} M\big(S(x,x,y_n),0,S(x,x,y_n),0,S(x,x,y_n)\big).$$

Note that $M \in \mathcal{M}$, hence taking the limit as $n \to \infty$ we get $S(x, x, Ty_n) \to 0$. This proves that $Ty_n \to x = Tx$. ∎

Next, we give some analogues of fixed point theorems in metric spaces for S-metric spaces by combining Theorem 2,6 with examples of $M \in \mathcal{M}$ and M satisfies conditions (C1), (C2) and (C3). The following corollary is an analogue of Banach's contraction principle.

Corollary 2.7. [13, Theorem 3.1] *Let T be a self-map on a complete S-metric space (X, S) and*

$$S(Tx, Tx, Ty) \leq L\, S(x, x, y)$$

for some $L \in [0, 1)$ and all $x, y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x, y, z, s, t) = L\, x$ for some $L \in [0, 1)$ and all $x, y, z, s, t \in \mathbb{R}_+$. ∎

The following corollary is an analogue of R. Kannan's result in [8].

Corollary 2.8. *Let T be a self-map on a complete S-metric space (X, S) and*

$$S(Tx, Tx, Ty) \leq a \left(S(Tx, Tx, x) + S(Ty, Ty, y) \right)$$

for some $a \in [0, 1/2)$ and all $x, y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x, y, z, s, t) = a\,(y + t)$ for some $a \in [0, 1/2)$ and all $x, y, z, s, t \in \mathbb{R}_+$. Indeed, M is continuous. First, we have $M(x, x, 0, z, y) = a(x + y)$. So, if $y \leq M(x, x, 0, z, y)$ with $z \leq 2x + y$, then $y \leq a/(1 - a)', x$ with $a/(1 - a) < 1$. Therefore, T satisfies the condition (C1).

Next, if $y \leq M(y, 0, y, y, 0) = 0$, then $y = 0$. Therefore, T satisfies the condition (C2).

Finally, if $x_i \leq y_i + z_i$ for $i \leq 5$, then

$$\begin{aligned} M(x_1, \ldots, x_5) &= a(x_2 + x_5) \leq a[(y_2 + z_2) + (y_5 + z_5)] \\ &= a(y_2 + z_2) + a.(y_5 + z_5) = M(y_1, \ldots, y_5) + M(z_1, \ldots, z_5). \end{aligned}$$

Moreover

$$M(0, 0, 0, y, 2y) = a(0 + 2y) = 2ay$$

where $2a < 1$. Therefore, T satisfies the condition (C3). ∎

Example 2.9. Let $\mathbb{R}$ be the usual S-metric space as in Example 1.7 and let

$$Tx = \begin{cases} 1/2 & \text{if } x \in [0, 1) \\ 1/4 & \text{if } x = 1. \end{cases}$$

Then T is a self-map on a complete S-metric space $[0, 1] \subset \mathbb{R}$. For all $x \in (3/4, 1)$ we have

$$\begin{aligned} S(Tx, Tx, T1) &= S(1/2, 1/2, 1/4) = |1/2 - 1/4| + |1/2 - 1/4| = 1/2 \\ S(x, x, 1) &= |x - 1| + |x - 1| = 2|x - 1| < 1/2. \end{aligned}$$

Then T does not satisfy the condition of Corollary 2.7. We also have

$$S(Tx,Tx,x) = \begin{cases} 2|1/2-x| & \text{if } x\in[0,1) \\ 3/2 & \text{if } x=1. \end{cases}$$

It implies that

$$5/12\big((S(Tx,Tx,x)+S(Ty,Ty,y)\big) = \begin{cases} 5/6\big(|1/2-x|+|1/2-y|\big) & \text{if } x,y\in[0,1) \\ 5/12|1/2-x|+5/8 & \text{if } x\in[0,1), y=1. \end{cases}$$

Then we get $S(Tx,Tx,Ty) \le 5/12\big((S(Tx,Tx,x)+S(Ty,Ty,y)\big)$. Therefore, T satisfies the condition of Corollary 2.8. It is clear that $x=1/2$ is the unique fixed point of T.

The following corollary is an analogue of R. M. T. Bianchini's result in [3].

COROLLARY 2.10. *Let T be a self-map on a complete S-metric space (X,S) and*

$$S(Tx,Tx,Ty) \le h\max\{S(Tx,Tx,x),S(Ty,Ty,y)\}$$

for some $h\in[0,1)$ and all $x,y\in X$. Then T has a unique fixed point in X. Moreover, if $h\in[0,1/2)$, then T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x,y,z,s,t) = h\max\{y,t\}$ for some $h\in[0,1)$ and all $x,y,z,s,t\in\mathbb{R}_+$. Indeed, M is continuous. First, we have $M(x,x,0,z,y)=h\max\{x,y\}$. So, if $y\le M(x,x,0,z,y)$ with $z\le 2x+y$, then $y\le h\,x$ or $y\le h\,y$. Therefore, $y\le h\,x$. Therefore, T satisfies the condition (C1).

Next, if $y\le M(y,0,y,y,0)=h\max\{y,0\}=h\,y$, then $y=0$ since $h<1/2$. Therefore, T satisfies the condition (C2).

Finally, if $x_i\le y_i+z_i$ for $i\le 5$, then

$$\begin{aligned} M(x_1,\dots,x_5) &= h\max\{x_2,x_5\} \le h\max\{y_2+z_2,y_5+z_5\} \\ &\le h\max\{y_2,y_5\}+h\max\{z_2,z_5\} = M(y_1,\dots,y_5)+M(z_1,\dots,z_5). \end{aligned}$$

Moreover, if $h\in[0,1/2)$, then $2h<1$ and $M(0,0,0,y,2y)=h\max\{0,2y\}=2h\,y$ where $2h<1$. Therefore, T satisfies the condition (C3). ■

EXAMPLE 2.11. Let $\mathbb{R}$ be the usual S-metric space as in Example 1.7 and let $Tx=x/3$ for all $x\in[0,1]$. We have

$$\begin{aligned} S(Tx,Tx,Ty) &= S(x/3,x/3,y/3) = |x/3-y/3|+|x/3-y/3| = 2/3|x-y| \\ S(Tx,Tx,x) &= S(x/3,x/3,x) = |x/3-x|+|x/3-x| = 4/3|x| \\ S(Ty,Ty,y) &= S(y/3,y/3,y) = |y/3-y|+|y/3-y| = 4/3|y| \\ S(Tx,Tx,x)+S(Ty,Ty,y) &= 4/3(|x|+|y|) \\ \max\{S(Tx,Tx,x),S(Ty,Ty,y)\} &= 4/3\max\{|x|,|y|\}. \end{aligned}$$

It implies that $S(T1,T1,T0)=2/3$, $S(T1,T1,1)+S(T0,T0,0)=4/3$. This proves that T does not satisfy the condition of Corollary 2.8. We also have that T satisfies the condition of Corollary 2.10 with $h=3/4$ and T has a unique fixed point $x=0$.

The following corollary is an analogue of S. Reich's result in [12].

Corollary 2.12. *Let T be a self-map on a complete S-metric space (X,S) and*

$$S(Tx,Tx,Ty) \le a\,S(x,x,y) + b\,S(Tx,Tx,x) + c\,S(Ty,Ty,y)$$

for some $a,b,c \ge 0$, $a+b+c<1$, and all $x,y \in X$. Then T has a unique fixed point in X. Moreover, if $c<1/2$, then T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x,y,z,s,t) = ax + by + ct$ for some $a,b,c \ge 0$, $a+b+c<1$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, M is continuous. First, we have $M(x,x,0,z,y) = ax+bx+cy$. So, if $y \le M(x,x,0,z,y)$ with $z \le 2x+y$, then $y \le (a+b)/(1-c)\,x$ with $(a+b)/(1-c) < 1$. Therefore, T satisfies the condition (C1).

Next, if $y \le M(y,0,y,y,0) = ay$, then $y=0$ since $a<1$. Therefore, T satisfies the condition (C2).

Finally, if $x_i \le y_i + z_i$ for $i \le 5$, then

$$\begin{aligned} M(x_1,\dots,x_5) &= ax_1 + bx_2 + cx_5 \\ &\le a(y_1+z_1) + b(y_2+z_2) + c(y_5+z_5) \\ &= (ay_1+by_2+cy_5) + (az_1+bz_2+cz_5) \\ &= M(y_1,\dots,y_5) + M(z_1,\dots,z_5). \end{aligned}$$

Moreover $M(0,0,0,y,2y) = 2cy$ where $2c<1$. Therefore, T satisfies the condition (C3). ∎

Example 2.13. Let $\mathbb{R}$ be the usual S-metric space as in Example 1.7 and let $Tx = x/2$ for all $x \in [0,1]$. We have

$$\begin{aligned} S(Tx,Tx,Ty) &= |x/2-y/2| + |x/2-y/2| = |x-y| \\ S(x,x,y) &= |x-y| + |x-y| = 2|x-y| \\ S(Tx,Tx,x) &= |x/2-x| + |x/2-x| = |x|. \end{aligned}$$

Then $S(Tx,Tx,T0) = |x|$, $\max\{S(Tx,Tx,x), S(T0,T0,0)\} = |x|$. This proves that T does not satisfy the condition of Corollary 2.10. We also have

$$S(Tx,Tx,Ty) \le 1/2S(x,x,y) + 1/3S(Tx,Tx,x) + 1/3S(Ty,Ty,y).$$

Then T satisfy the condition of Corollary 2.12. It is clear that T has a unique fixed point $x=0$.

The following corollary is an analogue of S. K. Chatterjee's result in [4].

Corollary 2.14. *Let T be a self-map on a complete S-metric space (X,S) and*

$$S(Tx,Tx,Ty) \le h\max\{S(Tx,Tx,y), S(Ty,Ty,x)\}$$

for some $h \in [0,1/3)$ and all $x,y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x,y,z,s,t) = h\max\{z,s\}$ for some $h \in [0,1/3)$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, M is continuous. First, we have $M(x,x,0,z,y) = h\max\{0,z\}$. So, if $y \le M(x,x,0,z,y)$ with $z \le 2x+y$, then $y \le 2hx + hy$. So $y \le 2h/(1-h)x$ with $2h/(1-h) < 1$. Therefore, T satisfies the condition (C1).

Next, if $y \le M(y,0,y,y,0) = hy$, then $y = 0$ since $h < 1/3$. Therefore, T satisfies the condition (C2).

Finally, if $x_i \le y_i + z_i$ for $i \le 5$, then

$$\begin{aligned} M(x_1,\dots,x_5) &= h\max\{x_3,x_4\} \le h\max\{y_3+z_3, y_4+z_4\} \\ &\le h\max\{y_3,y_4\} + h\max\{z_3,z_4\} = M(y_1,\dots,y_5) + M(z_1,\dots,z_5). \end{aligned}$$

Moreover

$$M(0,0,0,y,2y) = h\max\{0,y\} = hy$$

where $h < 1$. Therefore, T satisfies the condition (C3). ∎

COROLLARY 2.15. *Let T be a self-map on a complete S-metric space (X,S) and*

$$S(Tx,Tx,Ty) \le a.\big(S(Tx,Tx,y) + S(Ty,Ty,x)\big)$$

for some $a \in [0,1/3)$ and all $x,y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x,y,z,s,t) = a(z+s)$ for some $a \in [0,1/3)$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, M is continuous. First, we have $M(x,x,0,z,y) = a(0+z) = az$. So, if $y \le M(x,x,0,z,y)$ with $z \le 2x+y$, then $y \le 2ax + ay$. So $y \le 2a/(1-a)\,x$ with $2a/(1-a) < 1$. Therefore, T satisfies the condition (C1).

Next, if $y \le M(y,0,y,y,0) = a(y+y) = 2ay$ then $y = 0$ since $2a < 2/3$. Therefore, T satisfies the condition (C2).

Finally, if $x_i \le y_i + z_i$ for $i \le 5$, then

$$\begin{aligned} M(x_1,\dots,x_5) &= a(x_3+x_4) \le a(y_3+z_3+y_4+z_4) \\ &= a(y_3+y_4) + a(z_3+z_4) = M(y_1,\dots,y_5) + M(z_1,\dots,z_5). \end{aligned}$$

Moreover $M(0,0,0,y,2y) = a(0+y) = ay$ where $a < 1$. Therefore, T satisfies the condition (C3). ∎

EXAMPLE 2.16. Let $\mathbb{R}$ be the usual S-metric space as in Example 1.7 and let $Tx = x/3$ for all $x \in [0,1]$. Then we have $S(Tx,Tx,Ty) = 2|x/3 - y/3| = 2/3|x-y|$, $S(Tx,Tx,y) = 2|x/3 - y|$, $S(Ty,Ty,x) = 2|y/3 - x|$. It implies that $S(T1,T1,T0) = 2/3$, $S(T1,T1,0) = 2/3$, $S(T0,T0,1) = 2$. This proves that T does not satisfy the condition of Corollary 2.14. We also have

$$S(Tx,Tx,y) + S(Ty,Ty,x) = 2|x/3 - y| + 2|y/3 - x| \ge 8/3|x-y|.$$

Therefore, T satisfies the condition of Corollary 2.15. It is clear that T has a unique fixed point $x = 0$.

Corollary 2.17. *Let T be a self-map on a complete S-metric space (X,S) and*

$$S(Tx,Tx,Ty) \leq aS(x,x,y) + bS(Tx,Tx,y) + cS(Ty,Ty,x)$$

for some $a,b,c \geq 0$, $a+b+c<1$, $a+3c<1$ and all $x,y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x,y,z,s,t) = ax+bz+cs$ for some $a,b,c \geq 0$, $a+b+c<1$, $a+3c<1$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, M is continuous. First, we have $M(x,x,0,z,y) = ax+cz$. So, if $y \leq M(x,x,0,z,y)$ with $z \leq 2x+y$, then $y \leq ax+2cx+cy$. So $y \leq (a+2c)/(1-c)\, x$ with $(a+2c)/(1-c) < 1$. Therefore, T satisfies the condition (C1).

Next, if $y \leq M(y,0,y,y,0) = ay+by+cy = (a+b+c)y$ then $y=0$ since $a+b+c<1$. Therefore, T satisfies the condition (C2).

Finally, if $x_i \leq y_i + z_i$ for $i \leq 5$, then

$$\begin{aligned} M(x_1,\dots,x_5) &= ax_1 + bx_3 + cx_4 \leq a(y_1+z_1) + b(y_3+z_3) + c(y_4+z_4) \\ &= (ay_1+by_3+cy_4) + (az_1+bz_3+cz_4) \\ &= M(y_1,\dots,y_5) + M(z_1,\dots,z_5). \end{aligned}$$

Moreover $M(0,0,0,y,2y) = cy$ where $c<1$. Therefore, T satisfies the condition (C3). ∎

Example 2.18. Let $\mathbb{R}$ be the usual S-metric space as in Example 1.7 and let $Tx = 3/4(1-x)$ for all $x \in [0,1]$. Then we have $S(Tx,Tx,Ty) = 3/2|x-y|$, $S(Tx,Tx,y) = 2|3/4(1-x)-y|$. It implies that $S(T1,T1,T0) = 3/2$, $\max\{S(T1,T1,0), S(T0,T0,1)\} = \max\{0,1/2\} = 1/2$. This proves that T does not satisfy the condition of Corollary 2.14. We also have

$$4/5S(x,x,y) + 0\cdot S(Tx,Tx,y) + 0\cdot S(Ty,Ty,x) = (8/5)|x-y| \geq S(Tx,Tx,Ty).$$

Therefore, T satisfies the condition of Corollary 2.17. It is clear that T has a unique fixed point $x = 3/7$.

The following corollary is an analogue of G. E. Hardy and T. D. Rogers' result in [6].

Corollary 2.19. *Let T be a self-map on a complete S-metric space (X,S) and*

$$\begin{aligned} S(Tx,Tx,Ty) \leq a_1S(x,x,y) + a_2S(Tx,Tx,x) + a_3S(Tx,Tx,y) \\ + a_4S(Ty,Ty,x) + a_5S(Ty,Ty,y) \end{aligned}$$

for some $a_1,\dots,a_5 \geq 0$ such that $\max\{a_1+a_2+3a_4+a_5, a_1+a_3+a_4, a_4+2a_5\} < 1$ and all $x,y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x,y,z,s,t) = a_1x + a_2y + a_3z + a_4s + a_5t$ for some $a_1,\dots,a_5 \geq 0$ such that $\max\{a_1+a_2+3a_4+a_5, a_1+a_3+a_4, a_4+2a_5\} < 1$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, M is continuous. First, we have $M(x,x,0,z,y) = a_1x + a_2x + a_4z + a_5y$. So, if $y \le M(x,x,0,z,y)$ with $z \le 2x + y$, then

$$y \le a_1x + a_2x + a_4z + a_5y \le a_1x + a_2x + a_4(2x + y) + a_5y.$$

Then $y \le (a_1 + a_2 + 2a_4)/(1 - a_4 - a_5)\, x$ with $(a_1 + a_2 + 2a_4)/(1 - a_4 - a_5) < 1$. Therefore, T satisfies the condition (C1).

Next, if $y \le M(y,0,y,y,0) = a_1y + a_3y + a_4y = (a_1 + a_3 + a_4)y$ then $y = 0$ since $a_1 + a_3 + a_4 < 1$. Therefore, T satisfies the condition (C2).

Finally, if $x_i \le y_i + z_i$ for $i \le 5$, then

$$\begin{aligned} M(x_1,\dots,x_5) &= a_1x_1 + \cdots + a_5x_5 \\ &\le a_1(y_1 + z_1) + \cdots + a_5(y_5 + z_5) \\ &= (a_1y_1 + \cdots + a_5y_5) + (a_1z_1 + \cdots + a_5z_5) \\ &= M(y_1,\dots,y_5) + M(z_1,\dots,z_5). \end{aligned}$$

Moreover $M(0,0,0,y,2y) = a_4y+2a_5y = (a_4+2a_5)y$ where $a_4+2a_5 < 1$. Therefore, T satisfies the condition (C3). ■

EXAMPLE 2.20. Let T be the map in Example 2.16. Then we have

$$S(T1,T1,T1/2) = 1,$$
$$aS(1,1,1/2) + bS(T1,T1,1/2) + cS(T1/2,T1/2,1) = a + 2c.$$

This proves that T does not satisfy the condition of Corollary 2.17. We also have

$$\begin{aligned} 0\cdot S(x,x,y)+(3/4)S(Tx,Tx,x)+(3/4)S(Tx,Tx,y)+0\cdot S(Ty,Ty,x)+0\cdot S(Ty,Ty,y) \\ = (3/4)S(Tx,Tx,x) + (3/4)S(Tx,Tx,y) \ge S(Tx,Tx,Ty). \end{aligned}$$

Therefore, T satisfies the condition of Corollary 2.19. It is clear that T has a unique fixed point $x = 0$.

The following corollary is an analogue of L. B. Ćirić's result in [5].

COROLLARY 2.21. *Let T be a self-map on a complete S-metric space (X,S) and*

$$\begin{aligned} S(Tx,Tx,Ty) \le h \max\bigl\{S(x,x,y), S(Tx,Tx,x), S(Tx,Tx,y), \\ S(Ty,Ty,x), S(Ty,Ty,y)\bigr\} \end{aligned}$$

for some $h \in [0,1/3)$ and all $x,y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 2.6 with $M(x,y,z,s,t) = h\max\{x,y,z,s,t\}$ for some $h \in [0,1/3)$ and all $x,y,z,s,t \in \mathbb{R}_+$. Indeed, M is continuous. First, we have $M(x,x,0,z,y) = h\max\{x,x,0,z,y\}$. So, if $y \le M(x,x,0,z,y)$ with $z \le 2x+y$, then $y \le hx$ or $y \le hz \le h(2x+y)$. Then $y \le kx$ with $k = \max\{h, 2h/(1-h)\} < 1$. Therefore, T satisfies the condition (C1).

Next, if $y \le M(y,0,y,y,0) = h.y$, then $y = 0$ since $h < 1/3$. Therefore, T satisfies the condition (C2).

Finally, if $x_i \leq y_i + z_i$ for $i \leq 5$, then

$$\begin{aligned} M(x_1, \dots, x_5) &= h \max\{x_1, \dots, x_5\} \leq h \max\{y_1 + z_1, \dots, y_5 + z_5\} \\ &\leq h \max\{y_1, \dots, y_5\} + h \max\{z_1, \dots, z_5\} \\ &= M(y_1, \dots, y_5) + M(z_1, \dots, z_5). \end{aligned}$$

Moreover $M(0, 0, 0, y, 2y) = 2hy$ where $2h < 1$. Therefore, T satisfies the condition (C3). ∎

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