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COMPACT-LIKE PROPERTIES IN HYPERSPACES

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Abstract. $\mathcal{CL}(X)$ and $\mathcal{K}(X)$ denote the hyperspaces of non-empty closed and non-empty compact subsets of X, respectively, with the Vietoris topology. For an infinite cardinal number α, a space X is α-hyperbounded if for every family $\{S_\xi : \xi < \alpha\}$ of non-empty compact subsets of X, $Cl_X(\bigcup_{\xi<\alpha} S_\xi)$ is a compact set, and a space X is pseudo-ω-bounded if for each countable family $\mathcal{U}$ of non-empty open subsets of X, there exists a compact set $K \subseteq X$ such that each element in $\mathcal{U}$ has a non-empty intersection with K. We prove that X is α-hyperbounded if and only if $\mathcal{K}(X)$ is α-hyperbounded, if and only if $\mathcal{K}(X)$ is initially α-compact. Moreover, $\mathcal{K}(X)$ is pseudocompact if and only if X is pseudo-ω-bounded. Also, we show than if $\mathcal{K}(X)$ is normal and C^*-embbeded in $\mathcal{CL}(X)$, then X is ω-hyperbounded, and X is α-bounded if and only if X is α-hyperbounded, for every infinite cardinal number α.

1. Notations, basic definitions and introduction

Every space in this article is a Tychonoff space with more than one point. The letters ξ, ζ, γ and η represent ordinal numbers and the letters α, τ, κ and θ represent infinite cardinal numbers; ω is the first infinite cardinal, ω_1 is the first non-countable cardinal and $cf(\xi)$ is the cofinality of the ordinal ξ. Given a set X and a cardinal number κ, $[X]^{\leq\kappa}$ and $[X]^{\kappa}$ represent the sets $\{A \subseteq X : |A| \leq \kappa\}$ and $\{A \subseteq X : |A| = \kappa\}$, respectively. $\mathbb{R}$ is the space of real numbers with its usual topology and $\mathbb{N}$ is the subspace of $\mathbb{R}$ constituted by the natural numbers. Given two spaces X, Y, $C(X,Y)$ denotes the set of continuous functions from X to Y; if $Y = \mathbb{R}$, we write $C^*(X)$ for the set of bounded continuous functions with real values. For $X = \prod_{s\in S} X_s$ and $s \in S$, π_s is the projection from X onto X_s. $\beta(X)$ is the *Stone-Čech* compactification of the space X and X^* denotes the set $\beta(X) \setminus X$.

We will denote an ordinal number η with its discrete topology simply as η. The ordinal number η with its order topology will be symbolized by $[0, \eta)$.

Let X be a space, κ a cardinal number, $p \in \beta(\kappa)$ an ultrafilter, and $(x_\xi)_{\xi<\kappa}$ (respectively $(S_\xi)_{\xi<\kappa}$) a sequence of points (resp., a sequence of non-empty subsets)

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of X, we say that the point $z \in X$ is the p-limit of $(x_\xi)_{\xi<\kappa}$ (resp., of $(S_\xi)_{\xi<\kappa}$) if for all open neighbourhoods W of z,

$$\{\xi < \kappa : x_\xi \in W\} \in p \quad (\text{resp., } \{\xi < \kappa : S_\xi \cap W \neq \emptyset\} \in p).$$

A space X is p-compact (p-pseudocompact; see [4] and [7]) if, for every countable sequence of points (resp., of non-empty open sets) of X, there exists $x \in X$ which is a p-limit of $(x_\xi)_{\xi<\kappa}$ (resp., of $(S_\xi)_{\xi<\kappa}$). p-limits of sequences of points are unique (when they exist), if x is the p-limit point of the sequence $(x_\xi)_{\xi<\kappa}$, then we write $x = p{-}lim\ x_\xi$. The p-limits of sequences of sets are not necessarily unique, then, we denote with $L(p,(S_\xi)_{\xi<\kappa})$ the set of p-limit points of the sequence $(S_\xi)_{\xi<\kappa}$. Given an infinite cardinal κ, we say that X is *κ-compact* if X can be written as the union of κ compact subspaces of X; X is *initially κ-compact* if every set $A \in [X]^{\leq\kappa}$ has a complete cluster point and X is *κ-bounded* if every set $A \in [X]^{\leq\kappa}$ has a compact closure in X (see [15]). It is easy to see that every κ-bounded space is initially κ-compact.

For a topological space $(X, \mathcal{T})$, $CL(X)$, $K(X)$ and $F_n(X)$ denote the sets of closed, compact and finite subsets of cardinality less or equal than n of X, respectively. $\mathcal{CL}(X)$ denotes the hyperspace of non-empty closed subsets of X with the Vietoris topology. $\mathcal{K}(X)$ and, for each $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ are the subspaces of $\mathcal{CL}(X)$ formed by compact and finite subsets of cardinality less or equal to n, respectively. Remember that, the Vietoris topology has the sets of the form

$$V^+ = \{A \in CL(X) : A \subseteq V\} \quad \text{and} \quad V^- = \{A \in CL(X) : A \cap V \neq \emptyset\}$$

as a subbase, where V is an open subset of X. Given open sets $U_1, \ldots, U_n$ of X, we define

$$\langle U_1, \ldots, U_n \rangle = \{T \in CL(X) : T \in (\bigcup_{1\leq k\leq n} U_k)^+ \text{ and } T \in U_k^- \text{ for each } 1 \leq k \leq n\}.$$

So, the collection

$$\{\langle U_1, \ldots, U_n \rangle : n \in \mathbb{N}, U_1, \ldots, U_n \in \mathcal{T}\}$$

constitutes a base for $\mathcal{CL}(X)$.

It is known that the space X is homeomorphic to the subspace $\mathcal{F}_1(X)$ of $\mathcal{CL}(X)$, that $\mathcal{K}(X)$ is dense in $\mathcal{CL}(X)$ and $\mathcal{CL}(X)$ is compact if and only if X is compact, if and only if $\mathcal{CL}(X) = \mathcal{K}(X)$. On the other hand, for each $A \in CL(X)$, $\mathcal{CL}(A)$ can be considered as a subspace of $\mathcal{CL}(X)$; in particular, if $A \in K(X)$, then $\mathcal{K}(A)$ is a compact subspace of $\mathcal{CL}(X)$.

The class of hyperspaces has been widely studied and continues to generate significant results and problems. In 1985 Dušan Milovančević showed in [12] that countable compactness, ω-boundedness (strongly countable compactness according to the terminology of Milovančević) and hypercountable compactness (the closure of every σ-compact subset is compact) coincide in the class of spaces of the form $\mathcal{K}(X)$. In Section 1, we continue in this mood by introducing and studying the concept of α-hyperboundedness, and we prove that for every space X, $\mathcal{K}(X)$ is initially α-compact if and only if $\mathcal{K}(X)$ is α-bounded, if and only if $\mathcal{K}(X)$ is α-hyperbounded. Moreover, we show that for every infinite cardinal number α, there is an α-bounded space X which is not ω-hyperbounded. We finish Section 1 by analyzing the relationship between the maximal α-hyperbounded extension of a space X and its maximal α-bounded extension.

In Section 2 we give some equivalent conditions to that of pseudocompactness in the class of spaces of the form $\mathcal{K}(X)$. In particular, we show that $\mathcal{K}(X)$ is pseudocompact if and only if X is pseudo-ω-bounded. Also, we obtain some results about $\mathcal{K}(X)$ and X when $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$. In particular, we show than if $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$ then X is ω-hyperbounded, and X is α-bounded if and only if X is α-hyperbounded, for every infinite cardinal number α.

For those concepts which appear in this article without definition consult [6].

2. $\mathcal{K}(X)$ and α-hyperbounded spaces.

We begin this section by introducing the concept of α-hyperboundedness.

Definition 2.1. Let X be a space and let α be an infinite cardinal. We say that X is *α-hyperbounded* if for each family $\{S_\xi : \xi < \alpha\}$ of compact sets of X, $Cl_X(\bigcup_{\xi<\alpha} S_\xi)$ is a compact subspace.

It is clear that if κ and α are cardinals such that $\omega \leq \kappa \leq \alpha$, then every α-hyperbounded space is κ-hyperbounded; also every compact space is α-hyperbounded and every α-hyperbounded space is α-bounded.

Theorem 2.2. *Let X, Y be two spaces and let α be an infinite cardinal. Then:*

(1) *The α-hyperboundedness is inherited by closed subsets.*

(2) *A topological product $\prod_{s\in S} X_s$ is α-hyperbounded if and only if each factor X_s is α-hyperbounded.*

(3) *If Y is a continuous and perfect image of X then Y is α-hyperbounded if and only if X is α-hyperbounded.*

Proof. (1) Let X be an α-hyperbounded space, A a closed set of X and $\{S_\xi : \xi < \alpha\}$ a family of non-empty compact subspaces of A. Of course, $\{S_\xi : \xi < \alpha\}$ is a family of non-empty compact sets of X, so $Cl_X(\bigcup_{\xi<\alpha} S_\xi)$ is a compact set of X. Since A is closed, $Cl_A(\bigcup_{\xi<\alpha} S_\xi) = Cl_X(\bigcup_{\xi<\alpha} S_\xi)$ is a compact set of A.

(2) Let $\{X_s : s \in S\}$ be a family of spaces and let X be the topological product of this family. If X is α-hyperbounded, then, because of (1), for each $s \in S$, X_s is α-hyperbounded.

Now, suppose that each X_s is α-hyperbounded. Let $\{S_\xi : \xi < \alpha\}$ be a family of non-empty compact subsets of X. Then, for each $s \in S$, $(\pi_s[S_\xi])_{\xi<\alpha}$ is a family of non-empty compact subsets of X_s. Therefore, $L_s = Cl(\bigcup_{\xi<\alpha} \pi_s[S_\xi])$ is a compact set of X_s. To finish the proof, since

$$Cl_X(\bigcup_{\xi<\alpha} S_\xi) \subseteq \prod_{s\in S} L_s,$$

then $Cl_X(\bigcup_{\xi<\alpha} S_\xi)$ is a compact subspace of X.

(3) Let $f : X \longrightarrow Y$ be a continuous, perfect and onto function. Suppose that X is α-hyperbounded and let $\{S_\xi : \xi < \alpha\}$ be a family of non-empty compact subsets of Y. Since f is perfect, $(f^{-1}[S_\xi])_{\xi<\alpha}$ is a family of non-empty compact subsets of X. Since X is α-hyperbounded, $Cl_X(\bigcup_{\xi<\alpha} f^{-1}[S_\xi])$ is a compact subset of X and $L = f[Cl_X(\bigcup_{\xi<\alpha} f^{-1}[S_\xi])]$ is a compact subset of Y; since $Cl_Y(\bigcup_{\xi<\alpha} S_\xi) \subseteq L$, $Cl_Y(\bigcup_{\xi<\alpha} S_\xi)$ is compact.

On the other hand, if Y is α-hyperbounded, then X is α-hyperbounded by (1), (2) and Theorem 3.7.26 in [6]. ■

Theorem 2.3. (Theorem 2.2 in [12]) *Let X be a space. Then the following statements are equivalent:*

(1) *Every σ-compact set of X has a compact closure in X (X is ω-hyperbounded);*

(2) *$\mathcal{K}(X)$ is countably compact;*

(3) *$\mathcal{K}(X)$ is ω-bounded; and*

(4) *Every σ-compact subspace of $\mathcal{K}(X)$ has a compact closure in $\mathcal{K}(X)$ ($\mathcal{K}(X)$ is ω-hyperbounded).*

Now, we are going to generalize Theorem 2.3. First, we will prove some lemmas. As usual, $U(\kappa)$ will denote the set of uniform ultrafilters on κ.

Lemma 2.4. *Let X be a space, let κ be an infinite cardinal and let $\{S_\xi : \xi < \kappa\}$ be a subcollection of $\mathcal{CL}(X)$. Let $S = \bigcup_{\xi<\kappa} S_\xi$ and, for each $\xi < \kappa$, we define the set $T_\xi = Cl_X(\bigcup_{\zeta\leq\xi} S_\xi)$. Then the sequence $(T_\xi)_{\xi<\kappa}$ of $\mathcal{CL}(X)$ converges to $Cl_X(S)$ in $\mathcal{CL}(X)$.*

Proof. Let $\mathcal{U} = <U_1, \ldots, U_n>$ be such that $Cl_X(S) \in \mathcal{U}$. Note that, for each $\xi < \tau$, $T_\xi \subseteq Cl_X(S)$. Since $Cl_X(S) \subseteq \bigcup_{i\leq n} U_i$, for every $\xi < \alpha$, $T_\xi \subseteq \bigcup_{i\leq n} U_i$. On the other hand, for each $i \leq n$, there exists $\xi_i < \alpha$ such that $T_{\xi_i} \cap U_i \neq \emptyset$. Let $\eta = max\{\xi_i : i \leq n\}$. Then, for every $\xi \geq \eta$ and all $i \leq n$, $T_\xi \cap U_i \neq \emptyset$. So, we conclude that, for every $\xi \geq \eta$, $T_\xi \in \mathcal{U}$. ■

Lemma 2.5. *Let T be a compact subspace of $\mathcal{K}(X)$. Then $A = \cup T$ is a compact subspace of X and $T \subseteq \mathcal{K}(A)$.*

Proof. Let $\mathcal{U}$ be an open cover of A. Since $T \subseteq \mathcal{K}(X)$, for every $F \in T$, there exists a finite subcollection $\mathcal{U}_F$ of $\mathcal{U}$ which covers F. For each $F \in T$, take $V_F = \bigcup \mathcal{U}_F$. Then, the family $\{V_F^+ : F \in T\}$ is an open cover of T in $\mathcal{CL}(X)$. Since T is compact, there are sets $F_1, \ldots, F_n \in T$ such that $T \subseteq \bigcup_{k=1}^n V_{F_k}^+$, which implies that for each $F \in T$, there exists $1 \leq k \leq n$ such that $F \subseteq V_{F_k}$. Then $A \subseteq \bigcup_{k=1}^n V_{F_k}$. Thus, A is compact. Moreover, each $F \in T$ is a compact subspace of A, so $T \subseteq \mathcal{K}(A)$. ■

Theorem 2.6. *Let X be an space and let α be an infinite cardinal. Then the following statements are equivalent:*

(1) *X is α-hyperbounded;*

(2) *for every infinite cardinal $\kappa \leq \alpha$, every transfinite sequence $(S_\xi)_{\xi<\kappa}$ of non-empty compact subspaces of X and every uniform ultrafilter $p \in U(\kappa)$, $L(p,(S_\xi)_{\xi<\kappa})$ is a non-empty compact subspace of X;*

(3) *for every infinite cardinal $\kappa \leq \alpha$ and every transfinite sequence $(S_\xi)_{\xi<\kappa}$ of non-empty compact subspaces of X, there exists a uniform ultrafilter $p \in U(\kappa)$ such that $L(p,(S_\xi)_{\xi<\kappa})$ is a non-empty compact subspace of X;*

(4) *$\mathcal{K}(X)$ is α-bounded;*

(5) *$\mathcal{K}(X)$ is initially α-compact; and*

(6) *$\mathcal{K}(X)$ is α-hyperbounded.*

Proof. (1) $\Rightarrow$ (2). Let $p \in U(\kappa)$ and let $(S_\xi)_{\xi<\kappa}$ be a transfinite sequence of non-empty compact subspaces of X. For each $\xi < \kappa$, choose $x_\xi \in S_\xi$. Since $(x_\xi)_{\xi<\kappa}$ is contained in the compact space $Cl_X(\bigcup_{\xi<\kappa} S_\xi)$, there exists $x \in X$ such that $x = p - lim\ x_\xi$. That is, $x \in L(p,(S_\xi)_{\xi<\kappa})$. Moreover, it is known that, for every infinite cardinal $\kappa \leq \alpha$, every transfinite sequence $(S_\xi)_{\xi<\kappa}$ of non-empty subsets of X and all uniform ultrafilter $p \in U(\kappa)$, $L(p,(S_\xi)_{\xi<\kappa})$ is closed in X and $L(p,(S_\xi)_{\xi<\kappa}) \subseteq Cl(\bigcup_{\xi<\kappa} S_\xi)$. Then, $L(p,(S_\xi)_{\xi<\kappa})$ is a non-empty compact subspace of X.

The implication (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). We will show this implication by transfinite induction. We begin by proving that if the ω-version of (3) holds, then X is ω-hyperbounded.

Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of compact subspaces of X. For each $n \in \mathbb{N}$, put $D_n = \bigcup_{i=1}^n S_i$. Since every S_n is a compact subspace of X, $(D_n)_{n\in\mathbb{N}}$ is an increasing sequence of non-empty compact subsets of X such that $S = \bigcup_{n\in\mathbb{N}} D_n$. Because of our hypotheses, there is $p \in \omega^*$ such that $L(p,(D_n)_{n\in\mathbb{N}})$ is a non-empty compact subspace of X.

We will show that $Cl_X(S) \subseteq L(p,(D_n)_{n\in\mathbb{N}})$. Take $x \in Cl_X(S)$ and let V be an open neighbourhood of x. By Lemma 2.4, the sequence $(D_n)_{n\in\mathbb{N}}$ converges to $Cl_X(S)$ in $\mathcal{CL}(X)$. Since V^- is an open neighbourhood of $Cl_X(S)$, there exists $m \in \mathbb{N}$ such that $D_n \in V^-$ for every $n \geq m$. Then

$$\{n : D_n \cap V \neq \emptyset\} \supseteq \{n : m \leq n\} \in p.$$

So, we conclude that $x \in L(p,(D_n)_{n\in\mathbb{N}})$ and $Cl_X(S)$ is compact.

Given a cardinal $\alpha > \omega$, assume that for all cardinals $\omega \leq \theta < \alpha$, condition (3) with θ in place of α implies that X is θ-hyperbounded. Let $(S_\xi)_{\xi<\alpha}$ be a transfinite sequence of compact subsets of X with union S. Assume (3), which, by inductive hypothesis, implies that $D_\xi = Cl_X(\bigcup_{\eta\leq\xi} S_\eta)$ is compact for each $\xi < \alpha$. By (3), there is $p \in U(\alpha)$ with $L(p,(D_\xi)_{\xi<\alpha}) \in \mathcal{K}(X)$, and we will be done if we show that $D = Cl_X(\bigcup_{\xi<\alpha} D_\xi) \subseteq L(p,(D_\xi)_{\xi<\alpha})$. Indeed, let $x \in D$ and V be an open neighbourhood of x. Since by Lemma 2.4 $(D_\xi)_{\xi<\alpha}$ converges to $D \in \mathcal{CL}(X)$, there exists $\xi_0 < \alpha$ such that $D_\xi \in V^-$ for all $\xi \geq \xi_0$. Then

$$\{\xi : D_\xi \cap V \neq \emptyset\} \supseteq \{\xi : \xi_0 \leq \xi < \alpha\} \in p;$$

thus, $x \in L(p,(D_\xi)_{\xi<\alpha})$.

(1) $\Rightarrow$ (4). Assuming (1), and given a collection $\{S_\xi : \xi < \alpha\} \subseteq \mathcal{K}(X)$, we have that $D = Cl_X(\bigcup_{\xi<\alpha} S_\xi) \in \mathcal{K}(X)$. Since $\mathcal{K}(D)$ is a compact subspace of $\mathcal{K}(X)$, then $Cl_{\mathcal{K}(X)}(\{S_\xi : \xi < \alpha\}) \subseteq \mathcal{K}(D)$ is compact.

The implication (4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (6). For $\alpha = \omega$, the implication follows from Theorem 2.3. Given a cardinal $\alpha > \omega$, assume that for all cardinals $\omega \leq \theta < \alpha$, if $\mathcal{K}(X)$ is initially θ-compact, then $\mathcal{K}(X)$ is θ-hyperbounded. Let $\mathcal{K}(X)$ be initially α-compact. Then $\mathcal{K}(X)$, and by Theorem 2.2.(1), also X is θ-hyperbounded for all cardinals $\omega \leq \theta < \alpha$. If $\{\mathcal{S}_\xi : \xi < \alpha\}$ is a collection of compact subsets of $\mathcal{K}(X)$, then by Lemma 2.5, $\bigcup \mathcal{S}_\xi \in \mathcal{K}(X)$, moreover, by our assumptions $D_\xi = Cl_X(\bigcup_{\eta\leq\xi} \bigcup \mathcal{S}_\eta) \in \mathcal{K}(X)$ for each $\xi < \alpha$. If $Z \in \mathcal{K}(X)$ is a complete cluster point of $(D_\xi)_{\xi<\alpha}$, we will be done if we show that $D = \bigcup\{D_\xi : \xi < \alpha\} \subseteq Z$ (because then $Cl_{\mathcal{K}(X)}(\{\mathcal{S}_\xi : \xi < \alpha\}) \subseteq \mathcal{K}(Cl_X(D))$, and the latter is a compact subspace of $\mathcal{K}(X)$): suppose, on the contrary, that there exists $x \in D_{\xi_0} \setminus Z$ for some $\xi_0 < \alpha$. By compactness of Z we can find an open neighbourhood U of Z that misses x. Since $(D_\xi)_{\xi<\alpha}$ is nondecreasing, it follows that $D_\xi \notin U^+$ for every $\xi_0 \leq \xi < \alpha$, which contradicts the fact U^+ is an open neighbourhood of Z, and hence, should contain α-many members of $(D_\xi)_{\xi<\alpha}$.

(6) $\Rightarrow$ (1) is a consequence of (1) from Theorem 2.2. ∎

PROPOSITION 2.7. *Let X be a space with an α-compact dense subspace. Then X is α-hyperbounded if and only if it is compact.*

In [12], Milovančević gives examples of ω-bounded spaces that are not ω-hyperbounded. The following result shows an example of an α-bounded space which is not ω-hyperbounded for each $\alpha \geq \omega$; we use the Σ_κ-product of a cube $\{0,1\}^S$. Given a family of spaces $\{X_s : s \in S\}$, take the Tychonoff product $X = \prod_{s\in S} X_s$ and a fixed point $x \in X$. For each $z \in X$, we define the support of z with respect to x as the set

$$\operatorname{supp}_x(z) = \{s \in S : \pi_s(x) \neq \pi_s(z)\}.$$

For an infinite cardinal κ, we define the Σ_κ-product of X with respect to x as the subspace

$$\Sigma_\kappa(x, X) = \{z \in X : |\operatorname{supp}_x(z)| < \kappa\}$$

of X.

THEOREM 2.8. *For every cardinal $\kappa \geq \omega$, there exists a κ-bounded space X which is not ω-hyperbounded.*

Proof. Let κ be an infinite cardinal and S a set such that $\kappa < |S|$. Consider the space $X = \Sigma_{\kappa^+}(\underline{0}, \{0,1\}^S)$. It is easy to show that X is a proper dense κ-bounded subspace of $\{0,1\}^S$ (it is even C^*-embedded in $\{0,1\}^S$, see [1]). We will show that X has a dense σ-compact subspace. Let $S_n = \{x \in X : |\operatorname{supp}_{\underline{0}}(x)| \leq n\}$ and let $z \in \{0,1\}^S \setminus S_n$. Let $F \in [\operatorname{supp}_{\underline{0}}(z)]^{n+1}$ and let $U = \cap_{s\in F} \pi_s^{-1}(1)$. It is apparent that $z \in U$ and $U \cap S_n = \emptyset$, so $z \notin Cl(S_n)$. Then S_n is a closed subset of $\{0,1\}^S$. Therefore S_n is a compact subspace of X for all $n \in \mathbb{N}$.

We have:

$$\bigcup_{n\in\mathbb{N}} S_n = \{x \in \{0,1\}^S : |\operatorname{supp}_{\underline{0}}(x)| < \omega\} = \Sigma_\omega(\underline{0}, \{0,1\}^S).$$

This last set is dense in $\{0,1\}^S$, so it is dense in X. By Proposition 2.7, X cannot be ω-hyperbounded because X is not a compact space. ∎

Another important observation is the following: given a space X, and a closed-hereditary, productive topological property P, there exists the maximal extension $\beta_P(X)$ of X having property P (see Proposition 4.1(k) in [13]). It follows by Theorem 2.2, that X has a maximal α-hyperbounded (resp., α-bounded) extension $\beta_{h\alpha}(X)$ (resp., $\beta_\alpha(X)$). Such extension is the only α-hyperbounded (resp., α-bounded) space, up to topological equivalence, with the property that for all α-hyperbounded (resp., α-bounded) space Y and every $f \in C(X,Y)$, there exists a continuous extension $F \in C(\beta_{h\alpha}(X), Y)$ (resp., $F \in C(\beta_\alpha(X), Y)$) such that $F|_X = f$. It is known that such extension can be considered as the intersection of all α-hyperbounded (resp., α-bounded) subspaces of $\beta(X)$ containing X (see Chapter 5 of [13]). We finish this section by providing an example of a space X for which $X \subsetneq \beta_\alpha(X) \subsetneq \beta_{h\alpha}(X) \subsetneq \beta(X)$ (see Theorem 2.11 below). In order to obtain this example we need to recall the following known results.

Lemma 2.9. (Theorem 1 in [9]) *Let $\{X_s : s \in S\}$ be a family of spaces such that for all $s_0 \in S$, $\prod_{s\in S\setminus\{s_0\}} X_s$ is infinite. Then*

$$\beta(\prod_{s\in S} X_s) = \prod_{s\in S} \beta(X_s)$$

if and only if $\prod_{s\in S} X_s$ is a pseudocompact space.

We say that a topological property P is a *Tychonoff extension property* (see 5.6 in [13]) if it is inherited by closed subsets, productive and every compact space has property P.

Lemma 2.10. (Theorem 3.2 in [5]) *Let P be a Tychonoff extension property which implies pseudocompactness. Then the following statements are equivalent for any pair of spaces X and Y.*

(1) *$X \times Y$ is pseudocompact.*

(2) *$\beta_P(X \times Y) = \beta_P(X) \times \beta_P(Y)$.*

Theorem 2.11. *For each infinite cardinal α there exists a space X such that $X \subsetneq \beta_\alpha(X) \subsetneq \beta_{h\alpha}(X) \subsetneq \beta(X)$.*

Proof. Let κ and τ be infinite cardinals such that $max\{\omega_1, \alpha\} < \kappa < \tau$, define $Y = \Sigma_{\kappa^+}(\underline{0}, \{0,1\}^\tau)$ and $Z = Y \setminus \{\underline{0}\}$. In Theorem 2.8 we showed that Y is an α-bounded and non-α-hyperbounded space. Furthermore, it is easy to check that $[0, \alpha^+)$ is α-hyperbounded, and Z is not α-bounded. Let $S \in [\tau]^{\omega_1}$, and $p \in \{0,1\}^S$

be such that $\mathrm{supp}_{\underline{0}}(p) = S$. The Σ-product $\Sigma(p) = \Sigma_{\omega_1}(p, \{0,1\}^\tau)$ is a subspace of Z, and by a theorem of Glicksberg (Theorem 2 in [9]), $\beta(\Sigma(p)) = \{0,1\}^\tau$. Also, since $\Sigma(p)$ and $[0,\alpha^+)$ are ω-bounded, so is $\Sigma(p) \times [0,\alpha^+)$, thus, $\Sigma(p) \times [0,\alpha^+)$ is pseudocompact. Then, using Lemma 2.9 for $X = Z \times [0,\alpha^+)$, we have:

$$\beta(X) \subseteq \beta(Z) \times \beta([0,\alpha^+)) \subseteq \{0,1\}^\tau \times [0,\alpha^+] = \beta(\Sigma(p) \times [0,\alpha^+)) \subseteq \beta(X),$$

therefore, $\beta(X) = \beta(Z) \times \beta([0,\alpha^+)) = \{0,1\}^\tau \times [0,\alpha^+]$, so, by Lemma 2.9, X is pseudocompact. Then, by Lemma 2.10, and since $Y = Z \cup \{\underline{0}\}$ is the smallest α-bounded superset of Z, we get

$$\beta_\alpha(X) = \beta_\alpha(Z) \times \beta_\alpha([0,\alpha^+)) = Y \times [0,\alpha^+).$$

Since every α-hyperbounded space is α-bounded, $\beta_\alpha(X) \subseteq \beta_{h\alpha}(X)$, moreover, $\beta_\alpha(X) \neq \beta_{h\alpha}(X)$ (otherwise, $\beta_\alpha(X) = Y \times [0,\alpha^+)$ is α-hyperbounded, which implies that Y is α-hyperbounded). In conclusion, observe that

$$X \subsetneq \beta_\alpha(X) \subsetneq \beta_{h\alpha}(X) \subseteq \beta_{h\alpha}(Z) \times [0,\alpha^+) \subsetneq \{0,1\}^\tau \times [0,\alpha^+] = \beta(X). \quad \blacksquare$$

Note that if $\alpha < \kappa$ are infinite cardinals, and X is the space from Theorem 2.8, then $\beta_\kappa(\beta_{h\alpha}(X)) = \beta_{h\alpha}(\beta_\kappa(X))$: indeed, $\beta_\kappa(X) = X$, furthermore, $\beta_{h\alpha}(X)$ contains a dense σ-compact subspace, so by Proposition 2.7, $\beta_{h\alpha}(X)$ is compact, so $\beta_{h\alpha}(\beta_\kappa(X)) = \beta_{h\alpha}(X) = \beta(X)$; on the other side, $\beta_\kappa(\beta_{h\alpha}(X)) = \beta_\kappa(\beta(X)) = \beta(X)$. Moreover, if we take Z from Theorem 2.11, and define $X = Z \times [0,\kappa^+)$, we can argue analogously as in Theorem 2.11 that $\beta_\kappa(X) = Y \times [0,\kappa^+)$, and that it is pseudocompact, so using Lemma 2.10 and the above argument for the space from Theorem 2.8, we get $\beta_{h\alpha}(\beta_\kappa(X)) = \beta_{h\alpha}(Y) \times \beta_{h\alpha}([0,\kappa^+)) = \beta(Y) \times [0,\kappa^+) = \{0,1\}^\tau \times [0,\kappa^+)$; similarly, $\beta_\kappa(\beta_{h\alpha}(X)) = \beta_\kappa(\beta_{h\alpha}(Z) \times \beta_{h\alpha}([0,\kappa^+))) = \beta_\kappa(\beta(Z) \times [0,\kappa^+)) = \beta_\kappa(\{0,1\}^\tau \times [0,\kappa^+)) = \{0,1\}^\tau \times [0,\kappa^+)$, thus, again $\beta_\kappa(\beta_{h\alpha}(X)) = \beta_{h\alpha}(\beta_\kappa(X))$.

So, the following question arises:

Question 2.12. Are there two infinite cardinals α and κ with $\alpha < \kappa$ and a space X such that $\beta_\kappa(\beta_{h\alpha}(X)) \neq \beta_{h\alpha}(\beta_\kappa(X))$?

3. Pseudocompactness of $\mathcal{K}(X)$ and consequences when $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$.

In this section we are going to give some equivalent conditions to that of pseudocompactness in $\mathcal{K}(X)$. Also we are going to obtain some results about $\mathcal{K}(X)$ and X when $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$. We begin with some definitions:

Definition 3.1. Let X be a space and let $p \in \mathcal{D} \subseteq \mathbb{N}^*$. We say that:

(1) X is *p-pseudocompact* if for each sequence $(U_n)_{n\in\mathbb{N}}$ of non-empty open subsets of X, $L(p,(U_n)_{n\in\mathbb{N}}) \neq \emptyset$.

(2) X is *strongly p-pseudocompact* if for each sequence $(U_n)_{n\in\mathbb{N}}$ of non-empty open subsets of X there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ and there exists $z \in X$ such that, for each $n \in \mathbb{N}$, $x_n \in U_n$ and $z = p-\lim\ x_n$.

(3) X is *pseudo-$\mathcal{D}$-bounded* if for each sequence $(U_n)_{n\in\mathbb{N}}$ of non-empty open subsets of X there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, $x_n \in U_n$ and for each $p \in \mathcal{D}$, there exists $z_p \in X$ such that $z_p = p - \lim\ x_n$.

(4) X is pseudo-ω-bounded if for each countable family $\mathcal{U}$ of non-empty open subsets of X, there exists a compact set $K \subseteq X$ such that, for each $U \in \mathcal{U}$, $K \cap U \neq \emptyset$.

It is clear that the property of pseudo-ω-boundedness is equal to the property of pseudo-$\mathbb{N}^*$-boundedness and, in general, the property of pseudo-$\mathcal{D}$-boundedness is stronger than that of strongly p-pseudocompactness and weaker than pseudo-ω-boundedness.

Now we determine when $\mathcal{K}(X)$ is pseudocompact in terms of some properties of X.

Theorem 3.2. *Let X be a space. Then the following statements are equivalent:*

(1) *X is pseudo-ω-bounded;*

(2) *$\mathcal{K}(X)$ is pseudo-ω-bounded;*

(3) *$\mathcal{K}(X)$ is pseudo-$\mathcal{D}$-bounded for some $\mathcal{D} \subseteq \mathbb{N}^*$;*

(4) *$\mathcal{K}(X)$ is strongly-p-pseudocompact for some $p \in \mathbb{N}^*$;*

(5) *$\mathcal{K}(X)$ is p-pseudocompact for some $p \in \mathbb{N}^*$; and*

(6) *$\mathcal{K}(X)$ is pseudocompact.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are easy to see (cf. Section 3 of [2]).

(1) $\Rightarrow$ (2). Suppose that X is pseudo-ω-bounded and let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of non-empty canonical open sets of $\mathcal{K}(X)$; that is, for each $n \in \mathbb{N}$, the open set $\mathcal{U}_n$ is of the form $\langle U_1^n, \ldots, U_{k_n}^n \rangle$ where U_j^n is a non-empty open subset of X. Then the set

$$\{U_j^n : n \in \mathbb{N} \ \text{ and } \ 1 \leq j \leq k_n\}$$

is a countable family of non-empty open subsets of X. So, there exists a compact set C in X such that $C \cap U_j^n \neq \emptyset$ for every $n \in \mathbb{N}$, $1 \leq j \leq k_n$. Because of C is compact in X, $\mathcal{K}(C)$ is a compact subspace of $\mathcal{K}(X)$; so, it is enough to show that, for each $n \in \mathbb{N}$, $\mathcal{U}_n \cap \mathcal{K}(C) \neq \emptyset$. For each $n \in \mathbb{N}$ and $1 \leq j \leq k_n$, take $x_j^n \in C \cap U_j^n$. Take $F_n = \{x_1^n, \ldots, x_{k_n}^n\}$. Then, it is clear that, for every $n \in \mathbb{N}$, $F_n \in \mathcal{K}(C) \cap \mathcal{U}_n$. Therefore, $\mathcal{K}(X)$ is pseudo-ω-bounded.

(6) $\Rightarrow$ (1). Suppose that $\mathcal{K}(X)$ is pseudocompact and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty open subsets of X. For each $n \in \mathbb{N}$, let V_n be a non-empty open subset of X such that $Cl(V_n) \subseteq U_n$ and let $\mathcal{V}_n = \langle V_1, \ldots, V_n \rangle$. It is enough to show that there exists a compact space T such that, for each $n \in \mathbb{N}$, $Cl(V_n) \cap T \neq \emptyset$. Since $\mathcal{K}(X)$ is a pseudocompact space, there exists $T \in \mathcal{K}(X)$ such that T is a cluster point of the sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$. Suppose that there exists a natural number $n \in \mathbb{N}$ such that $T \cap Cl(V_n) = \emptyset$. Then, there exists an open set W such that $T \subseteq W$ and $W \cap Cl(V_n) = \emptyset$. So $T \in W^+$ and, for each $m \geq n$, $\mathcal{V}_m \cap W^+ = \emptyset$. This last equality contradicts the fact that T is a cluster point of the sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$. ■

Below, we are going to obtain some interesting consequences on X and $\mathcal{K}(X)$ when $\mathcal{K}(X)$ is normal. First some known results:

Theorem 3.3. (Theorem 4.9.13 in [11]) *A space X is metrizable if and only if $\mathcal{K}(X)$ is metrizable.*

Lemma 3.4. *Let η be an ordinal number with $cf(\eta) = \omega$. Then $\mathcal{K}(\eta)$ is σ-compact.*

Proof. Let $(\xi_n)_{n<\omega}$ be an increasing and cofinal sequence of ordinal numbers in η. Then $\mathcal{K}(\eta)$ is equal to $\bigcup_{n<\omega} \mathcal{K}(\xi_n + 1)$. For each $n < \omega$, $\mathcal{K}(\xi_n + 1)$ is compact, so $\mathcal{K}(\eta)$ is σ-compact. ∎

Theorem 3.5. (Corollary 10 in [10]) *Let η be an ordinal number. Then $\mathcal{K}(\eta)$ is normal if and only if $cf(\eta) \in \{1, \omega, \eta\}$.*

On the other hand, G. Artico and R. Moresco show in [3] that if X is the Sorgenfrey line then the hyperspace $\mathcal{K}(X)$ is not normal. Remember that the Sorgenfrey line is a submetrizable, perfectly normal and hereditarily Lindelöf space.

We say that a space X is *$<\alpha$-bounded* if X is θ-bounded for each infinite cardinal $\theta < \alpha$. Recall that a space X is *α-pseudocompact* if $f[X]$ is a compact subset of $\mathbb{R}^\alpha$ for every continuous function $f : X \to \mathbb{R}^\alpha$. ω-pseudocompactness is equivalent to pseudocompactnes, and if $\theta < \alpha$, every α-pseudocompact space is θ-pseudocompact. Moreover, every initially α-compact space is α-pseudocompact.

Lemma 3.6. (Theorem 1.7 in [8]) *Let α be an infinite cardinal. Then every α-pseudocompact, $<\alpha$-bounded and normal space is initially α-compact.*

Theorem 3.7. *Let α be an infinite cardinal number and let X be a topological space such that $\mathcal{K}(X)$ is normal. Then $\mathcal{K}(X)$ is α-pseudocompact if and only if X is α-hyperbounded.*

Proof. We will prove this theorem by transfinite induction. It is known that a normal pseudocompact space is countably compact, so by Theorem 2.3, X is ω-hyperbounded. Now, suppose that $\mathcal{K}(X)$ is α-pseudocompact and suppose that our statement is true for every $\theta < \alpha$. Then, by Theorem 2.6, $\mathcal{K}(X)$ is $<\alpha$-bounded. By Lemma 3.6, $\mathcal{K}(X)$ is initially α-compact and again, by Theorem 2.6, X is α-hyperbounded. ∎

Corollary 3.8. *Let X be the space defined in Theorem 2.8. Then X is a normal space such that $\mathcal{K}(X)$ is not normal.*

Proof. From Theorem 2.8, we know that X is ω-bounded. So, X is pseudo-ω-bounded. Then, by Theorem 3.2, $\mathcal{K}(X)$ is pseudocompact. If $\mathcal{K}(X)$ is a normal space then by Theorem 3.7, X is ω-hyperbounded. But, by Theorem 2.8, the last statement is not true. ∎

Theorem 3.9. (R. Hernández) *Let X be a space. If $\mathcal{K}(X)$ is a normal space which is C^*-embedded in $\mathcal{CL}(X)$ then X is ω-hyperbounded.*

Proof. Suppose that X is not ω-hyperbounded. Let $\{S_n : n \in \mathbb{N}\}$ be a collection of compact subsets of X such that $Cl_X(S)$ is not a compact set, where $S = \bigcup_{n\in\mathbb{N}} S_n$. We will define a strictly increasing sequence $(A_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $\bigcup_{n\in\mathbb{N}} A_n = S$ and $A_k \subsetneq A_{k+1} \subsetneq S$ for each $k \in \mathbb{N}$. We take $A_1 = S_1$. Suppose we have defined all compact sets $A_1, \dots, A_m$ such that, for each $1 \le k < m$, $A_k \subsetneq A_{k+1} \subsetneq S$. Since $Cl_X(S)$ is not a compact set of X, there exists $N(m) > m$ such that $S_{N(m)} \setminus A_m \neq \emptyset$. Let $A_{m+1} = A_m \cup (\bigcup_{n\le N(m)} S_n)$. Then the sequence $(A_n)_{n\in\mathbb{N}}$ satisfies our conditions. By Lemma 2.4, the sequence $(A_n)_{n\in\mathbb{N}}$ converges to $Cl_X(S)$ in $\mathcal{CL}(X)$. Thus, $\{A_n : n \in \mathbb{N}\}\bigcup\{Cl_X(S)\}$ is a compact subspace of $\mathcal{CL}(X)$. Since $Cl_X(S)$ is the only cluster point of the set $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ in $\mathcal{CL}(X)$, $\mathcal{A}$ is a discrete subset of $\mathcal{CL}(X)$. Moreover, $Cl_X(S) \in \mathcal{CL}(X) \setminus \mathcal{K}(X)$, hence $\mathcal{A}$ is closed and discrete in $\mathcal{K}(X)$. On the other hand, $\mathcal{K}(X)$ is a normal space, then $\{A_n : n \in \mathbb{N}\}$ is C^*-embedded in $\mathcal{K}(X)$. But $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$, so $\{A_n : n \in \mathbb{N}\}$ is C^*-embedded in $\mathcal{CL}(X)$. This last assertion contradicts the fact that $(A_n)_{n\in\mathbb{N}}$ is a non-trivial convergent sequence in $\mathcal{CL}(X)$. Therefore, we have to conclude that X is ω-hyperbounded. ■

Recall that a point x in a space X is a *butterfly* point of X ([14]) if there are two disjoint subsets A and B of $X \setminus \{x\}$ such that $\{x\} = Cl_X(A) \cap Cl_X(B)$. These kind of points have been useful in solving problems about normality, as we can see in the following known result.

Lemma 3.10. *Let Y be a C^*-embedded subspace of X. If there is a point $x \in X \setminus Y$ which is a butterfly point of $Y \cup \{x\}$ then Y is not normal.*

Lemma 3.11. (Theorem 2.3 in [15]) *Let X be a space and let κ be a singular cardinal number. If X is an initially θ-compact space for every cardinal number $\theta < \kappa$, then X is an initially κ-compact space.*

Theorem 3.12. *Let α be an infinite cardinal number. Let X be a space such that $\mathcal{K}(X)$ is normal and C^*-embedded in $\mathcal{CL}(X)$. Then, X is α-bounded if and only if X is α-hyperbounded.*

Proof. Of course, if X is α-hyperbounded then X is α-bounded. Now, suppose that X is α-bounded and assume that X is not α-hyperbounded. Let τ be the minimum cardinal κ such that X is not κ-hyperbounded. By Lemma 3.9, $\tau > \omega$ and by Theorem 2.6 and Lemma 3.11, τ is regular. Note that X is $< \tau$-hyperbounded and $\tau \le \alpha$. Since X is not τ-hyperbounded, there exists a transfinite sequence $(S_\xi)_{\xi<\tau}$ of non-empty compact subspaces of X such that $Cl_X(S)$ is a non-compact space, where $S = \bigcup_{\xi<\tau} S_\xi$. We construct two sets $\mathcal{B}$ and $\mathcal{C}$ contained in $\mathcal{K}(X)$ with disjoint closures in $\mathcal{K}(X)$ each of them converging to $Cl_X(S)$ in $\mathcal{CL}(X)$. For each $\xi < \tau$, let $A_\xi = \bigcup_{\zeta\le\xi} S_\xi$ and $B_\xi = Cl_X(A_\xi)$. For each $\xi < \tau$, X is $|\xi|$-hyperbounded, so $B_\xi \in \mathcal{K}(X)$. Thus, for each $\xi < \tau$, $B_\xi \in \mathcal{K}(X)$. Moreover, for each $\xi < \tau$, there exists a point $x_\xi \in S \setminus B_\xi$. Let $R = Cl_X(\{x_\xi : \xi < \tau\})$. Since X is α-bounded, R is a compact subspace of X. For each $\xi < \tau$, let $C_\xi = B_\xi \cup R$. Then, for each $\xi < \tau$, $C_\xi \in \mathcal{K}(X)$. Let $\mathcal{B} = \{B_\xi : \xi < \tau\}$ and $\mathcal{C} = \{C_\xi : \xi < \tau\}$. Then $\mathcal{B}$, $\mathcal{C}$ are clearly disjoint subsets of $\mathcal{K}(X)$ and, by Lemma 2.4, they both converge to

$Cl_X(S) \in \mathcal{CL}(X) \setminus \mathcal{K}(X)$. It follows that $\{Cl_X(S)\} = CL_{\mathcal{CL}(X)}(\mathcal{B}) \cap CL_{\mathcal{CL}(X)}(\mathcal{C})$, so $Cl_X(S)$ is a butterfly point of $\mathcal{CL}(X)$ and, by Lemma 3.10, $\mathcal{K}(X)$ cannot be normal. This contradiction says that X must be α-hyperbounded.

The transfinite sequences $\mathcal{B}$ and $\mathcal{C}$ converge to $Cl_X(S)$.

Claim 1. $Cl_{\mathcal{K}(X)}(\mathcal{B}) \cap Cl_{\mathcal{K}(X)}(\mathcal{C}) = \emptyset$.

Let $T \in Cl_{\mathcal{K}(X)}(\mathcal{B})$. Since T is a compact space, we can take the minimum ordinal of the set $\{\xi < \tau : B_\xi \nsubseteq T\}$. Call this number η. Let $x \in B_\eta \setminus T$ and let U_1 be an open set of X such that $T \subseteq U_1$ and $x \notin U_1$. Of course, for each $\xi < \eta$, $B_\xi \subseteq T$.

Claim 2. $T = Cl_X(\bigcup_{\xi<\tau} B_\eta)$

Indeed, assume the contrary, so there exists a point $y \in T \setminus Cl_X(\bigcup_{\xi<\eta} B_\xi)$ and an open set U_2 of X such that $y \in U_2$ and $U_2 \cap Cl_X(\bigcup_{\xi<\eta} B_\xi) = \emptyset$. But this is not possible because $T \in U_1^+ \cap U_2^-$ and $(U_1^+ \cap U_2^-) \cap \mathcal{B} = \emptyset$. Then, $T \subseteq B_\eta$ and our Claim 2 is proved.

As $x_\eta \notin B_\eta$, there exists an open subset U_3 of X such that $B_\eta \subseteq U_3$ and $x_\eta \notin U_3$. It is apparent that $T \in U_3^+$ and $U_3^+ \cap \mathcal{C} = \emptyset$, so the proof of our claim is complete.

By Claim 1, $\{Cl_X(S)\} = Cl_{\mathcal{CL}(X)}(\mathcal{B}) \cap Cl_{\mathcal{CL}(X)}(\mathcal{C})$. Then $Cl_X(S)$ is a butterfly point of $\mathcal{CL}(X)$. By Lemma 3.10, $\mathcal{K}(X)$ cannot be normal. This contradiction says that X must be α-hyperbounded. ∎

Theorem 3.13. *Let X be a space such that $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$. Then the next statements are equivalent:*

(1) *X is compact,*
(2) *X is σ-compact,*
(3) *$\mathcal{K}(X)$ is compact,*
(4) *$\mathcal{K}(X)$ is σ-compact,*
(5) *$\mathcal{K}(X)$ is Lindelöf, and*
(6) *$\mathcal{K}(X)$ is paracompact.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) and (1) $\Leftrightarrow$ (3) are well known. Suppose that $\mathcal{K}(X)$ is a paracompact space, then $\mathcal{K}(X)$ is normal. By Theorems 3.9 and 2.6, $\mathcal{K}(X)$ is countably compact. It is known that every paracompact and countably compact space is compact. So $\mathcal{K}(X)$ is a compact space and then X is compact. ∎

Corollary 3.14. *Let X be a metrizable space. Then $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$ if and only if X is a compact space.*

Proof. Let X be a metrizable space. By Theorem 3.3, $\mathcal{K}(X)$ is metrizable. It is clear that if X is a compact space, then $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$. On the other hand, if $\mathcal{K}(X)$ is C^*-embedded in $\mathcal{CL}(X)$, then, by Theorem 3.13, X is a compact space because every metrizable space is a paracompact space. ∎

Corollary 3.15. *For every infinite discrete space X, $\mathcal{K}(X)$ is not C^*-embedded in $\mathcal{CL}(X)$.*

Corollary 3.16. *Let η be an ordinal such that $cf(\eta) = \omega$. Then $\mathcal{K}([0,\eta))$ is not C^*-embedded in $\mathcal{CL}([0,\eta))$.*

Proof. Follows from Lemma 3.4 and Theorem 3.13. ∎

Question 3.17. For which ordinal numbers η, $\mathcal{K}([0,\eta))$ is C^*-embedded in $\mathcal{CL}([0,\eta))$?

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