

ON VITALI SETS AND THEIR UNIONS

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Abstract. It is well known that any Vitali set on the real line \mathbb{R} does not possess the Baire property. In this article we prove the following:

Let S be a Vitali set, S_r be the image of S under the translation of \mathbb{R} by a rational number r and $\mathcal{F} = \{S_r : r \text{ is rational}\}$. Then for each non-empty proper subfamily \mathcal{F}' of \mathcal{F} the union $\cup \mathcal{F}'$ does not possess the Baire property.

Our starting point is the classical theorem by Vitali [3] stating the existence of subsets of the real line \mathbb{R} , called below Vitali sets, which are non-Lebesgue measurable. Recall that if one considers a Vitali set S on the real line \mathbb{R} then each its image S_x under the translation of \mathbb{R} by a real number x is also a Vitali set. Moreover, the family $\mathcal{F} = \{S_r : r \text{ is rational}\}$ is disjoint and its union is \mathbb{R} . Such decompositions of the real line \mathbb{R} are used in measure theory to show the nonexistence of a certain type of measure.

Recall that a subset A of \mathbb{R} possesses *the Baire property* if and only if there is an open set O and two meager sets M, N such that $A = (O \setminus M) \cup N$. Note that the Vitali sets do not possess the Baire property. This can be easily seen from the following known properties of Vitali sets:

- (A) each Vitali set is not meager;
- (B) each Vitali set does not contain the set $O \setminus M$ for any non-empty open subset O of \mathbb{R} and any meager set M .

Really, if a Vitali set S possesses the Baire property then there is an open set O and two meager sets M, N such that $S = (O \setminus M) \cup N$. If $O = \emptyset$ then $S = N$, i.e. S is meager. But this is impossible by (A). So $O \neq \emptyset$ and $S \supset O \setminus M$. However this inclusion is impossible by (B). Hence S does not possess the Baire property. The first proof of (B) belongs to Vitali (cf. [2]), another one the reader can find in Proposition 3.1.

Let us say now that a subset A of \mathbb{R} possesses *Vitali property* if there exist a non-empty open set O and a meager set M such that $A \supset O \setminus M$.

We continue with the following question.

For what proper subfamilies \mathcal{F}' of \mathcal{F} do the unions $\bigcup \mathcal{F}'$ not possess the Baire property (resp. the Vitali property)?

Here we present characterizations for countable union of nonmeager sets of a hereditarily Lindelöf space R :

- (1) to be the union $(O \setminus M) \cup N$, where O is an open set and M, N are meager sets of R (Theorem 2.1); as a corollary we show that for each non-empty proper subfamily \mathcal{F}' of \mathcal{F} the union $\bigcup \mathcal{F}'$ does not possess the Baire property (see Theorem 3.1); and
- (2) to have a complement in R which contains the set $O \setminus M$, where O is a non-empty open set and M is a meager set of R (Theorem 2.2); as a corollary we get a characterization of those non-empty proper subfamilies \mathcal{F}' of \mathcal{F} which unions $\bigcup \mathcal{F}'$ possess the Vitali property (see Theorem 3.2);

1. Preliminary notations

Let \mathbb{R} be the real line. For $x \in \mathbb{R}$ denote by T_x the translation of \mathbb{R} by x , i.e. $T_x(y) = y + x$ for each $y \in \mathbb{R}$.

The equivalence relation E on \mathbb{R} is defined as follows. For $x, y \in \mathbb{R}$, let xEy if and only if $x - y \in \mathbb{Q}$, where \mathbb{Q} is the set of rational numbers. Let us denote the equivalence classes by $E_\alpha, \alpha \in I$. Note that $|I| = c$. It is evident that for each $\alpha \in I$ and each $x \in E_\alpha$, $E_\alpha = T_x(\mathbb{Q})$. Hence every equivalence class E_α is dense in \mathbb{R} .

A *Vitali set* here is any subset S of \mathbb{R} such that $|S \cap E_\alpha| = 1$ for each $\alpha \in I$.

Let S be a Vitali set. For each $x \in \mathbb{R}$ put $S_x = T_x(S)$. Note that S_x is a Vitali set. Observe also that $S_0 = S$ and $S_{x_2} = S_{x_2 - x_1}(S_{x_1})$ for any $x_1, x_2 \in \mathbb{R}$. Moreover, if $r_1, r_2 \in \mathbb{Q}$ and $r_1 \neq r_2$ then $S_{r_1} \cap S_{r_2} = \emptyset$ and

$$\bigcup_{r \in \mathbb{Q}} S_r = \mathbb{R}. \quad (*)$$

Recall also that a subset A of a topological space R is called *meager in R* if A is the union of countably many nowhere dense in R sets, otherwise A is called *nonmeager in R* . It is evident that if $B \subset A \subset R$ then B is meager in R whenever A is meager in R , and A is nonmeager in R whenever B is nonmeager in R .

Our terminology follows [1] and [2].

2. Countable unions of nonmeager sets in a hereditarily Lindelöf space

Let R be a topological space.

DEFINITION 2.1. Let X be a nonmeager subset in R . Put $O_X = \text{Int}_R(\text{Cl}_R(X))$. Note that $O_X \neq \emptyset$. Define further

$X' = \{x \in X \cap O_X : \text{there is an open nbd } V \text{ of } x \text{ such that } V \cap X \text{ is meager}\}$
and $X'' = (X \cap O_X) \setminus X'$.

LEMMA 2.1. *Let X be a nonmeager set in R . Then the set $Y = X \cap (R \setminus \text{Cl}_R O_X)$ is nowhere dense in R .*

Proof. Since $Y \subset X$ (resp. $Y \subset R \setminus \text{Cl}_R O_X$), we have $\text{Int}_R(\text{Cl}_R(Y)) \subset O_X$ (resp. $\text{Int}_R(\text{Cl}_R(Y)) \subset R \setminus O_X$). Hence, $\text{Int}_R(\text{Cl}_R(Y)) = \emptyset$. ■

PROPOSITION 2.1. *Let X be a nonmeager set in R and X' be Lindelöf. Then*

- (i) *the set X' is meager;*
- (ii) *the set $X \setminus X''$ is meager and $X'' \neq \emptyset$;*
- (iii) *for any open subset V of R such that $V \cap X'' \neq \emptyset$, the set $V \cap X''$ is nonmeager;*
In particular, the set $X'' = X'' \cap R$ is nonmeager and so $O_{X''} \neq \emptyset$.

Proof. (i) For each $x \in X'$ there is an open neighbourhood V_x of x such that $V_x \cap X$ is meager. Note that $X' = \bigcup \{V_x \cap X' : x \in X'\}$. Since X' is Lindelöf, there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of points of X' such that $\bigcup \{V_x \cap X' : x \in X'\} = \bigcup_{i=1}^{\infty} (V_{x_i} \cap X')$. Note that the set $V_{x_i} \cap X' \subset V_{x_i} \cap X$ is meager for each i . So the set X' is also meager.

(ii) Let us note that $X \setminus X'' = (X \cap (R \setminus \text{Cl}_R O_X)) \cup (X \cap \text{Bd}_R O_X) \cup X'$ and each term of the union is meager. Hence, the union $X \setminus X''$ is meager. Since X is nonmeager, we have $X'' \neq \emptyset$.

(iii) Since $V \cap X'' \neq \emptyset$, the set $V \cap X = V \cap (X'' \cup (X \setminus X'')) = (V \cap X'') \cup (V \cap (X \setminus X''))$ is nonmeager. By (ii) the set $V \cap (X \setminus X'')$ is meager. Hence, the set $(V \cap X'')$ is nonmeager. ■

COROLLARY 2.1. *Let X be a nonmeager set in R and X' be Lindelöf. Then*

- (i) *$X'' \subset \text{Cl}_R(O_{X''})$;*
- (ii) *the set $X \setminus O_{X''}$ is meager;*
- (iii) *for any non-empty open subset V of $O_{X''}$ the set $V \cap X''$ is nonmeager.*

Proof. Let us note that by Proposition 2.1 (iii) the set X'' is nonmeager and $O_{X''} \neq \emptyset$.

(i) Assume that $Y = X'' \setminus \text{Cl}_R(O_{X''}) = X'' \cap (R \setminus \text{Cl}_R(O_{X''})) \neq \emptyset$. It follows from Lemma 2.1 that the set Y is nowhere dense in R . Observe also that the set $R \setminus \text{Cl}_R(O_{X''})$ is open. Hence by Proposition 2.1 (iii), Y must be nonmeager. This is a contradiction.

(ii) By (i) we have $X \cap (R \setminus \text{Cl}_R(O_{X''})) \subset X \setminus X''$. Now it follows from Proposition 2.1 (ii) that $X \cap (R \setminus \text{Cl}_R(O_{X''}))$ is meager. Hence, the set $X \setminus O_{X''} = (X \cap (R \setminus \text{Cl}_R(O_{X''}))) \cup (X \cap \text{Bd}_R(O_{X''}))$ is also meager.

(iii) Since $V \cap X'' \neq \emptyset$, it follows from Proposition 2.1 (iii) that $V \cap X''$ is nonmeager. ■

THEOREM 2.1. *Let R be a hereditarily Lindelöf topological space, \mathcal{A} be a non-empty set with $|\mathcal{A}| \leq \aleph_0$ and $X(\alpha)$ be nonmeager in R for each $\alpha \in \mathcal{A}$. Then the*

set $U = \bigcup\{(X(\alpha)) : \alpha \in \mathcal{A}\}$ is equal to the union $(O \setminus M) \cup N$ for some open set O and some meager sets M, N iff $O_{X''(\alpha)} \setminus U$ is meager for each $\alpha \in \mathcal{A}$.

Proof. “ \Rightarrow ”. Let us assume that $U = (O \setminus M) \cup N$ for some open set O and some meager sets M, N , and the set $A(\alpha^*) = O_{X''(\alpha^*)} \setminus U$ is nonmeager for some $\alpha^* \in \mathcal{A}$. Since U is a nonmeager set, we have $O \neq \emptyset$.

It follows from Proposition 2.1 (iii) that $A''(\alpha^*)$ is nonmeager and so the set $V = O_{A''(\alpha^*)}$ is non-empty. Since $A''(\alpha^*) \subset A(\alpha^*) \subset O_{X''(\alpha^*)}$, we have $V = \text{Int}_R(\text{Cl}_R(A''(\alpha^*))) \subset \text{Int}_R(\text{Cl}_R(O_{X''(\alpha^*)}))$. But $\text{Int}_R(\text{Cl}_R(O_{X''(\alpha^*)})) = \text{Int}_R(\text{Cl}_R(\text{Int}_R(\text{Cl}_R(X''(\alpha^*)))))) = \text{Int}_R(\text{Cl}_R(X''(\alpha^*))) = O_{X''(\alpha^*)}$. So we get that $V \subset O_{X''(\alpha^*)}$. It follows from Corollary 2.1 (iii) that the set $X''(\alpha^*) \cap V$ is nonmeager.

Put $W = O \cap V$. Note that $W \neq \emptyset$. Really, if $O \cap V = \emptyset$ then we have that $X''(\alpha^*) \cap V \subset X(\alpha^*) \cap V \subset U \cap V = ((O \setminus M) \cup N) \cap V \subset N$. Recall that the set N is meager and the set $X''(\alpha^*) \cap V$ is nonmeager. This is a contradiction.

Since W is a non-empty subset of V , it follows from Corollary 2.1 (iii) that the set $A''(\alpha^*) \cap W$ is nonmeager. Note that $A''(\alpha^*) \subset A(\alpha^*) \subset R \setminus U$ and $W \subset O$. So $A''(\alpha^*) \cap W \subset O \setminus U = O \setminus ((O \setminus M) \cup N) \subset M$. Recall that the set M is meager. Consequently, the set $A''(\alpha^*) \cap W$ is also meager. This is again a contradiction. The necessity is proved.

“ \Leftarrow ”. Let us put $O = \bigcup\{O_{X''(\alpha)} : \alpha \in \mathcal{A}\}$, $M = \bigcup\{O_{X''(\alpha)} \setminus U : \alpha \in \mathcal{A}\}$ and $N = \bigcup\{X(\alpha) \setminus O_{X''(\alpha)} : \alpha \in \mathcal{A}\}$. Note that the set O is non-empty, open and M, N are meager sets (see Corollary 2.1 (ii) for N). Observe also that $U = U \cap R = (U \cap O) \cup ((R \setminus O) \cap U)$, $U \cap O = O \setminus (O \setminus U)$, $M_1 = O \setminus U \subset M$, $N_1 = (R \setminus O) \cap U \subset N$. Thus $U = (O \setminus M_1) \cup N_1$, where the set O is non-empty and open, and the sets M_1, N_1 are meager. The sufficiency is also proved. ■

THEOREM 3.2. *Let R be a hereditarily Lindelöf topological space, \mathcal{A} be a non-empty set with $|\mathcal{A}| \leq \aleph_0$ and $X(\alpha)$ be nonmeager in R for each $\alpha \in \mathcal{A}$. Then the set $R \setminus (\bigcup\{X(\alpha) : \alpha \in \mathcal{A}\})$ contains the difference $O \setminus M$ for some non-empty open set O and some meager set M if and only if $\text{Cl}_R(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \neq R$.*

Proof. Let us put $U = \bigcup\{(X(\alpha)) : \alpha \in \mathcal{A}\}$.

“ \Rightarrow ”. Assume that $R \setminus U$ contains the set $O \setminus M$, where O is a non-empty open subset of R and M is a meager set.

So $\bigcup\{X(\alpha) \cap O : \alpha \in \mathcal{A}\} = (\bigcup\{X(\alpha) : \alpha \in \mathcal{A}\}) \cap O = U \cap O \subset M$. Note that for each $\alpha \in \mathcal{A}$ we have $X''(\alpha) \cap O \subset X(\alpha) \cap O \subset U \cap O$. Since M is meager, for each $\alpha \in \mathcal{A}$ the set $X''(\alpha) \cap O$ must be meager. But it follows from Proposition 2.1 (iii) that $\bigcup\{X''(\alpha) \cap O : \alpha \in \mathcal{A}\} = (\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \cap O = \emptyset$.

So $\text{Cl}_R(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \neq R$. The necessity is proved.

“ \Leftarrow ”. Note that

$$\begin{aligned} R \setminus U &= R \setminus \bigcup\{X(\alpha) : \alpha \in \mathcal{A}\} = R \setminus \bigcup\{X''(\alpha) \cup (X(\alpha) \setminus X''(\alpha)) : \alpha \in \mathcal{A}\} = \\ &= R \setminus ((\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \cup (\bigcup\{X(\alpha) \setminus X''(\alpha) : \alpha \in \mathcal{A}\})) = \end{aligned}$$

$$(R \setminus \bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \setminus \bigcup\{X(\alpha) \setminus X''(\alpha) : \alpha \in \mathcal{A}\} \supset \\ (R \setminus \text{Cl}_{\mathbb{R}}(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\})) \setminus \bigcup\{X(\alpha) \setminus X''(\alpha) : \alpha \in \mathcal{A}\}.$$

Since $\text{Cl}_{\mathbb{R}}(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \neq R$, the set $O = R \setminus \text{Cl}_{\mathbb{R}}(\bigcup\{X''(\alpha) : \alpha \in \mathcal{A}\}) \neq \emptyset$. Put $M = \bigcup\{X \setminus X''(\alpha) : \alpha \in \mathcal{A}\}$. Since $|\mathcal{A}| \leq \aleph_0$ and for each $\alpha \in \mathcal{A}$ the set $X \setminus X''(\alpha)$ is meager (see Proposition 2.1 (ii)), we have that even M is meager. Moreover, $R \setminus U \supset O \setminus M$. The sufficiency is also proved. ■

3. Unions of Vitali sets and the Vitali (resp. Baire) property

PROPOSITION 3.1. *Let $S(i)$ be a Vitali set for each $i \leq n$, where n is some integer ≥ 1 . Then $\bigcup_{i=1}^n S(i)$ does not possess the Vitali property. (Hence $\bigcup_{i=1}^n S(i)$ does not possess the Baire property.)*

Proof. Put $U = \bigcup_{i=1}^n S(i)$. Assume that there is a non-empty open set O and a meager set M such that $U \supset O \setminus M$. Since $O \neq \emptyset$ and U consists of finitely many Vitali sets, $E_\alpha \cap (O \setminus U) \neq \emptyset$ for each $\alpha \in I$. So we can find a Vitali set S such that $S \subset O$ and $S \cap U = \emptyset$. Hence $S \subset M$. Note that S is nonmeager and M is meager. This is a contradiction. ■

REMARK 3.1. Since equality (*) is valid, Proposition 3.1 can not be extended in general onto countable unions. However we can prove the following statement.

THEOREM 3.1. *If S is a Vitali set and \mathcal{A} is a non-empty proper subset of \mathbb{Q} then the set $U(S, \mathcal{A}) = \bigcup\{S_r : r \in \mathcal{A}\}$ does not possess the Baire property.*

Proof. Put $V = \text{Int}_{\mathbb{R}}(\text{Cl}_{\mathbb{R}} S'')$ and $V_r = T_r(V)$ for each $r \in \mathbb{Q}$. Note that $V_r = \text{Int}_{\mathbb{R}}(\text{Cl}_{\mathbb{R}}(S_r)'')$ for each $r \in \mathbb{Q}$. Recall that $V \neq \emptyset$ (see Proposition 2.1 (iii)) and hence $V_r \neq \emptyset$ for each $r \in \mathbb{Q}$.

Assume that $U(S, \mathcal{A})$ possesses the Baire property.

CLAIM. *For any $r_1 \in \mathcal{A}$ and $r_2 \in \mathbb{Q} \setminus \mathcal{A}$ we have $V_{r_1} \cap V_{r_2} = \emptyset$.*

Proof. Suppose that $W = V_{r_1} \cap V_{r_2} \neq \emptyset$. Since W is a non-empty open subset of V_{r_2} it follows from Corollary 2.1 (iii) that the set $W \cap (S_{r_2})''$ is nonmeager. Note also that $W \cap (S_{r_2})'' \subset V_{r_1} \cap S_{r_2} \subset V_{r_1} \setminus U(S, \mathcal{A})$. Hence by Theorem 2.1 the set $U(S, \mathcal{A})$ does not possess the Baire property. This is a contradiction. So $W = \emptyset$. The claim is proved.

We continue with the proof of the theorem. It follows from the claim that $(\bigcup\{V_{r_1} : r_1 \in \mathcal{A}\}) \cap (\bigcup\{V_{r_2} : r_2 \in \mathbb{Q} \setminus \mathcal{A}\}) = \emptyset$. But the family $\{V_r : r \in \mathbb{Q}\}$ is evidently an open cover of \mathbb{R} . Thus we have a contradiction with the connectedness of \mathbb{R} . The theorem is proved. ■

THEOREM 3.2. *If S is a Vitali set and \mathcal{A} is a non-empty proper subset of \mathbb{Q} then the set $\bigcup\{S_r : r \in \mathcal{A}\}$ possesses the Vitali property if and only if $\text{Cl}_{\mathbb{R}}(\bigcup\{(S_r)'' : r \in \mathbb{Q} \setminus \mathcal{A}\}) \neq \mathbb{R}$.*

Proof. Note that $\bigcup\{S_r : r \in \mathcal{A}\} = \mathbb{R} \setminus \bigcup\{S_r : r \in \mathbb{Q} \setminus \mathcal{A}\}$. Apply then Theorem 2.2. ■

4. Concluding remarks

Some of our results can be considered as particular answers to the following general problem: *Let G be a topological group acting on a topological space X . Assume that $A \subset G$, $B \subset X$, and let \mathcal{P} be a topological property. For what A, B does $A \cdot B$ possess (or not possess) the property \mathcal{P} ?*

Really, in this paper we have studied the additive group $G = \mathbb{R}$ of the real numbers acting on the real line $X = \mathbb{R}$ by the natural way: $x \cdot y = x + y$, where $x \in G$ and $y \in X$, and the Baire property (respectively, the Vitali property). It is easy to see that some of our results can be rewritten as follows.

- (a) If A is finite and B is the union of finitely many Vitali sets then $A \cdot B$ does not possess the Baire property (respectively, the Vitali property) (Proposition 3.1);
- (b) if A is a non-empty proper subset of \mathbb{Q} and B is a Vitali set then $A \cdot B$ does not possess the Baire property (Theorem 3.1);
- (c) if A is a non-empty proper subset of \mathbb{Q} and B is a Vitali set then $A \cdot B$ possess the Vitali property iff $\text{Cl}_{\mathbb{R}}(\bigcup\{(r \cdot B)'' : r \in \mathbb{Q} \setminus \mathcal{A}\}) \neq \mathbb{R}$ (Theorem 3.2).

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