МАТЕМАТИЧКИ ВЕСНИК
62, 3 (2010), 199–214
September 2010

оригинални научни рад
research paper

ON s-CLOSEDNESS AND $\mathcal{S}$-CLOSEDNESS IN TOPOLOGICAL SPACES

Zbigniew Duszyński

Abstract. Some properties of sets s-closed or $\mathcal{S}$-closed relative to a space, and s-closed or $\mathcal{S}$-closed subspaces, are obtained. Relationships between some of them are indicated. New characterizations of Hausdorff spaces in terms of s-closedness and α-compactness relative to a space, are obtained.

1. Preliminaries

Throughout the paper (X,τ) (or (Y,σ)) denotes a topological space. For a subset S of (X,τ), $\mathrm{int}\,(S)$ (or $\mathrm{int}_X(S)$), $\mathrm{cl}\,(S)$ (or $\mathrm{cl}_X(S)$, or $\mathrm{cl}_\tau(X)$) stand for the interior of S and the closure of S, respectively. If $X_0 \subset X$, then (X_0,τ_{X_0}) denotes a subspace of (X,τ), and $\mathrm{int}_{X_0}(.)$, $\mathrm{cl}_{X_0}(.)$ are interior and closure operators (respectively) in (X_0,τ_{X_0}). $\mathrm{CO}\,(X,\tau)$ is the intersection of τ and $\{X\setminus S : S\in\tau\}$. A subset S of (X,τ) is said to be *regular open* (resp. *regular closed*) if $S = \mathrm{int}\,(\mathrm{cl}\,(S))$ (resp. $S = \mathrm{cl}\,(\mathrm{int}\,(S))$). A set S is said to be *α-open* [28] (resp. *semi-open* [22], *semi-closed* [8], *preopen* [25], *semi-preopen* (or *β-open*) [2,1]) in (X,τ), if $S \subset \mathrm{int}\,(\mathrm{cl}\,(\mathrm{int}\,(S)))$ (resp. $S \subset \mathrm{cl}\,(\mathrm{int}\,(S))$, $S \supset \mathrm{int}\,(\mathrm{cl}\,(S))$, $S \subset \mathrm{int}\,(\mathrm{cl}\,(S))$, $S \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S)))$). A subset S of (X,τ) is semi-open if and only if there exists a $U\in\tau$ such that $U \subset S \subset \mathrm{cl}\,(U)$ [22]. The collection of all regular open (resp. regular closed, α-open, semi-open, semi-closed, preopen, semi-preopen) subsets of (X,τ) is denoted by $\mathrm{RO}\,(X,\tau)$ (resp. $\mathrm{RC}\,(X,\tau)$, τ^α, $\mathrm{SO}\,(X,\tau)$, $\mathrm{SC}\,(X,\tau)$, $\mathrm{PO}\,(X,\tau)$, $\mathrm{SPO}\,(X,\tau)$). The family τ^α forms a topology on X such that $\tau \subset \tau^\alpha$. An S is said to be *semi-regular* [10] (see also [5] and [41]) if it is both semi-closed and semi-open in (X,τ). We denote $\mathrm{SO}\,(X,\tau)\cap\mathrm{SC}\,(X,\tau) = \mathrm{SR}\,(X,\tau)$. We have in each (X,τ), $\mathrm{RO}\,(X,\tau)\cup\mathrm{RC}\,(X,\tau) \subset \mathrm{SR}\,(X,\tau)$ [41, Lemma 2.3], and $\mathrm{RO}\,(X,\tau)\cap\mathrm{RC}\,(X,\tau) = \mathrm{CO}\,(X,\tau)$ (see for instance [11,]). The *semi-closure* [8] (resp. the *semi-interior* [8]) of an $S \subset X$ is the intersection of all semi-closed subsets of (X,τ) containing S (resp. the union of all semi-open subsets of (X,τ) contained in S), and is denoted

2010 AMS Subject Classification: 54G99.

Keywords and phrases: α-open, semi-open, preopen, semi-preopen, semi-regular sets; s-closed, $\mathcal{S}$-closed.

respectively by $\mathrm{scl}(S)$ (or $\mathrm{scl}_X(S)$) and $\mathrm{sint}_X(S)$. The union of any family of semi-open subsets of (X,τ) is semi-open as well [22].

A space (X,τ) is said to be *extremally disconnected* (briefly *e.d.*) if $\mathrm{cl}(S) \in \tau$ for any $S \in \tau$.

A subset A of a space (X,τ) is said to be *s-closed* [10] (resp. *$\mathcal{S}$-closed* [32], *$\mathcal{N}$-closed* [7], *quasi-$\mathcal{H}$-closed* [38]) relative to (X,τ), if every cover $\{V_\alpha\}_{\alpha\in\nabla} \subset \mathrm{SO}(X,\tau)$ (resp. $\{V_\alpha\}_{\alpha\in\nabla} \subset \mathrm{SO}(X,\tau)$, $\{V_\alpha\}_{\alpha\in\nabla} \subset \tau$, $\{V_\alpha\}_{\alpha\in\nabla} \subset \tau$) of A admits a finite subfamily $\nabla_0 \subset \nabla$ such that $A \subset \bigcup_{\alpha\in\nabla_0} \mathrm{scl}(V_\alpha)$ (resp. $A \subset \bigcup_{\alpha\in\nabla_0} \mathrm{cl}(V_\alpha)$, $A \subset \bigcup_{\alpha\in\nabla_0} \mathrm{int}(\mathrm{cl}(V_\alpha))$, $A \subset \bigcup_{\alpha\in\nabla_0} \mathrm{cl}(V_\alpha)$). In the case $A = X$, (X,τ) is said to be s-closed [10] (resp. $\mathcal{S}$-closed [42]). (X_0,τ_{X_0}) is called an s-closed (resp. $\mathcal{S}$-closed) subspace of (X,τ) if it is s-closed (resp. $\mathcal{S}$-closed) as a space.

The following results are useful in the sequel:

1. Let $S \subset A \in \mathrm{SO}(X,\tau)$. Then $S \in \mathrm{SO}(X,\tau)$ if and only if $S \in \mathrm{SO}(A,\tau_A)$ [29, Theorem 5].
2. In any space (X,τ),

$$\mathrm{scl}(S) = S \cup \mathrm{int}(\mathrm{cl}(S)) \qquad [2, \text{Theorem } 1.5(\mathrm{a})],$$
$$\mathrm{cl}_{\tau^\alpha}(S) = S \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))) \qquad [2, \text{Theorem } 1.5(\mathrm{c})]$$

3. In any space (X,τ), $\mathrm{cl}_{\tau^\alpha}(V) = \mathrm{cl}_\tau(V)$ for each $V \in \mathrm{SO}(X,\tau)$ [17, Lemma 1(i)].
4. In any e.d. space (X,τ), $\tau^\alpha = \mathrm{SO}(X,\tau)$ [19, Theorem 2.9].

2. *s*-closedness

In [4] the following two results have been stated.

THEOREM 1. [4, Theorem 1] *Let $A \in \mathrm{PO}(X,\tau)$. Then (A,τ_A) is s-closed if and only if A is s-closed relative to (X,τ).*

THEOREM 2. [4, Theorem 2] *Let $A \subset B \subset X$, where $B \in \mathrm{PO}(X,\tau)$. Then, the set A is s-closed relative to (B,τ_B) if and only if it is s-closed relative to (X,τ).*

Proofs for these theorems are based on [12, Theorem 2.7], which states that $\mathrm{SR}(A,\tau_A) = \mathrm{SR}(X,\tau) \cap A$ (i.e., $\mathrm{SR}(A,\tau_A) = \{S \cap A : S \in \mathrm{SR}(X,\tau)\}$) for any space (X,τ) and any $A \in \mathrm{PO}(X,\tau)$. Unfortunately, the proof for $\mathrm{SR}(A,\tau_A) \subset \mathrm{SR}(X,\tau) \cap A$ given in [12] is far from clear (it is worth to see [20, Lemma 3]). We shall give a proof for [12, Theorem 2.7]. It will make use of the subsequent lemmas.

LEMMA 1. [37, Teorema 3.2] *Let X_0 be an arbitrary subset of a space (X,τ). If $A \in \mathrm{SO}(X_0,\tau_{X_0})$, then $A = X_0 \cap B$ for some $B \in \mathrm{SO}(X,\tau)$.*

LEMMA 2. *Let (X,τ) be a space and $X_0 \in \mathrm{PO}(X,\tau)$.*

(a) [34, Lemma 2.2] *One has $B \cap X_0 \in \mathrm{SO}(X_0,\tau_{X_0})$ for every $B \in \mathrm{SO}(X,\tau)$.*

(b) [34, Lemma 2.3] *One has $B \cap X_0 \in \mathrm{SC}(X_0,\tau_{X_0})$ for every $B \in \mathrm{SC}(X,\tau)$.*

Corollary 1. *If* $A \in \mathrm{PO}(X,\tau)$ *then* $\mathrm{SR}(X,\tau) \cap A \subset \mathrm{SR}(A,\tau_A)$.

Lemma 3. [34, Theorem 2.4]. *If* $A \subset X_0 \in \mathrm{PO}(X,\tau)$ *then* $X_0 \cap \mathrm{scl}_X(A) = \mathrm{scl}_{X_0}(A)$.

Lemma 4. [33, Lemma 3.5] *If either* $A \in \mathrm{SO}(X,\tau)$ *or* $B \in \mathrm{SO}(X,\tau)$ *then*

$$\mathrm{int}(\mathrm{cl}(A \cap B)) = \mathrm{int}(\mathrm{cl}(A)) \cap \mathrm{int}(\mathrm{cl}(B)).$$

Lemma 5. *Let* (X,τ) *be any space. The following statements are equivalent:*

(a) $S \in \mathrm{SR}(X,\tau)$.

(b) [10, Proposition 2.1(c)] *There exists a set* $U \in \mathrm{RO}(X,\tau)$ *such that* $U \subset S \subset \mathrm{cl}_X(U)$.

(c) [41, Lemma 2.2(iii)] $S = \mathrm{scl}_X\big(\mathrm{sint}_X(S)\big)$.

Lemma 6. (compare [10, Proposition 2.2]) *If* $S \in \mathrm{SPO}(X,\tau)$ *then* $\mathrm{scl}(S) \in \mathrm{SR}(X,\tau)$.

Proof. By the use of [2, Theorem 1.5(a)] we obtain

$$\mathrm{int}(\mathrm{cl}(S)) \subset \mathrm{scl}(S) = S \cup \mathrm{int}(\mathrm{cl}(S)) \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))) \cup \mathrm{int}(\mathrm{cl}(S)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S))).$$

Thus, by Lemma 5(b), $\mathrm{scl}(S) \in \mathrm{SR}(X,\tau)$. ∎

Theorem 3. [12, Theorem 2.7] *For any space* (X,τ), *if* $X_0 \in \mathrm{PO}(X,\tau)$ *then*

$$\mathrm{SR}(X_0,\tau_{X_0}) = \mathrm{SR}(X,\tau) \cap X_0.$$

Proof. In view of Corollary 1 only the inclusion $\mathrm{SR}(X_0,\tau_{X_0}) \subset \mathrm{SR}(X,\tau) \cap X_0$ requires a proof. Let $S \in \mathrm{SR}(X_0,\tau_{X_0})$ be arbitrarily chosen. By Lemmas 5(c) and 3 we have $\mathrm{scl}_{X_0}\big(\mathrm{sint}_{X_0}(S)\big) = X_0 \cap \mathrm{scl}_X\big(\mathrm{sint}_{X_0}(S)\big)$.

Obviously $\mathrm{sint}_{X_0}(S) \in \mathrm{SO}\left(X_0,\tau_{X_0}\right)$, so by Lemma 1, $\mathrm{sint}_{X_0}(S) = X_0 \cap B$ for some set $B \in \mathrm{SO}(X,\tau)$. We are to show that $X_0 \cap B \in \mathrm{SPO}(X,\tau)$. Indeed, by Lemma 4 we have the following inclusions:

$$\begin{aligned} X_0 \cap B &\subset \mathrm{int}(\mathrm{cl}(X_0)) \cap \mathrm{cl}(\mathrm{int}(B)) \subset \\ &\subset \mathrm{cl}\big(\mathrm{int}(\mathrm{cl}(X_0)) \cap \mathrm{int}(\mathrm{cl}(B))\big) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(X_0 \cap B))). \end{aligned}$$

Finally, $\mathrm{scl}_X(X_0 \cap B) \in \mathrm{SR}(X,\tau)$, by Lemma 6, and the proof is complete. ∎

Remark 1. Theorems 1 and 2 may be proved independently of Theorem 3 by using Lemmas 1, 2(a), 3, and Lemma 7 below. Details are omitted (it is worth to see for instance [32, Theorems 3.1 and 3.2] and left to the reader.

Lemma 7. *Let* $B \in \mathrm{PO}(X,\tau)$ *and* $V \in \mathrm{SO}(X,\tau)$. *Then* $B \cap \mathrm{scl}(V) \subset \mathrm{scl}(B \cap V)$.

Proof. By [2, Theorem 1.5(a)] and Lemma 4 we have $B \cap \mathrm{scl}(V) = B \cap \big(V \cup \mathrm{int}(\mathrm{cl}(V))\big) = (B \cap V) \cup \big(B \cap \mathrm{int}(\mathrm{cl}(V))\big) \subset (B \cap V) \cup \big(\mathrm{int}(\mathrm{cl}(B)) \cap \mathrm{int}(\mathrm{cl}(V))\big) = \mathrm{scl}(B \cap V)$. ∎

Remark 2. It is interesting to recall that if $B \in \mathrm{PO}\,(X,\tau)$ and $V \in \mathrm{SO}\,(X,\tau)$, then $B \cap \mathrm{cl}\,(V) \subset \mathrm{cl}\,(B \cap V)$ [35, Lemma 2.1]. The latter inclusion is equivalent the following: $B \cap \mathrm{cl}_{\tau^\alpha}(V) \subset \mathrm{cl}_{\tau^\alpha}(B \cap V)$ for every $B \in \mathrm{PO}\,(X,\tau^\alpha)$ and $V \in \mathrm{SO}\,(X,\tau^\alpha)$. It is so since $\mathrm{SO}\,(X,\tau^\alpha) = \mathrm{SO}\,(X,\tau)$ [28, Proposition 3], $\mathrm{PO}\,(X,\tau^\alpha) = \mathrm{PO}\,(X,\tau)$ [20, Corollary 2.5(a)], $\mathrm{cl}_{\tau^\alpha}(V) = \mathrm{cl}_\tau(V)$ [17, Lemma 1(i)], and $\mathrm{cl}_{\tau^\alpha}(B \cap V) \supset \mathrm{cl}_\tau(B \cap V)$ (to prove this one use Lemma 4 and [2, Theorem 1.5(c)]).

We omit details in the proofs of the next three corollaries.

Corollary 2. *Let $A \subset X_0 \subset X_1 \subset X$ and $X_0, X_1 \in \mathrm{PO}\,(X,\tau)$. Then A is s-closed relative to (X_0, τ_{X_0}) if and only if A is s-closed relative to (X_1, τ_{X_1}).*

Proof. Theorem 2. ■

Corollary 3. *Let $A \in \mathrm{PO}\,(X_0, \tau_{X_0})$ and $X_0 \in \mathrm{PO}\,(X,\tau)$. Then A is an s-closed subspace of (X_0, τ_{X_0}) if and only if A is an s-closed subspace of (X,τ).*

Proof. This follows from Theorems 1–2 and [26, Lemma 2.2]: if $A \in \mathrm{PO}\,(X_0, \tau_{X_0})$ and $X_0 \in \mathrm{PO}\,(X,\tau)$ then $A \in \mathrm{PO}\,(X,\tau)$. ■

Corollary 3 improves [4, Corollary 1].

Corollary 4. *Let $A \in \mathrm{PO}\,(X_0, \tau_{X_0})$, $X_0 \in \mathrm{PO}\,(X_1, \tau_{X_1})$, and $X_1 \in \mathrm{PO}\,(X,\tau)$. Then A is an s-closed subspace of (X_0, τ_{X_0}) if and only if it is an s-closed subspace of (X_1, τ_{X_1}).*

Proof. By Corollary 2 and [26, Lemma 2.2]. ■

Definition 1. A subset S of a space (X,τ) is said to be *sspo-closed relative to* (X,τ) if, for every cover $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}\,(X,\tau)$ of S there is a finite set of indices $\nabla_0 \subset \nabla$ such that $S \subset \bigcup_{\alpha \in \nabla_0} \mathrm{scl}_X(V_\alpha)$. If $S = X$, then (X,τ) is called an *sspo-closed space*.

Theorem 4. *In any space (X,τ) and for any subset S of it, the following statements are equivalent:*

(a) *S is sspo-closed relative to (X,τ),*

(b) *S is s-closed relative to (X,τ).*

Proof. (a)⇒(b). Obvious, since $\mathrm{SO}\,(X,\tau) \subset \mathrm{SPO}\,(X,\tau)$.

(a)⇐(b). Let $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}\,(X,\tau)$ cover a set S. Then, $S \subset \bigcup_{\alpha \in \nabla} \mathrm{scl}_X(V_\alpha)$. Since S is s-closed relative to (X,τ) if and only if each semi-regular cover of S admits a finite subcover [10, Proposition 4.1], application of Lemma 6 completes the proof. ■

Lemma 8. *Let A be an arbitrary subset of a space (X,τ). If $U \in \mathrm{SPO}\,(A, \tau_A)$ then*

$$\mathrm{int}_X(A) \cap U \subset \mathrm{cl}_X\big(\mathrm{int}_X\big(\mathrm{cl}_X(U)\big)\big).$$

Proof. Using the equality $\operatorname{int}_X(E) = \operatorname{int}_A(E) \cap \operatorname{int}_X(A)$ that holds for any subset $E \subset A$ [36, Exercise 7(vi)], we calculate as follows:

$$\begin{aligned}\operatorname{int}_X(A) \cap U &\subset \operatorname{int}_X(A) \cap \operatorname{cl}_A\big(\operatorname{int}_A\big(\operatorname{cl}_A(U)\big)\big) \subset \operatorname{int}_X(A) \cap \operatorname{cl}_X\big(\operatorname{int}_A\big(\operatorname{cl}_A(U)\big)\big) \subset \\ &\subset \operatorname{cl}_X\big(\operatorname{int}_X(A) \cap \operatorname{int}_A\big(\operatorname{cl}_A(U)\big)\big) = \operatorname{cl}_X\big(\operatorname{int}_X\big(\operatorname{cl}_A(U)\big)\big) \\ &\subset \operatorname{cl}_X\big(\operatorname{int}_X\big(\operatorname{cl}_X(U)\big)\big). \quad \blacksquare\end{aligned}$$

COROLLARY 5. *If $A \in \tau$ and $U \in \mathrm{SPO}\,(A, \tau_A)$, then $U \in \mathrm{SPO}\,(X, \tau)$.*

COROLLARY 6. *If $A \in \tau$ and $U \in \mathrm{SPO}\,(A, \tau_A)$, then $\operatorname{cl}_A(U) \in \mathrm{SPO}\,(X, \tau)$.*

LEMMA 9. *If $A \in \tau$ and $V \in \mathrm{SPO}\,(X, \tau)$, then $A \cap V \in \mathrm{SPO}\,(A, \tau_A)$.*

Proof. We have

$$\begin{aligned}A \cap V &\subset A \cap \operatorname{cl}_X\big(\operatorname{int}_X\big(\operatorname{cl}_X(V)\big)\big) \subset \operatorname{cl}_A\big(A \cap \operatorname{int}_X\big(\operatorname{cl}_X(V)\big)\big) = \\ &= \operatorname{cl}_A\big(\operatorname{int}_A\big(A \cap \operatorname{cl}_X(V)\big)\big) \subset \operatorname{cl}_A\big(\operatorname{int}_A\big(\operatorname{cl}_A(A \cap V)\big)\big). \quad \blacksquare\end{aligned}$$

THEOREM 5. *Let (X, τ) be a space and $A \in \tau$. The following are equivalent:*

(a) (A, τ_A) *is sspo-closed,*

(b) (A, τ_A) *is s-closed.*

Proof. (a)$\Rightarrow$(b). Making use of Theorems 1 and 4 we will show A is *sspo*-closed relative to (X, τ). Suppose $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}\,(X, \tau)$ is a cover of A. By Lemma 9, $\{A \cap V_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}\,(A, \tau_A)$ covers A and hence we get $A = \bigcup_{\alpha \in \nabla_0} \operatorname{scl}_A(A \cap V_\alpha)$ for some finite $\nabla_0 \subset \nabla$. It is easy to see that by Lemma 3, $A \subset \bigcup_{\alpha \in \nabla_0} \operatorname{scl}_X(V_\alpha)$. Thus (A, τ_A) is s-closed.

(a)$\Leftarrow$(b). Suppose A is s-closed relative to (X, τ) (utilize Theorem 1). Let $\{U_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}\,(A, \tau_A)$ be a cover of A. We have $\{U_\alpha : \alpha \in \nabla\} \subset \mathrm{SPO}\,(X, \tau)$ (Corollary 2) and $A \subset \bigcup_{\alpha \in \nabla} \operatorname{scl}_X(U_\alpha)$, where $\{\operatorname{scl}_X(U_\alpha) : \alpha \in \nabla\} \subset \mathrm{SR}\,(X, \tau)$. By [10, Proposition 4.1], $A \subset \bigcup_{\alpha \in \nabla_0} \operatorname{scl}_X(U_\alpha)$ for some finite $\nabla_0 \subset \nabla$. Hence, using Lemma 3 we get that $A = \bigcup_{\alpha \in \nabla_0} \operatorname{scl}_A(U_\alpha)$. This completes the proof. $\blacksquare$

LEMMA 10. [12] (compare also [24, Example 3.3(ii)]). *If $V \in \mathrm{SO}\,(X, \tau)$ and $W \subset X$, the following holds:*

$$V \cap \operatorname{scl}(W) \subset \operatorname{cl}\big(\operatorname{scl}(V \cap W)\big).$$

THEOREM 6. *Let $A, B \in \mathrm{SC}\,(X, \tau)$ and $A \cap B \in \mathrm{SO}\,(X, \tau)$. If A and B are both s-closed relative to (X, τ), then $A \cap B$ is also s-closed relative to (X, τ).*

Proof. Let $A \cap B \subset \bigcup_{\alpha \in \nabla} V_\alpha$ where $V_\alpha \in \mathrm{SO}\,(X, \tau)$ for each $\alpha \in \nabla$. We have $A \subset (X \setminus B) \cup \bigcup_{\alpha \in \nabla} V_\alpha$ and $B \subset (X \setminus A) \cup \bigcup_{\alpha \in \nabla} V_\alpha$, where $X \setminus A, X \setminus B \in \mathrm{SO}\,(X, \tau)$. By hypothesis there are finite subfamilies $\nabla_1, \nabla_2 \subset \nabla$ with

$$A \subset \operatorname{scl}(X \setminus B) \cup \bigcup_{\alpha \in \nabla_1} \operatorname{scl}(V_\alpha) \quad \text{and}$$

$$B \subset \operatorname{scl}(X \setminus A) \cup \bigcup_{\alpha \in \nabla_2} \operatorname{scl}(V_\alpha).$$

It follows easily from Lemma 10 that

$$A \cap B = (A \cap B) \cap (A \cup B) \subset \bigcup_{\alpha \in \nabla_1} \mathrm{scl}\,(V_\alpha) \cup \bigcup_{\alpha \in \nabla_2} \mathrm{scl}\,(V_\alpha).$$

Thus, $A \cap B$ is s-closed relative to (X, τ). ■

Corollary 7. *If $A, B \in \mathrm{SC}\,(X, \tau)$, $A \cap B \in \mathrm{SO}\,(X, \tau)$, and A, B are both s-closed relative to (X, τ), then $A \cap B$ is an s-closed subspace of (X, τ).*

Proof. Follows from Theorem 6 and [20, Theorem 4]. ■

It is of worth to compare Corollary 7 with [14, Theorem 2.2].

Theorem 7. *Let $A, B \in \mathrm{SO}\,(X, \tau)$ and $A \cap B = \emptyset$. If a set $A \cup B$ is s-closed relative to (X, τ), then B and A are s-closed relative to (X, τ).*

Proof. Similar to that of Theorem 28 below—one uses Lemma 10. ■

The notion of $\mathcal{S}$-connectedness has been introduced by Pipitone and Russo in [37]: *(X, τ) is $\mathcal{S}$-connected if there are no two nonempty sets $A_1, A_2 \in \mathrm{SO}\,(X, \tau)$ such that $X = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$.* A space that is not $\mathcal{S}$-connected is said to be $\mathcal{S}$-disconnected.

Corollary 8. *Let (X, τ) be an $\mathcal{S}$-disconnected and s-closed space. Then there exists a nonempty set $B \in \mathrm{SO}\,(X, \tau)$ which is s-closed relative to (X, τ) and is an s-closed subspace of (X, τ).*

Proof. By Theorem 7 and [21, Theorem 4]. ■

Theorem 8. *Let (X, τ) be s-closed and $A \in \mathrm{SR}\,(X, \tau)$. Then $X \setminus A$ is an s-closed subspace of (X, τ).*

Proof. Let $X \setminus A \subset \bigcup_{\alpha \in \nabla} V_\alpha$ where $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SR}\,(X, \tau)$. Then $X = A \cup \bigcup_{\alpha \in \nabla} V_\alpha$, and by [10, Proposition 3.1] there exists some finite $\nabla_0 \subset \nabla$ with $X = A \cup \bigcup_{\alpha \in \nabla_0} V_\alpha$. So, $X \setminus A$ is s-closed relative to (X, τ) and by [21, Theorem 4] it is an s-closed subspace. ■

Theorem 9. *Let $A \in \mathrm{CO}\,(X, \tau)$ be a set s-closed relative to (X, τ). Then (X, τ) is s-closed if and only if $X \setminus A$ is an s-closed subspace of it.*

Proof. Necessity. Theorem 8. *Sufficiency.* By Theorem 1, $X \setminus A$ is s-closed relative to (X, τ). Hence $X = A \cup (X \setminus A)$ is s-closed relative to (X, τ) [4, Theorem 4]; i.e., (X, τ) is s-closed. ■

Lemma 11. *Let $B \in \mathrm{SR}\,(X, \tau)$, $A \subset X$, and $A \cup B$ be s-closed relative to (X, τ). Then, $A \setminus B$ is s-closed relative to (X, τ).*

Proof. Follows easily from [10, Proposition 4.1] and the identity $A \setminus B = (A \cup B) \cap (X \setminus B)$. ■

THEOREM 10. *Let, in a space* (X,τ), (A,τ_A) *and* (B,τ_B) *be s-closed subspaces. If* $A\in\tau^\alpha$ *and* $B\in \mathrm{CO}\,(X,\tau)$, *then* $(A\setminus B,\tau_{A\setminus B})$ *is an s-closed subspace of* (X,τ).

Proof. By Theorem 1, A and B are s-closed relative to (X,τ). Using [4, Theorem 4] and Lemma 11 we get that $A\setminus B$ is s-closed relative to (X,τ). It is enough now to recall that $\mathrm{CO}\,(X,\tau)=\mathrm{CO}\,(X,\tau^\alpha)$ ■

REMARK 3. The above Theorems 7 to 10 should be compared with respective Theorems 28 to 31 in the sequel (Section 4).

Recall the following notions [10, p.227]: a point x of a space (X,τ) is said to be a *semi* θ*-adherent point* of a subset $S\subset X$ if $S\cap \mathrm{scl}_X(U)\neq\emptyset$ for every set $U\in \mathrm{SO}\,(X,\tau)$ with $x\in U$. The set of all semi θ-adherent points of an S is called the *semi* θ*-closure* of S in (X,τ). A set $S\subset X$ is called *semi* θ*-closed* if the semi θ-closure of S is S.

THEOREM 11. *Let* $A\in \mathrm{SPO}\,(X,\tau)$. *If* $A\cup\big(X\setminus \mathrm{scl}_X(A)\big)$ *is s-closed relative to* (X,τ), *then* A *is s-closed relative to* (X,τ).

Proof. Let $A\subset\bigcup_{\alpha\in\nabla}V_\alpha$ where $\{V_\alpha : \alpha\in\nabla\}\subset \mathrm{SR}\,(X,\tau)$. By Lemma 6, $\mathrm{scl}_X(A)\in \mathrm{SR}\,(X,\tau)$ and hence $\mathrm{scl}_X(A)$ is semi θ-closed [12, Proposition 2.3(b)]. Thus, for each $x\in X\setminus \mathrm{scl}_X(A)$ there exists $V_x\in \mathrm{SO}\,(X,\tau)$ with $x\in V_x$, such that $\mathrm{scl}_X(V_x)\subset X\setminus \mathrm{scl}_X(A)$. The family $\{\mathrm{scl}_X(V_x) : x\in X\setminus \mathrm{scl}_X(A)\}\cup\{V_\alpha : \alpha\in\nabla\}$ covers the set $A\cup\big(X\setminus \mathrm{scl}_X(A)\big)$. Thus, by hypothesis, there exists a finite $\nabla_0\subset\nabla$ with $A\subset\bigcup_{\alpha\in\nabla_0}V_\alpha$. ■

COROLLARY 9. *Let* (X,τ) *be an s-closed space and* $A\in \mathrm{SPO}\,(X,\tau)$. *If* $\mathrm{scl}_X(A)\setminus A\in \mathrm{SR}\,(X,\tau)$ *then* A *is s-closed relative to* (X,τ).

Proof. By the proof of Theorem 8 the set $X\setminus\big(\mathrm{scl}_X(A)\setminus A\big)$ is s-closed relative to (X,τ). Apply now Theorem 11. ■

A space (X,τ) is said to be *weakly-*$\mathcal{T}_2$ [40], if each point of X can be expressed as an intersection of regular closed subsets of (X,τ). In [10, Proposition 4.3] the following is proved: *if* K *is s-closed relative to a weakly-*$\mathcal{T}_2$ *space, then* K *is semi* θ*-closed in* (X,τ).

THEOREM 12. *Let* $A\subsetneq X$ *be a set s-closed relative to* (X,τ). *Assume that*

for each $x\in X\setminus A$ *and* $y\in A$, *there exist sets*

$$V_x\in\tau^\alpha,\ V_y\in \mathrm{SO}\,(X,\tau),\ V_x\ni x,\ V_y\ni y,\ \text{with } V_x\cap V_y=\emptyset. \quad (1)$$

Then, A *is semi* θ*-closed in* (X,τ).

Proof. Pick an arbitrary $x_0\in X\setminus A$. For each $y\in A$, there exist sets $V_{x_0,y}\in\tau^\alpha$, $V_{x_0,y}\ni x_0$, and $V_y\in \mathrm{SO}\,(X,\tau)$, $V_y\ni y$, with $V_{x_0,y}\cap V_y=\emptyset$. Thus, $\{V_y : y\in A\}$ covers A and, as A is s-closed relative to (X,τ), we have $A\subset\bigcup_{i=1}^n \mathrm{scl}\,(V_{y_i})$ for some $y_1,\dots,y_n\in A$. Making use of Lemma 7 (or Lemma 10) we get $V_{x_0,y_i}\cap$

$\operatorname{scl}(V_{y_i}) = \emptyset$, $i = 1, \ldots, n$. We have also $A \subset \bigcup_{i=1}^{n} \operatorname{scl}(V_{y_i}) = V \in \mathrm{SO}(X,\tau)$ and $x_0 \in \bigcap_{i=1}^{n} V_{x_0,y_i} = B \in \tau^{\alpha}$. So, by [17, Lemma 1(i)],

$$B \cap \mathrm{cl}_{\tau}(V) = B \cap \mathrm{cl}_{\tau^{\alpha}}(V) \subset \mathrm{cl}_{\tau^{\alpha}}(B \cap V) = \emptyset,$$

where $\mathrm{cl}_{\tau}(V) \in \mathrm{SR}(X,\tau)$. This implies that $x_0 \in X \setminus \mathrm{cl}_{\tau}(V) \in \mathrm{SR}(X,\tau)$; i.e., there is a $U \in \mathrm{SO}(X,\tau)$ containing x_0 such that $\mathrm{scl}_X(U) \cap A = \emptyset$. Thus, x_0 is not a semi θ-adherent point of A and hence A is semi θ-closed. ∎

Example 1. There exist a space (X,τ) which is not weakly-$\mathcal{T}_2$, and a subset $A \subsetneq X$ such that (1) of Theorem T12 holds. Indeed, if $X = \{a,b,c,d,e\}$, $\tau = \{\emptyset, X, \{a,b\}, \{c,d\}, \{e\}\}$, then consider $A = \{c,d,e\}$.

Remark 4. Recall that (X,τ) is called a *semi-$\mathcal{T}_2$-space* [23], if for any distinct points $x_1, x_2 \in X$ there exist disjoint $V_1, V_2 \in \mathrm{SO}(X,\tau)$ with $V_1 \ni x_1$ and $V_2 \ni x_2$. Using [19, Theorem 2.9] and the fact that (X,τ) is $\mathcal{T}_2$ if and only if (X,τ^{α}) is $\mathcal{T}_2$ [11, Theorem 3], we obtain that every e.d. semi-$\mathcal{T}_2$ space is $\mathcal{T}_2$. So, directly from [10, Proposition 4.3] we infer what follows: in any e.d. semi-$\mathcal{T}_2$ space (X,τ), every subset s-closed relative to (X,τ) is semi θ-closed in (X,τ).

A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *semi-continuous* [22] (resp. *s-open* [6]) if $f^{-1}(V) \in \mathrm{SO}(X,\tau)$ (resp. $f(U) \in \sigma$) for every $V \in \sigma$ (resp. $U \in \mathrm{SO}(X,\tau)$). An f is semi-continuous if and only if for every $S \subset X$, $f(\mathrm{scl}_X(S)) \subset \mathrm{cl}_Y(f(S))$ [9, Theorem 1.16].

Theorem 13. *Consider a function $f : (X,\tau) \to (Y,\sigma)$ and a subset G s-closed relative to (X,τ).*

(a) *If f is semi-continuous and s-open then $f(G)$ is $\mathcal{N}$-closed relative to (Y,σ).*

(b) *If f is semi-continuous then $f(G)$ is quasi $\mathcal{H}$-closed relative to (Y,σ).*

Proof. (a) Let $\{V_{\alpha} : \alpha \in \nabla\} \subset \sigma$ be a cover of $f(G)$. Then $\{f^{-1}(V_{\alpha}) : \alpha \in \nabla\} \subset \mathrm{SO}(X,\tau)$ is a cover of G. There is a finite $\nabla_0 \subset \nabla$ such that $G \subset \bigcup_{\alpha \in \nabla_0} \mathrm{scl}_X(f^{-1}(V_{\alpha}))$. As f is semi-continuous and s-open, we obtain

$$\begin{aligned} f(G) &\subset \bigcup_{\alpha \in \nabla_0} f\big(\mathrm{scl}_X(f^{-1}(V_{\alpha}))\big) \subset \bigcup_{\alpha \in \nabla_0} \mathrm{int}_Y\big(\mathrm{cl}_Y\big(f\big(f^{-1}(V_{\alpha})\big)\big)\big) \\ &\subset \bigcup_{\alpha \in \nabla_0} \mathrm{int}_Y\big(\mathrm{cl}_Y(V_{\alpha})\big). \end{aligned}$$

Thus, $f(G)$ is $\mathcal{N}$-closed relative to (Y,σ).

(b) Similar to the case (a). ∎

Semi-continuity and s-openness are independent notions, as seen by the example below.

Example 2. (a). Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a,b\}, \{a\}\}$, $Y = \{a,b,c,d\}$, and $\sigma = \{\emptyset, Y, \{a\}, \{a,b\}, \{a,d\}, \{a,b,d\}\}$. Define $f : (X,\tau) \to (Y,\sigma)$ as the identity on X. One checks that f is semi-continuous. But, f is not s-open since $f(\{a,b\}) \notin \sigma$.

(b). Let $X=\{a,b,c\}$, $\tau=\big\{\emptyset,X,\{a,b\}\big\}$, and $\sigma=\big\{\emptyset,X,\{c\},\{a,b\}\big\}$. Let again $f:(X,\tau)\to(X,\sigma)$ be the identity on X. Then f is s-open not being semi-continuous as $f^{-1}(\{c\})\notin \mathrm{SO}\,(X,\tau)$.

Definition 2. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be SR-*open* (resp. R-*open*), if $f(U)\in\mathrm{SR}\,(Y,\sigma)$ (resp. $f(U)\in\mathrm{RO}\,(Y,\sigma)$) for every $U\in\mathrm{SR}\,(X,\tau)$ (resp. $U\in\mathrm{RO}\,(X,\tau)$).

Theorem 14. *Let a set B be s-closed relative to (Y,σ). If a bijection $f:(X,\tau)\to(Y,\sigma)$ is* SR-*open then $f^{-1}(B)$ is s-closed relative to (X,τ).*

Proof. Use [10, Proposition 4.1]. ■

A function $f:(X,\tau)\to(Y,\sigma)$ is called *a.c.H.* ([18, 25] and [39, Theorem 4]) if $f^{-1}(V)\in\mathrm{PO}\,(X,\tau)$ for every $V\in\sigma$.

Theorem 15. *If a function $f:(X,\tau)\to(Y,\sigma)$ is a.c.H. and* R-*open, then it is* SR-*open.*

Proof. Let $A\in\mathrm{SR}\,(X,\tau)$. There exists a set $U\in\mathrm{RO}\,(X,\tau)$ such that $U\subset A\subset \mathrm{cl}_X(U)$ [10, Proposition 2.1]. Since f is a.c.H., $f\big(\mathrm{cl}_X(S)\big)\subset\mathrm{cl}_Y(f(S))$ for every $S\in\tau$ [39, Theorem 6]. Thus, by R-openness of f and, again, by [10, Proposition 2.1] we obtain that f is SR-open. ■

3. Hausdorffness of spaces

In this section we offer some characterizations of $\mathcal{T}_2$ and semi-$\mathcal{T}_2$ spaces.

Theorem 16. *A space (X,τ) is $\mathcal{T}_2$ if and only if, for each $A\subsetneq X$ s-closed relative to (X,τ) and each point $x\in X\setminus A$ there exist disjoint sets $U_1,U_2\in\mathrm{RO}\,(X,\tau)$ with $U_1\ni x$ and $U_2\supset A$.*

Proof. Necessity. Let $x_0\in X\setminus A$ be arbitrary. By Hausdorffness of (X,τ), for each $y\in A$ there are disjoint $V_{x_0,y},V_y\in\tau^\alpha$ with $V_{x_0,y}\ni x_0$ and $V_y\ni y$ [17, Theorem 3]. Since A is s-closed relative to (X,τ), $A\subset\bigcup_{i=1}^n\mathrm{scl}\,(V_{y_i})$ for certain $y_1,\dots,y_n\in A$. It is enough to show that

$$\mathrm{scl}\left(\bigcap_{i=1}^n V_{x_0,y_i}\right)\cap\mathrm{scl}\left(\bigcup_{i=1}^n\mathrm{scl}\,(V_{y_i})\right)=\emptyset,$$

because $\mathrm{scl}\,(S)=\mathrm{int}\,(\mathrm{cl}\,(S))$ for any $S\in\tau^\alpha\subset\mathrm{PO}\,(X,\tau)$ [20, Proposition 2.7(a)]. Indeed, we get by Lemma 7 (for instance), [8, Theorem 1.7(4)], and Lemma 4:

$$\begin{aligned}\mathrm{scl}\left(\bigcap_{i=1}^n V_{x_0,y_i}\right)\cap\mathrm{scl}\left(\bigcup_{i=1}^n\mathrm{scl}\,(V_{y_i})\right)&\subset\mathrm{scl}\left(\mathrm{scl}\left(\bigcap_{i=1}^n V_{x_0,y_i}\right)\cap\bigcup_{i=1}^n\mathrm{scl}\,(V_{y_i})\right)\\&\subset\mathrm{scl}\left(\mathrm{scl}\left(\bigcap_{i=1}^n V_{x_0,y_i}\right)\cap\mathrm{scl}\left(\bigcup_{i=1}^n V_{y_i}\right)\right)\\&=\mathrm{scl}\left(\mathrm{int}\left(\mathrm{cl}\left(\bigcap_{i=1}^n V_{x_0,y_i}\cap\bigcup_{i=1}^n V_{y_i}\right)\right)\right)=\mathrm{scl}\,(\mathrm{int}\,(\mathrm{cl}\,(\emptyset)))=\emptyset.\end{aligned}$$

Thus, if we put

$$U_1 = \operatorname{scl}\left(\bigcap_{i=1}^{n} V_{x_0,y_i}\right) \in \mathrm{RO}\,(X,\tau), \quad U_2 = \operatorname{scl}\left(\bigcup_{i=1}^{n} \operatorname{scl}\,(V_{y_i})\right) \in \mathrm{RO}\,(X,\tau),$$

then

$$x_0 \in U_1, \quad A \subset U_2, \quad \text{and } U_1 \cap U_2 = \emptyset.$$

Sufficiency. This is clear as every singleton is s-closed relative to (X,τ) (compare [10, Proposition 4.1]). ■

Recall that a subset A of a space (X,τ) is said to be *α-compact relative to* (X,τ) [3], if every τ^α-cover of A admits a finite subcover.

THEOREM 17. *A space (X,τ) is $\mathcal{T}_2$ if and only if, for each $A \subsetneq X$, α-compact relative to (X,τ) and each point $x \in X \setminus A$, there exist disjoint sets $U_1, U_2 \in$ $\mathrm{RO}\,(X,\tau)$ with $U_1 \ni x$ and $U_2 \supset A$.*

Proof. Very similar to that of Theorem 16 (after few modifications—details left to the reader). ■

In [15] the author has proved that a space (X,τ) is semi-$\mathcal{T}_2$ if and only if, for any distinct $x, y \in X$, there are sets $U_x, U_y \in \mathrm{SR}\,(X,\tau)$ such that $x \in U_x$, $y \in U_y$, $U_x \cap U_y = \emptyset$. So, since every singleton is s-closed relative to (X,τ) [10, Proposition 4.1], we get as a corollary

THEOREM 18. *Assume that for each subset $A \subsetneq X$, s-closed relative to (X,τ), and for each point $x \in X \setminus A$, there exist disjoint $U_1, U_2 \in \mathrm{SR}\,(X,\tau)$ with $U_1 \ni x$ and $U_2 \supset A$. Then (X,τ) is semi-$\mathcal{T}_2$.*

Combining Theorem 18 with [21, Theorem 6] we obtain the following characterization of e.d. semi-$\mathcal{T}_2$ spaces.

THEOREM 19. *An e.d. space (X,τ) is semi-$\mathcal{T}_2$ if and only if, for any $A \subsetneq X$, s-closed relative to (X,τ), and each $x \in X \setminus A$, there exist disjoint semi-regular subsets U and V with $U \ni x$ and $V \supset A$.*

4. $\mathcal{S}$-closedness

The following result has been stated by Khan, Ahmad, and Noiri [21, Theorem 5]: *if every semi-regular subset of an e.d. space (X,τ) is an s-closed subspace of (X,τ), then (X,τ) is s-closed.* In this theorem '*(X,τ) is s-closed*' may be replaced by '*(X,τ) is $\mathcal{S}$-closed*' since in e.d. spaces these two notions coincide [27, Theorem 14]. Moreover, the next result we state shows that after this replacement, the assumption '*(X,τ) is e.d.*' becomes superfluous.

THEOREM 20. *If every semi-regular subset of (X,τ) is an s-closed subspace of (X,τ), then (X,τ) is $\mathcal{S}$-closed.*

Proof. Suppose $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{SO}\,(X,\tau)$ is a cover of (X,τ). Take into consideration a set $\mathrm{cl}_X(V_\beta) \neq X$ with $V_\beta \neq \emptyset$. Obviously, $\mathrm{cl}_X(V_\beta) \in \mathrm{SR}\,(X,\tau)$ and hence $X \setminus \mathrm{cl}_X(V_\beta) \in \mathrm{SR}\,(X,\tau)$ as well. By hypothesis $X \setminus \mathrm{cl}_X(V_\beta)$ is an s-closed subspace of (X,τ), and since it is open in (X,τ), we infer from Theorem 1 that $X \setminus \mathrm{cl}_X(V_\beta)$ is s-closed relative to (X,τ). We have $X \setminus \mathrm{cl}_X(V_\beta) \subset \bigcup_{\alpha\in\nabla} V_\alpha$ and there is a finite $\nabla_0 \subset \nabla$ such that

$$X \setminus \mathrm{cl}_X(V_\beta) \subset \bigcup_{\alpha\in\nabla_0} \mathrm{scl}_X(V_\alpha).$$

Thus, one gets $X = \bigcup_{\alpha\in\nabla_0\cup\{\beta\}} \mathrm{cl}_X(V_\alpha)$. This shows that (X,τ) is $\mathcal{S}$-closed. ∎

In [32, Theorem 3.1] Noiri proved that if $A \in \tau^\alpha$, then the subspace (A,τ_A) is $\mathcal{S}$-closed if and only if it is $\mathcal{S}$-closed relative to (X,τ). Combining this result with Theorem 1, it is easy to show that for $A \in \tau^\alpha$, if (A,τ_A) is s-closed then it is $\mathcal{S}$-closed. The theorem below is a strong improvement of this corollary.

Theorem 21. *Let A be an arbitrary subset of (X,τ). If (A,τ_A) is s-closed then it is $\mathcal{S}$-closed.*

Proof. Let $\{U_\alpha : \alpha\in\nabla\} \subset \mathrm{SO}\,(A,\tau_A)$ be a cover of A. By assumption, there is a finite $\nabla_0 \subset \nabla$ such that $A = \bigcup_{\alpha\in\nabla_0} \mathrm{scl}_A(U_\alpha)$. So, $A = \bigcup_{\alpha\in\nabla_0} \mathrm{cl}_A(U_\alpha)$. ∎

Theorem 22. *Let $A \in \tau^\alpha$ be a subset of an e.d. space (X,τ). Then, (A,τ_A) is $\mathcal{S}$-closed if and only if it is s-closed.*

Proof. Let (A,τ_A) be $\mathcal{S}$-closed. By [32, Theorem 3.1] it is equivalent A being $\mathcal{S}$-closed relative to (X,τ). By means of [27, Theorem 14] and Theorem 1, the latter is equivalent (A,τ_A) being s-closed. ∎

Remark 5. The following is an interesting consequence of [27, Theorem 14]: for any subset A of (X,τ) such that (A,τ_A) is e.d., (A,τ_A) is $\mathcal{S}$-closed if and only if A is s-closed.

In [14, Theorem 2.7] the author proved that if $A \in \tau^\alpha$ is an $\mathcal{S}$-closed subspace of (X,τ), then $\big(\mathrm{scl}_X(A), \tau_{\mathrm{scl}_X(A)}\big)$ is also $\mathcal{S}$-closed. Since $\mathrm{scl}_X(A) = \mathrm{int}_X(\mathrm{cl}_X(A))$ for any $A \in \mathrm{PO}\,(X,\tau)$ [20, Proposition 2.7(a)], by the use of Theorem 22 it follows that if $A \in \tau^\alpha$ is an s-closed subspace of an e.d. (X,τ), then $\big(\mathrm{scl}_X(A), \tau_{\mathrm{scl}_X(A)}\big)$ is s-closed too. This result shall be extended to $A \in \mathrm{PO}\,(X,\tau)$ (in e.d. spaces) in Theorem 23 below.

Lemma 12. *For any (X,τ) and $S_1, S_2 \subset X$,*

$$\mathrm{int}\,(\mathrm{cl}\,(S_1 \cup S_2)) = \mathrm{int}\,\big(\mathrm{cl}\,\big(\mathrm{int}\,(\mathrm{cl}\,(S_1)) \cup \mathrm{int}\,(\mathrm{cl}\,(S_2))\big)\big).$$

Proof. Clearly, $\mathrm{int}\,(\mathrm{cl}\,(S_1)) \cup \mathrm{int}\,(\mathrm{cl}\,(S_2)) \subset \mathrm{int}\,(\mathrm{cl}\,(S_1\cup S_2))$. Next, we calculate as follows: $\mathrm{int}\,(\mathrm{cl}\,(S_1\cup S_2)) \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1\cup S_2))) = \mathrm{cl}\,\big(\mathrm{int}\,\big(\mathrm{cl}\,(S_1)\cup \mathrm{cl}\,(S_2)\big)\big)$ $= \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1))) \cup \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_2)))$ by the dual to Lemma 4. So,

$$\mathrm{int}\,(\mathrm{cl}\,(S_1\cup S_2)) \subset \mathrm{cl}\,\big(\mathrm{int}\,(\mathrm{cl}\,(S_1)) \cup \mathrm{int}\,(\mathrm{cl}\,(S_2))\big) \subset \mathrm{cl}\,(\mathrm{int}\,(\mathrm{cl}\,(S_1\cup S_2))),$$

and this concludes the proof. ∎

Lemma 13. *Let* (X,τ) *be e.d. Then for every* $S_1, S_2 \subset X$,

$$\operatorname{int}(\operatorname{cl}(S_1 \cup S_2)) = \operatorname{int}(\operatorname{cl}(S_1)) \cup \operatorname{int}(\operatorname{cl}(S_2)).$$

Proof. Follows easily from Lemma 12. ∎

Lemma 14. *In any* (X,τ), *if* $A \subset X$ *and* $U \in \mathrm{SO}\big(\mathrm{scl}_X(A), \tau_{\mathrm{scl}_X(A)}\big)$ *then* $U \cap A \in \mathrm{SO}(A, \tau_A)$.

Proof. For a certain $O \in \tau$, $V = O \cap \mathrm{scl}_X(A) \subset U \subset \mathrm{cl}_{\mathrm{scl}_X(A)}(V)$. Then $V \subset U \subset \mathrm{cl}_X(V) \cap \mathrm{scl}_X(A) \subset \mathrm{cl}_X\big(O \cap \mathrm{cl}_X(A)\big) \cap \mathrm{scl}_X(A) \subset \mathrm{cl}_X(O \cap A) \cap \mathrm{scl}_X(A) \subset \mathrm{cl}_X(O \cap A)$. Therefore we obtain

$$O \cap A \subset U \cap A \subset \mathrm{cl}_X(O \cap A) \cap A = \mathrm{cl}_A(O \cap A). \quad ∎$$

Theorem 23. *Let* (A, τ_A) *be an s-closed subspace of e.d.* (X,τ), *where* $A \in \mathrm{PO}(X,\tau)$. *Then the subspace* $\big(\mathrm{scl}_X(A), \tau_{\mathrm{scl}_X(A)}\big)$ *is s-closed.*

Proof. Let $\{U_\alpha : \alpha \in \nabla\} \subset \mathrm{SO}\big(\mathrm{scl}_X(A), \tau_{\mathrm{scl}_X(A)}\big)$ cover $\mathrm{scl}_X(A)$. By Lemma 14 the family $\{U_\alpha \cap A : \alpha \in \nabla\} \subset \mathrm{SO}(A, \tau_A)$ forms a cover of A. Since (A, τ_A) is s-closed, $A = \bigcup_{\alpha \in \nabla_0} \mathrm{scl}_A(U_\alpha \cap A)$ for some finite $\nabla_0 \subset \nabla$. Hence by Lemma 3 and by [20, Proposition 2.7(a)] we get $A \subset \bigcup_{\alpha \in \nabla_0} \big(\mathrm{int}_X(\mathrm{cl}_X(A)) \cap \mathrm{scl}_X(U_\alpha)\big)$, and since (X,τ) is e.d. we have by Lemmas 13 and 4

$$\begin{aligned}\mathrm{scl}_X(A) &\subset \mathrm{int}_X\Big(\mathrm{cl}_X\Big(\bigcup_{\alpha \in \nabla_0} \big(\mathrm{int}_X(\mathrm{cl}_X(A)) \cap \mathrm{scl}_X(U_\alpha)\big)\Big)\Big) \\ &= \bigcup_{\alpha \in \nabla_0} \big(\mathrm{int}_X(\mathrm{cl}_X(A)) \cap \mathrm{int}_X(\mathrm{cl}_X(\mathrm{scl}_X(U_\alpha)))\big).\end{aligned}$$

So, as $\mathrm{scl}_X(U_\alpha) \in \mathrm{SC}(X,\tau)$, $\alpha \in \nabla_0$, we obtain $\mathrm{scl}_X(A) = \bigcup_{\alpha \in \nabla_0} \mathrm{scl}_{\mathrm{scl}_X(A)}(U_\alpha)$. Thus $\mathrm{scl}_X(A)$ is s-closed. ∎

Lemma 15. *Let* $A \in \mathrm{SO}(X,\tau)$. *If* $\big(\mathrm{int}_X(A), \tau_{\mathrm{int}_X(A)}\big)$ *is s-closed, then for any cover* $\{V_i : i \in \nabla\} \subset \mathrm{SPO}(X,\tau)$ *of* A *there is some finite* $\nabla_0 \subset \nabla$ *such that* $A \subset \bigcup_{i \in \nabla_0} \mathrm{cl}_{\tau^\alpha}(V_i)$.

Proof. Let $\emptyset \neq \mathrm{int}_X(A) \subset A \subset \bigcup_{i \in \nabla} V_i$, where $V_i \in \mathrm{SPO}(X,\tau)$ for each $i \in \nabla$. Then $\mathrm{int}_X(A) = \bigcup_{i \in \nabla}\big(\mathrm{int}_X(A) \cap V_i\big)$ and by Lemma 9 we have

$$\mathrm{int}_X(A) \cap V_i \in \mathrm{SPO}\big(\mathrm{int}_X(A), \tau_{\mathrm{int}_X(A)}\big)$$

for $i \in \nabla$. By hypothesis there exists a finite $\nabla_0 \subset \nabla$ with

$$\mathrm{int}_X(A) = \bigcup_{i \in \nabla_0} \mathrm{scl}_{\mathrm{int}_X(A)}\big(\mathrm{int}_X(A) \cap V_i\big)$$

(see Theorem 5). Making use of Lemmas 3 and 8 we get

$$\mathrm{int}_X(A) \subset \bigcup_{i \in \nabla_0} \mathrm{scl}_X\big(\mathrm{int}_X(A) \cap (\mathrm{int}_X(A) \cap V_i)\big) \subset \bigcup_{i \in \nabla_0} \mathrm{cl}_X(\mathrm{int}_X(\mathrm{cl}_X(V_i))).$$

On the other hand, by [2, Theorem 1.5(c)], $\mathrm{cl}_{\tau^\alpha}(V) = \mathrm{cl}_X(\mathrm{int}_X(\mathrm{cl}_X(V)))$ for each $V \in \mathrm{SPO}\,(X,\tau)$. Therefore, since $A \in \mathrm{SO}\,(X,\tau)$,

$$A \subset \bigcup_{i\in\nabla_0} \mathrm{cl}_{\tau^\alpha}(V_i). \quad \blacksquare$$

THEOREM 24. *Let $A \in \mathrm{SO}\,(X,\tau)$. If the subspace $\big(\mathrm{int}_X(A), \tau_{\mathrm{int}_X(A)}\big)$ is s-closed then (A,τ_A) is $\mathcal{S}$-closed.*

Proof. Let $A = \bigcup_{i\in\nabla} U_i$ where $U_i \in \mathrm{SO}\,(A,\tau_A)$ for each $i \in \nabla$. By [29, Theorem 5], $U_i \in \mathrm{SO}\,(X,\tau)$. Since $\mathrm{SO}\,(X,\tau) \subset \mathrm{SPO}\,(X,\tau)$, from Lemma 15 we infer that for some finite $\nabla_0 \subset \nabla$, $A \subset \bigcup_{i\in\nabla_0} \mathrm{cl}_{\tau^\alpha}(U_i) = \bigcup_{i\in\nabla_0} \mathrm{cl}_\tau(U_i)$ [17, Lemma 1(i)]. Consequently, $A = \bigcup_{i\in\nabla_0} \mathrm{cl}_A(U_i)$. ■

By [19, Theorem 2.9] we have for each subset S of X that $\mathrm{cl}_{\tau^\alpha}(S) = \mathrm{scl}_X(S)$. Thus, by [17, Lemma 1(i)], it leads to the following theorem.

THEOREM 25. *Let (X,τ) be an e.d. space. Any of the two conditions: 'for every semi-open (or open) cover $\mathcal{U}$ of $A \subset X$ there is a finite subfamily $\mathcal{U}_0$ with $A \subset \mathrm{scl}_X\big(\bigcup \mathcal{U}_0\big)$', coincides with any of the properties: 'A is $\mathcal{S}$-closed relative to (X,τ)', 'A is s-closed relative to (X,τ)', 'A is $\mathcal{N}$-closed relative to (X,τ)', 'A is quasi $\mathcal{H}$-closed relative to (X,τ)'.*

Proof. We use [27, Theorem 14] (the reader is advised to compare [27, Theorem 2]. ■

The following result has been stated in [5, Theorem 2]: *a space (X,τ) is $\mathcal{S}$-closed if and only if every cover $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{RC}\,(X,\tau)$ of X admits a finite subcover.* This fact is a particular case of our next theorem.

THEOREM 26. *A subset A of (X,τ) is $\mathcal{S}$-closed relative to (X,τ) if and only if every cover $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{RC}\,(X,\tau)$ of A admits a finite subcover.*

Proof. Necessity. Let $A \subset \bigcup_{\alpha\in\nabla} V_\alpha$ where $V_\alpha \in \mathrm{RC}\,(X,\tau) \subset \mathrm{SO}\,(X,\tau)$ for each $\alpha \in \nabla$. So, by our assumption, $A \subset \bigcup_{\alpha\in\nabla_0} \mathrm{cl}\,(V_\alpha) = \bigcup_{\alpha\in\nabla_0} V_\alpha$ for some finite $\nabla_0 \subset \nabla$.

Sufficiency. Let $A \subset \bigcup_{\alpha\in\nabla} V_\alpha$ where $V_\alpha \in \mathrm{SO}\,(X,\tau)$ for each $\alpha \in \nabla$. Obviously $A \subset \bigcup_{\alpha\in\nabla} \mathrm{cl}\,(V_\alpha)$ and since $\mathrm{cl}\,(S) = \mathrm{cl}\,(\mathrm{int}\,(S))$ for every $S \in \mathrm{SO}\,(X,\tau)$ [30, Lemma 2], we get by hypothesis that there exists a finite $\nabla_0 \subset \nabla$ with $A \subset \bigcup_{\alpha\in\nabla_0} \mathrm{cl}\,(V_\alpha)$. ■

LEMMA 16. *Let $A \in \mathrm{RO}\,(X,\tau)$. Then for each $G \subset A$, $G \in \mathrm{RO}\,(X,\tau)$ if and only if $G \in \mathrm{RO}\,(A,\tau_A)$.*

Proof. Strong necessity. Let $A \in \tau$. We have

$$G = A \cap \mathrm{int}_X(\mathrm{cl}_X(G)) = \mathrm{int}_X\big(A \cap \mathrm{cl}_X(G)\big) = \mathrm{int}_X(\mathrm{cl}_A(G)) = \mathrm{int}_A(\mathrm{cl}_A(G)).$$

Sufficiency. This has been shown in the proof of [4, Theorem 6]. ■

In [31, Theorem 1.3] the following was proved: (X,τ) is $\mathcal{S}$-closed if and only if its every proper subset $S \in \mathrm{RO}\,(X,\tau)$ is $\mathcal{S}$-closed.

Theorem 27. *Let $A \in \mathrm{RO}\,(X,\tau)$. Then, the subspace (A,τ_A) is $\mathcal{S}$-closed if and only if every proper subset $G \subset A$ with $G \in \mathrm{RO}\,(X,\tau)$ is $\mathcal{S}$-closed.*

Theorem 28. *Let $A \in \mathrm{SO}\,(X,\tau)$, $B \in \mathrm{PO}\,(X,\tau)$, $A \cap B = \emptyset$. If the union $A \cup B$ is $\mathcal{S}$-closed relative to (X,τ), then B is $\mathcal{S}$-closed relative to (X,τ).*

Proof. Let a family $\mathcal{F} \subset \mathrm{SO}\,(X,\tau)$ be a cover of B. Then, the family $\mathcal{F} \cup \{A\}$ covers $A \cup B$. There exist $V_1,\ldots,V_n \in \mathcal{F}$ such that $A \cup B \subset \mathrm{cl}\,(A) \cup \bigcup_{i=1}^n \mathrm{cl}\,(V_i)$. So, by [35, Lemma 2.1] (see Remark 2) we obtain $B \subset \bigcup_{i=1}^n \mathrm{cl}\,(V_i)$. This completes the proof. ■

By [13, Theorem 1] the author has proved that a space (X,τ) is $\mathcal{S}$-disconnected if and only if there exists nonempty $U_1 \in \mathrm{SO}\,(X,\tau)$, $U_2 \in \tau^\alpha$ such that $X = U_1 \cup U_2$ and $\emptyset = U_1 \cap U_2$. Directly from this result together with Theorem 28, follows

Corollary 10. *Let (X,τ) be an $\mathcal{S}$-disconnected and $\mathcal{S}$-closed space. Then there exists a nonempty set $B \in \tau^\alpha$ which is $\mathcal{S}$-closed relative to (X,τ) (hence it is also such a subspace of (X,τ)* [32, Theorem 3.1]*).*

Theorem 29. *Let (X,τ) be $\mathcal{S}$-closed and $A \in \mathrm{CO}\,(X,\tau)$. Then $X \setminus A$ is an $\mathcal{S}$-closed subspace of (X,τ).*

Proof. Let $X \setminus A \subset \bigcup_{\alpha\in\nabla} V_\alpha$ where $\{V_\alpha : \alpha \in \nabla\} \subset \mathrm{RC}\,(X,\tau)$. By [5, Theorem 2] there is a finite $\nabla_0 \subset \nabla$ such that $X \subseteq A \cup \bigcup_{\alpha\in\nabla_0} V_\alpha$. From Theorem 26 we infer that $X \setminus A$ is $\mathcal{S}$-closed relative to (X,τ). Therefore, in view of [32, Theorem 3.1], $X \setminus A$ is $\mathcal{S}$-closed as a subspace. ■

Theorem 30. *Let $A \in \mathrm{CO}\,(X,\tau)$ be an $\mathcal{S}$-closed subspace of (X,τ). Then, (X,τ) is $\mathcal{S}$-closed if and only if $X \setminus A$ is an $\mathcal{S}$-closed subspace of (X,τ).*

Proof. Necessity. Theorem 29.

Sufficiency. By [32, Theorem 3.1], the set $X \setminus A$ is $\mathcal{S}$-closed relative to (X,τ). Thus, by [32, Theorem 3.6], $X = A \cup (X \setminus A)$ is $\mathcal{S}$-closed relative to (X,τ); i.e., (X,τ) is $\mathcal{S}$-closed. ■

Lemma 17. *Let $A \subset X$ be arbitrary, $B \in \mathrm{RC}\,(X,\tau)$, an let $A \cup B$ be $\mathcal{S}$-closed relative to (X,τ). Then $A \setminus B$ is $\mathcal{S}$-closed relative to (X,τ).*

Proof. This follows from Theorem 26. ■

Theorem 31. *Let (A,τ_A) and (B,τ_B) be $\mathcal{S}$-closed subspaces of (X,τ). If $A, B \in \mathrm{CO}\,(X,\tau)$ then $(A \setminus B, \tau_{A\setminus B})$ is $\mathcal{S}$-closed too.*

Proof. Use [32, Theorems 3.1 and 3.6] and Lemma 17. ■

REFERENCES

[**1**] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.

[**2**] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik **38** (1986), 24–32.

[**3**] R. H. Atia, S. N. El-Deeb, A. S. Mashhour, *α-compactness and α-homeomorphism*, preprint.

[**4**] C. K. Basu, *On locally s-closed spaces*, Inter. J. Math. Math. Sci. **19** (1996), 67–74.

[**5**] D. E. Cameron, *Properties of $\mathcal{S}$-closed spaces*, Proc. Amer. Math Soc. **72** (1978), 581–586.

[**6**] D. E. Cameron, G. Woods, *s-continuous and s-open mappings*, preprint.

[**7**] D. Carnahan, *Locally nearly-compact spaces*, Bolletino U.M.I. (4)6 (1972), 146–153.

[**8**] C. G. Crossley, S. K. Hildebrand, *Semi-closure*, Texas J. Sci. **22** (1971), 99–112.

[**9**] C. G. Crossley, S. K. Hildebrand, *Semi-closed sets and semi-continuity in topological spaces*, Texas J. Sci. **22** (1971), 123–126.

[**10**] G. Di Maio, T. Noiri, *On s-closed spaces*, Indian J. Pure Appl. Math. **18** (1987), 226–233.

[**11**] K. Dlaska, N. Ergun, M. Ganster, *On the topology generated by semi-regular sets*, Indian J. Pure Appl. Math. **25** (1994), 1163–1170.

[**12**] Ch. Dorsett, *Semiregularization spaces and the semi closure operator, s-closed spaces, and quasi-irresolute functions*, Indian J. Pure Appl. Math. **21** (1990), 416–422.

[**13**] Z. Duszyński, *On some concepts of weak connectedness of topological spaces*, Acta Math. Hungar. **110** (2006), 81–90.

[**14**] Z. Duszyński, *Remarks on $\mathcal{S}$-closedness in topological spaces*, Bolletino U.M.I. (8)10-B (2007), 469–483.

[**15**] Z. Duszyński, *A note on sets s- or $\mathcal{S}$-closed relative to a space and some separation axioms*, Ann. Univ. Bucuresti, Seria Mat. **57** (2008), 31–38.

[**16**] Z. Duszyński, *On pre-semi-open mappings*, Inter. J. Math. Game Th. Algebra **17** (2009), 255–267.

[**17**] G. L. Garg, D. Sivaraj, *Semitopological properties*, Mat. Vesnik **36** (1984), 137–142.

[**18**] T. Husain, *Almost continuous mappings*, Prace Mat. **10** (1966), 1–7.

[**19**] D. S. Janković, *On locally irreducible spaces*, Ann. Soc. Sci. Bruxelles **97** (1983), 59–72.

[**20**] D. S. Janković, *A note on mappings of extremally disconnected spaces*, Acta Math. Hungar. **46** (1985), 83–92.

[**21**] M. Khan, B. Ahmad, T. Noiri, *On s-closed subspaces*, Indian J. Pure Appl. Math. **28** (1997), 175–179.

[**22**] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.

[**23**] S. N. Maheshwari, R. Prasad, *Some new separation axioms*, Ann. Soc. Sci. Bruxelles **89** (1975), 395-402.

[**24**] H. Maki, K. Chandrasekhara Rao, M. Nagoor Gani, *On generalizing semi-open sets and preopen sets*, Pure Appl. Math. Sci. **49** (1999), 17–29.

[**25**] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[**26**] A. S. Mashhour, I. A. Hasanein, S. N. El-Deeb, *A note on semi-continuity and precontinuity*, Indian J. Pure Appl. Math. **13** (1982), 1119–1123.

[**27**] M. N. Mukherjee, C. K. Basu, *On $\mathcal{S}$-closed and s-closed spaces*, Bull. Malaysian Math. Sci. (S.S.) **15** (1992), 1–7.

[**28**] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.

[**29**] T. Noiri, *Remarks on semi-open mappings*, Bull. Cal. Math. Soc. **65** (1973), 197–201.

[**30**] T. Noiri, *On semi-continuous mappings*, Lincei-Rend. Sc. fis. mat. e nat. **54** (1973), 210–214.

[**31**] T. Noiri, *On $\mathcal{S}$-closed spaces*, Ann. Soc. Sci. Bruxelles **91** (1977), 189–194.

[**32**] T. Noiri, *On $\mathcal{S}$-closed subspaces*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **64** (1978), 157–162.

[**33**] T. Noiri, *On α-continuous functions*, Čas. pěst. mat. **109** (1984), 118–126.

[**34**] T. Noiri, B. Ahmad, *A note on semi-open functions*, Math. Sem. Notes Kobe Univ. **10** (1982), 437–441.

[**35**] T. Noiri, A. S. Mashhour, I. A. Hasanein, S. N. El-Deeb, *A note on $\mathcal{S}$-closed subspaces*, Math. Sem. Notes Kobe Univ. **10** (1982), 431–435.

[**36**] W. J. Pervin, *Foundations of general topology*, Academic Press, New York–London, 1964.

[**37**] V. Pipitone, G. Russo, *Spazi semiconnessi a spazi semiaperti*, Rend. Circ. Mat. Palermo **24** (1975), 273–285.

[**38**] J. R. Porter, J. Thomas, *On $\mathcal{H}$-closed and minimal Haussdorff spaces (1)*, Trans. Amer. Math. Soc. **138** (1969), 159–170.

[**39**] D. A. Rose, *Weak continuity and almost continuity*, Inter. J. Math. Math. Sci. **7** (1984), 311–318.

[**40**] T. Soundararajan, *Weakly Hausdorff spaces and cardinality of topological spaces*, General Topology and its Application to Modern Analysis and Algebra III, Proc. Conf. Kampur 1968, Academia, Prague, (1971), 301–306.

[**41**] S. F. Tadros, A. B. Khalaf, *On regular semi-open sets and s^*-closed spaces*, Tamkang J. Math. **23** (1992), 337–348.

[**42**] T. Thompson, *$\mathcal{S}$-closed spaces*, Proc. Amer. Math. Soc. **60** (1976), 335–338.

(received 25.05.2009; in revised form 03.11.2009)

Institute of Mathematics, Casimirus the Great University, Pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland

E-mail: `imath@ukw.edu.pl`