МАТЕМАТИЧКИ ВЕСНИК
62, 2 (2010), 137–144
June 2010

оригинални научни рад
research paper

STRONG CONVERGENCE THEOREMS OF COMMON FIXED POINTS FOR A PAIR OF QUASI-NONEXPANSIVE AND ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

Gurucharan Singh Saluja

Abstract. The purpose of this paper is to give necessary and sufficient condition of modified three-step iteration scheme with errors to converge to common fixed points for a pair of quasi-nonexpansive and asymptotically quasi-nonexpansive mappings in Banach spaces. The results presented in this paper generalize, improve and unify the corresponding results in [1, 3, 4, 8, 9].

1. Introduction and preliminaries

Let E be a real Banach space, K be a nonempty subset of E and $S, T\colon K \to K$ be two mappings. $F(S,T)$ denotes the set of common fixed points of S and T. We recall the following definitions.

Definition 1.1. Let $T\colon K \to K$ be a mapping:

(1) T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$.

(2) T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for all $x \in K$, $p \in F(T)$.

(3) T is said to be asymptotically nonexpansive if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $\|T^n x - T^n y\| \le (1 + r_n)\|x - y\|$ for all $x, y \in K$ and $n \ge 1$.

(4) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $\|T^n x - p\| \le (1 + r_n)\|x - p\|$ for all $x \in K$, $p \in F(T)$ and $n \ge 1$.

(5) T is said to be uniformly L-Lipschitzian if there exists a positive constant L such that $\|T^n x - T^n y\| \le L\|x - y\|$ for all $x, y \in K$ and $n \ge 1$.

(6) T is said to be uniformly quasi-Lipschitzian if there exists $L \in [1, +\infty)$ such that $\|T^n x - p\| \le L\|x - p\|$ for all $x \in K$, $p \in F(T)$ and $n \ge 1$.

2010 AMS Subject Classification: 47H09, 47H10.

Keywords and phrases: Quasi-nonexpansive mappings, asymptotically quasi nonexpansive mapping, common fixed points, modified three-step iteration scheme with errors with respect to a pair of mappings, strong convergence, Banach space.

From the above definitions, it follows that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, a uniformly L-Lipschitzian mapping must be uniformly quasi-Lipschitzian and an asymptotically quasi-nonexpansive mapping must be uniformly quasi-Lipschitzian. But the converse does not hold.

In 1973, Petryshyn and Williamson [8] established a necessary and sufficient condition for a Mann [7] iterative sequence to converge strongly to a fixed point of a quasi-nonexpansive mapping. Subsequently, Ghosh and Debnath [1] extended the results of [8] and obtained some necessary and sufficient condition for an Ishikawa-type iterative sequence to converge to a fixed point of a quasi-nonexpansive mapping. In 2001, Liu in [3, 4] extended the results of Ghosh and Debnath [1] to the more general asymptotically quasi-nonexpansive mappings. In 2006, Shahzad and Udomene [9] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space.

Recently, Liu et al. in [5, 6] study the weak and strong convergence of common fixed points for modified two and modified three-step iteration sequence with errors with respect to a pair of mappings S and T.

Motivated and inspired by Liu et al. in [5, 6] and others, we study the following iteration scheme for a pair of quasi-nonexpansive and asymptotically quasi-nonexpansive mappings. Our scheme is as follows.

Definition 1.2. Let K be a nonempty convex subset of a normed linear space E and $S, T\colon K \to K$ be two mappings. For an arbitrary $x_1 \in K$, the modified three-step iteration sequence with errors $\{x_n\}_{n\geq 1}$ with respect to S and T defined by:

$$\begin{aligned} z_n &= \alpha_n'' S x_n + \beta_n'' T^n x_n + \gamma_n'' u_n, \\ y_n &= \alpha_n' S x_n + \beta_n' T^n z_n + \gamma_n' v_n, \\ x_{n+1} &= \alpha_n S x_n + \beta_n T^n y_n + \gamma_n w_n, \quad \forall n \geq 1, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are sequences in $[0,1]$ satisfying

$$\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$$

and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in K.

Remark 1.1. In case $S = I$ and $\beta_n'' = \gamma_n'' = 0$ for $n \geq 1$, then the sequence $\{x_n\}_{n\geq 1}$ generated in (1.1) reduces to the usual modified Ishikawa iterative sequence with errors.

The purpose of this paper is to give necessary and sufficient condition to converge to a common fixed point of modified three-step iterative sequence with errors for a pair of quasi-nonexpansive and asymptotically quasi-nonexpansive mappings in a real Banach space. The results presented in this paper generalize, improve and unify the corresponding results of [1, 3, 4, 8, 9] and many others.

In the sequel we need the following lemmas to prove our main results.

Lemma 1.1. [10, Lemma 1] *Let* $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ *and* $\{r_n\}_{n=1}^\infty$ *be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \le (1+\beta_n)\alpha_n + r_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^\infty \beta_n < \infty$ *and* $\sum_{n=1}^\infty r_n < \infty$, *then* $\lim_{n\to\infty} \alpha_n$ *exists. In particular,* $\{\alpha_n\}_{n=1}^\infty$ *has a subsequence which converges to zero, then* $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.2. *Let* K *be a nonempty convex subset of a normed linear space* E. *Let* $S\colon K \to K$ *be a quasi-nonexpansive mapping and* $T\colon K \to K$ *be an asymptotically quasi-nonexpansive mapping with a sequence* $\{r_n\} \subset [0,\infty)$ *satisfying* $\lim_{n\to\infty} r_n = 0$ *such that* $\sum_{n=1}^\infty r_n < \infty$ *and* $F(S,T) \ne \emptyset$. *Let the sequence* $\{x_n\}_{n\ge1}$ *defined by* (1.1) *with the restrictions* $\sum_{n=1}^\infty \gamma_n < \infty$, $\sum_{n=1}^\infty \gamma'_n < \infty$, $\sum_{n=1}^\infty \gamma''_n < \infty$. *Then:*

(*a*) $\lim_{n\to\infty} \|x_n - p\|$ *exists for any* $p \in F(S,T)$.

(*b*) *There exists a constant* $M > 0$ *such that*

$$\|x_{n+m} - p\| \le M\|x_n - p\| + M\sum_{k=n}^{n+m-1} A_k,$$

for all $n, m \ge 1$ *and* $p \in F(S,T)$, *where* $M = e^{3\sum_{k=n}^{n+m-1} r_k}$.

Proof. (a) Let $p \in F(S,T)$. Note that $\{u_n - p\}_{n\ge1}$, $\{v_n - p\}_{n\ge1}$ and $\{w_n - p\}_{n\ge1}$ are bounded. It follows that $M = \sup\{\|u_n - p\|, \|v_n - p\|, \|w_n - p\| : n \ge 1\} < \infty$. Since S is quasi-nonexpansive and T is asymptotically quasi-nonexpansive, by (1.1) we note that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n Sx_n + \beta_n T^n y_n + \gamma_n w_n - p\| \\ &\le \alpha_n \|Sx_n - p\| + \beta_n \|T^n y_n - p\| + \gamma_n \|w_n - p\| \\ &\le \alpha_n \|x_n - p\| + \beta_n(1+r_n)\|y_n - p\| + \gamma_n \|w_n - p\| \\ &\le \alpha_n \|x_n - p\| + \beta_n(1+r_n)\|y_n - p\| + \gamma_n M \end{aligned} \tag{1.2}$$

and

$$\begin{aligned}\|y_n - p\| &= \|\alpha'_n Sx_n + \beta'_n T^n z_n + \gamma'_n v_n - p\| \\ &\le \alpha'_n \|Sx_n - p\| + \beta'_n \|T^n z_n - p\| + \gamma'_n \|v_n - p\| \\ &\le \alpha'_n \|x_n - p\| + \beta'_n(1+r_n)\|z_n - p\| + \gamma'_n \|v_n - p\| \\ &\le \alpha'_n \|x_n - p\| + \beta'_n(1+r_n)\|z_n - p\| + \gamma'_n M \end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\|z_n - p\| &= \|\alpha''_n Sx_n + \beta''_n T^n x_n + \gamma''_n u_n - p\| \\ &\le \alpha''_n \|Sx_n - p\| + \beta''_n \|T^n x_n - p\| + \gamma''_n \|u_n - p\| \\ &\le \alpha''_n \|x_n - p\| + \beta''_n(1+r_n)\|x_n - p\| + \gamma''_n \|u_n - p\| \\ &\le (\alpha''_n + \beta''_n)(1+r_n)\|x_n - p\| + \gamma''_n \|u_n - p\| \\ &= (1-\gamma''_n)(1+r_n)\|x_n - p\| + \gamma''_n \|u_n - p\| \\ &\le (1+r_n)\|x_n - p\| + \gamma''_n M. \end{aligned} \tag{1.4}$$

Substituting (1.4) into (1.3), we have

$$\begin{aligned}
\|y_n - p\| &\le \alpha'_n \|x_n - p\| + (1+r_n)\beta'_n[(1+r_n)\|x_n - p\| + \gamma''_n M] + \gamma'_n M \\
&\le (1+r_n)^2(\alpha'_n + \beta'_n)\|x_n - p\| + \beta'_n(1+r_n)\gamma''_n M + \gamma'_n M \\
&= (1+r_n)^2(1-\gamma'_n)\|x_n - p\| + \beta'_n(1+r_n)\gamma''_n M + \gamma'_n M \\
&\le (1+r_n)^2\|x_n - p\| + (1+r_n)[\gamma'_n + \gamma'_n]M. \qquad (1.5)
\end{aligned}$$

Substituting (1.5) into (1.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n\|x_n - p\| + \beta_n(1+r_n)[(1+r_n)^2\|x_n - p\| \\
&\quad + (1+r_n)(\gamma'_n + \gamma'_n)M] + \gamma_n M \\
&\le (1+r_n)^3(\alpha_n + \beta_n)\|x_n - p\| + \beta_n(1+r_n)^2(\gamma'_n + \gamma'_n)M + \gamma_n M \\
&= (1+r_n)^3(1-\gamma_n)\|x_n - p\| + \beta_n(1+r_n)^2(\gamma'_n + \gamma'_n)M + \gamma_n M \\
&\le (1+r_n)^3\|x_n - p\| + (1+r_n)^2[\gamma_n + \gamma'_n + \gamma''_n]M \\
&= (1+r_n)^3\|x_n - p\| + A_n \qquad (1.6)
\end{aligned}$$

where $A_n = (1+r_n)^2[\gamma_n + \gamma'_n + \gamma''_n]M$. Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma'_n < \infty$ and $\sum_{n=1}^{\infty}\gamma''_n < \infty$, so that $\sum_{n=1}^{\infty} A_n < \infty$, thus by Lemma 1.1, we have $\lim_{n\to\infty}\|x_n - p\|$ exists. This completes the proof of part (a).

(b) Since $1 + x \le e^x$ for all $x > 0$. Then from part (a) it can be obtained that

$$\begin{aligned}
\|x_{n+m} - p\| &\le (1+r_{n+m-1})^3\|x_{n+m-1} - p\| + A_{n+m-1} \\
&\le e^{3r_{n+m-1}}\|x_{n+m-1} - p\| + A_{n+m-1} \\
&\le e^{3r_{n+m-1}}[e^{3r_{n+m-2}}\|x_{n+m-2} - p\| + A_{n+m-2}] + A_{n+m-1} \\
&\le e^{3(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2} - p\| + e^{3r_{n+m-1}}A_{n+m-2} + A_{n+m-1} \\
&\le e^{3(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2} - p\| + e^{3r_{n+m-1}}[A_{n+m-2} + A_{n+m-1}] \\
&\le \cdots \\
&\le e^{3\sum_{k=n}^{n+m-1} r_k}.\|x_n - p\| + e^{3\sum_{k=n}^{n+m-1} r_k}.\sum_{k=n}^{n+m-1} A_k \\
&\le M\|x_n - p\| + M\sum_{k=n}^{n+m-1} A_k, \quad \textit{where} \quad M = e^{3\sum_{k=n}^{n+m-1} r_k}.
\end{aligned}$$

This completes the proof of part (b). ■

2. Main results

Theorem 2.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S\colon K \to K$ be a quasi-nonexpansive mapping and $T\colon K \to K$ be an asymptotically quasi-nonexpansive mapping with a sequence $\{r_n\} \subset [0,\infty)$ satisfying $\lim_{n\to\infty} r_n = 0$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(S,T) \ne \emptyset$. Let the sequence $\{x_n\}_{n\ge1}$ defined by (1.1) with the restrictions $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma'_n < \infty$, $\sum_{n=1}^{\infty}\gamma''_n < \infty$. Then $\{x_n\}_{n\ge1}$ converges strongly to a common fixed point of the mappings S and T if and only if $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$, where $d(x, F(S,T))$ denotes the distance between x and the set $F(S,T)$.*

Proof. The necessity is obvious. Thus we only prove the sufficiency. For all $p \in F(S,T)$, by equation (1.6) of Lemma 1.2, we have

$$\|x_{n+1} - p\| \le (1 + r_n)^3 \|x_n - p\| + A_n, \quad \forall n \in N \tag{2.1}$$

where $A_n = (1 + r_n)^2[\gamma_n + \gamma'_n + \gamma''_n]M$. Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma''_n < \infty$, so that $\sum_{n=1}^{\infty} A_n < \infty$, so from equation (2.1), we obtain

$$d(x_{n+1}, F(S,T)) \le (1 + r_n)^3 d(x_n, F(S,T)) + A_n \tag{2.2}$$

Since $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$ and from Lemma 1.1, we have $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$.

Next we will show that $\{x_n\}$ is a Cauchy sequence. For all $\varepsilon_1 > 0$, from Lemma 1.2, it can be known there must exists a constant $M > 0$ such that

$$\|x_{n+m} - p\| \le M \|x_n - p\| + M \sum_{k=n}^{n+m-1} A_k, \quad \forall n, m \in N, \ \forall p \in F(S,T). \tag{2.3}$$

Since $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$ and $\sum_{k=n}^{\infty} A_k < \infty$, then there must exists a constant N_1, such that when $n \ge N_1$

$$d(x_n, F(S,T)) < \frac{\varepsilon_1}{4M}, \tag{2.4}$$

and

$$\sum_{k=n}^{\infty} A_k < \tfrac{\varepsilon_1}{2M}. \tag{2.5}$$

So there must exists $p^* \in F(S,T)$, such that

$$d(x_{N_1}, F(S,T)) = \|x_{N_1} - p^*\| < \frac{\varepsilon_1}{4M}. \tag{2.6}$$

From (2.3), (2.5) and (2.6) it can be obtained that when $n \ge N_1$

$$\begin{aligned} \|x_{n+m} - x_n\| &\le \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\le M \|x_{N_1} - p^*\| + M \sum_{k=N_1}^{\infty} A_k + M \|x_{N_1} - p^*\| \\ &\le 2M \|x_{N_1} - p^*\| + M \sum_{k=N_1}^{\infty} A_k \\ &< 2M \cdot \frac{\varepsilon_1}{4M} + M \cdot \frac{\varepsilon_1}{2M} < \varepsilon_1 \end{aligned} \tag{2.7}$$

that is $\|x_{n+m} - x_n\| < \varepsilon_1$.

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = y^*$. Then $y^* \in K$. It remains to show that $y^* \in F(S,T)$. Let $\varepsilon_2 > 0$ be given. Then there exists a natural number N_2 such that

$$\|x_n - y^*\| < \frac{\varepsilon_2}{2(L+1)}, \quad \forall n \ge N_2. \tag{2.8}$$

Since $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$, there must exists a natural number $N_3 \ge N_2$ such that for all $n \ge N_3$, we have

$$d(x_n, F(S,T)) < \frac{\varepsilon_2}{3(L+1)}, \tag{2.9}$$

and in particular, we have

$$d(x_{N_3}, F(S,T)) < \frac{\varepsilon_2}{3(L+1)}. \tag{2.10}$$

Therefore, there exists $z^* \in F(S,T)$ such that

$$\|x_{N_3} - z^*\| < \frac{\varepsilon_2}{2(L+1)}. \tag{2.11}$$

Consequently, we have

$$\begin{aligned}
\|Ty^* - y^*\| &= \|Ty^* - z^* + z^* - x_{N_3} + x_{N_3} - y^*\| \\
&\le \|Ty^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le L\,\|y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le L\,\|y^* - x_{N_3} + x_{N_3} - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le L[\|y^* - x_{N_3}\| + \|x_{N_3} - z^*\|] + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le (L+1)\,\|y^* - x_{N_3}\| + (L+1)\,\|z^* - x_{N_3}\| \\
&< (L+1)\cdot\frac{\varepsilon_2}{2(L+1)} + (L+1)\cdot\frac{\varepsilon_2}{2(L+1)} < \varepsilon_2.
\end{aligned} \tag{2.12}$$

This implies that $y^* \in F(T)$. Similarly, we can show that $y^* \in F(S)$. Since S is quasi-nonexpansive, so it is uniformly quasi-1 Lipschitzian, so here taking $L = 1$, we have

$$\begin{aligned}
\|Sy^* - y^*\| &= \|Sy^* - z^* + z^* - x_{N_3} + x_{N_3} - y^*\| \\
&\le \|Sy^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le \|y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le \|y^* - x_{N_3} + x_{N_3} - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le [\|y^* - x_{N_3}\| + \|x_{N_3} - z^*\|] + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\le 2\,\|y^* - x_{N_3}\| + 2\,\|z^* - x_{N_3}\| < 2\cdot\frac{\varepsilon_2}{4} + 2\cdot\frac{\varepsilon_2}{4} < \varepsilon_2.
\end{aligned} \tag{2.13}$$

This shows that $y^* \in F(S)$. Thus $y^* \in F(S,T)$, that is, y^* is a common fixed point of the mappings S and T. This completes the proof. ∎

Theorem 2.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S\colon K \to K$ be a quasi-nonexpansive mapping and $T\colon K \to K$ be an asymptotically quasi-nonexpansive mapping with a sequence $\{r_n\} \subset [0,\infty)$ satisfying $\lim_{n\to\infty} r_n = 0$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(S,T) \neq \emptyset$. Let the sequence $\{x_n\}_{n\ge1}$ defined by (1.1) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$. Then $\{x_n\}_{n\ge1}$ converges strongly to a common fixed point p of the mappings S and T if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.*

Proof. The proof of Theorem 2.2 follows from Lemma 1.1 and Theorem 2.1. ∎

Theorem 2.3. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S\colon K \to K$ be a quasi-nonexpansive mapping and $T\colon K \to K$*

be an asymptotically quasi-nonexpansive mapping with a sequence $\{r_n\} \subset [0, \infty)$ *satisfying* $\lim_{n\to\infty} r_n = 0$ *such that* $\sum_{n=1}^{\infty} r_n < \infty$ *and* $F(S,T) \neq \emptyset$. *Let the sequence* $\{x_n\}_{n\geq 1}$ *defined by* (1.1) *with the restrictions* $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, $\sum_{n=1}^{\infty} \gamma_n'' < \infty$. *Suppose that the mapping* S *and* T *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$;

(ii) *there exists a constant* $A > 0$ *such that* $\{\|x_n - Sx_n\| + \|x_n - Tx_n\|\} \geq Ad(x_n, F(S,T))$, $\forall n \geq 1$.

Then $\{x_n\}_{n\geq 1}$ *converges strongly to a common fixed point of the mappings* S *and* T.

Proof. From conditions (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$, it follows as in the proof of Theorem 2.1, that $\{x_n\}_{n\geq 1}$ must converges strongly to a common fixed point of the mappings S and T. ∎

Example 2.1. Let E be the real line with the usual norm $|\cdot|$ and $K = [0,1]$. Define S and $T\colon K \to K$ by

$$Tx = \sin x, \quad x \in [0,1] \quad \text{and} \quad Sx = x, \quad x \in [0,1],$$

for $x \in K$. Obviously $T(0) = 0$ and $S(0) = 0$, that is, 0 is a common fixed point of S and T, that is, $F(S,T) = \{0\}$. Now we check that T is asymptotically quasi-nonexpansive. In fact, if $x \in [0,1]$ and $p = 0 \in [0,1]$, then

$$|Tx - p| = |Tx - 0| = |\sin x - 0| = |\sin x| \leq |x| = |x - 0| = |x - p|,$$

that is $|Tx - p| \leq |x - p|$. That is, T is quasi-nonexpansive. It follows that T is uniformly quasi-Lipschitzian and asymptotically quasi-nonexpansive with $k_n = 1$ for each $n \geq 1$. Similarly, we can verify that S is quasi-nonexpansive, for if $x \in [0,1]$ and $p = 0 \in [0,1]$, then $|Sx - p| = |Sx - 0| = |x - 0| = |x - p|$.

Remark 2.1. Theorem 2.1 extends, improves and unifies the corresponding results of [1, 3, 4, 8, 9]. Especially Theorem 2.1 extends, improves and unifies Theorems 1 and 2 in [4], Theorem 1 in [3] and Theorem 3.2 in [9] in the following ways:

(1) The identity mapping in [3, 4, 9] is replaced by a more general quasi-nonexpansive mapping.

(2) The usual Ishikawa iteration scheme in [3], the usual modified Ishikawa iteration scheme with errors in [4] and the usual modified Ishikawa iteration scheme with errors for two mappings are extended to the modified three-step iteration scheme with errors with respect to a pair of mappings.

Remark 2.2. Theorem 2.2 extends, improves and unifies Theorem 3 in [4] and Theorem 2.3 extends, improves and unifies Theorem 3 in [3] in the following aspects:

(1) The identity mapping in [3] and [4] is replaced by a more general quasi-nonexpansive mapping.

(2) The usual Ishikawa iteration scheme in [3] and the usual modified Ishikawa iteration scheme with errors in [4] are extended to the modified three-step iteration scheme with errors with respect to a pair of mappings.

Remark 2.3. Recently, Zhao and Wang in [12] and Xiao et al. in [11] have studied respectively a finite family of asymptotically nonexpansive and a finite family of asymptotically quasi-nonexpansive mappings and have proved some strong convergence theorems while in this paper we have taken a pair of different mappings, one is quasi-nonexpansive and other is asymptotically quasi-nonexpansive mapping and have given a necessary and sufficient condition of strong convergence of common fixed points for the above mappings.

Acknowledgement. The author thanks the referees for their valuable suggestions and comments.

REFERENCES

[**1**] M.K. Ghosh, L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings*, J. Math. Anal. Appl. **207** (1997), 96–103.

[**2**] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[**3**] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl. **259** (2001), 1–7.

[**4**] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl. **259** (2001), 18–24.

[**5**] Z. Liu, R.P. Agarwal, C. Feng, S. M. Kang, *Weak and strong convergence theorems of common fixed points for a pair of nonexpansive and asymptotically nonexpansive mappings*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **44** (2005), 83–96.

[**6**] Z. Liu, C. Feng, J.S. Ume, S. M. Kang, *Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings*, Taiw. J. Math. **11** (2007), 27–42.

[**7**] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[**8**] W.V. Petryshyn, T.E. Williamson, *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, J. Math. Anal. Appl. **43** (1973), 459–497.

[**9**] N. Shahzad, A. Udomene, *Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., Vol. 2006, Article ID 18909, Pages 1–10.

[**10**] K.K. Tan, H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301–308.

[**11**] J.Z. Xiao, J. Sun, X. Huang, *Approximating common fixed points of asymptotically quasi-nonexpansive mappings by a* $(k+1)$*- step iterative scheme with error terms*, J. Comp. Appl. Math. (2009), doi:10.1016/j.cam.2009.09.039.

[**12**] P. Zhao, L. Wang, *Strong convergence theorems of common fixed points for a finite family of asymptotically nonexpansive mappings*, Math. Sci. Res. J. **12** (2008), 225–231.

(received 17.04.2009; in revised form 07.12.2009)

Department of Mathematics & Information Technology, Govt. Nagarjun P.G. College of Science, Raipur (C.G.), India

E-mail: saluja_1963@rediffmail.com