МАТЕМАТИЧКИ ВЕСНИК
62, 1 (2010), 85–93
March 2010

оригинални научни рад
research paper

ON A CLASS OF SEQUENCES RELATED TO THE l^p SPACE DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

Ayhan Esi

Abstract. In this article we introduce the space $m(\Omega, \phi, q)$ on generalizing the sequence space $m(\phi)$ using the sequence of Orlicz functions. We study its different properties and obtain some inclusion results involving the space $m(\Omega, \phi, q)$.

1. Introduction

Let l_∞ be the set of all real or complex sequences $x = (x_k)$ with the norm $\|x\| = \sup_k |x_k| < \infty$. A linear functional L on l_∞ is said to be a Banach limit [1] if it has the properties: (a) $L(x) \geq 0$ if $x \geq 0$ (i.e., $x_k \geq 0$ for all $k \in \mathbb{N}$), (b) $L(e) = 1$, where $e = (1, 1, 1, \dots)$, (c) $L(Sx) = L(x)$, where the shift operator S is defined on l_∞ by $(Sx)_k = x_{k+1}$.

Let $\mathbf{B}$ be the set of all Banach limits on l_∞. A sequence x is said to be almost convergent to a number l if $L(x) = l$ for all $L \in \mathbf{B}$. Lorentz [5] has shown that x is almost convergent to l if and only if

$$t_{mn} = t_{mn}(x) = \frac{x_n + x_{n+1} + \cdots + x_{n+m}}{m+1} \to l \quad \text{as } m \to \infty, \text{ uniformly in } n.$$

Throughout the article $w(X)$, $l_\infty(X)$ and $l^p(X)$ denote respectively the spaces of all, bounded and p-absolutely summable sequences with the elements in X, where (X, q) is a seminormed space. By $\theta = (0, 0, \dots)$, we denote the zero element in X.

The sequence space $m(\phi)$ was introduced by Sargent [10]. He studied some of its properties and obtained its relationship with the space l^p. Later on it was investigated from sequence space point of view and related with summability theory by Rath and Tripathy [10], Tripathy [13],Tripathy and Sen [11], Tripathy and Mahanta [12], Esi [4] and others.

An Orlicz function is a function $M : (0, \infty] \to (0, \infty]$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

2010 AMS Subject Classification: 40A05, 46A45, 46E30.
Keywords and phrases: Seminorm; Orlicz function.

An Orlicz function is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u), u \geq 0$.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda \leq 1$.

2. Definitions and background

Throughout the article P_s denotes the class of subsets of $\mathbb{N}$, the natural numbers, those do not contain more than s elements. Further (ϕ_s) will denote non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$. The class of all the sequences (ϕ_s) satisfying this property is denoted by Φ.

The sequence space $m(\phi)$ introduced and studied by Sargent [10] which is defined by:

$$m(\phi) = \Big\{ x = (x_k) : \|x\| = \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty \Big\}.$$

Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to construct the sequence space

$$l_M = \Big\{ (x_k) : \sum_k M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0 \Big\}.$$

The space l_M becomes a Banach space with the norm

$$\|x\| = \inf\Big\{ \rho > 0 : \sum_k M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}$$

which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$, $1 \leq p < \infty$.

A generalization of Orlicz sequence space is due to Woo [14]. Let $\Omega = (M_k)$ be a sequence of Orlicz functions. Define the sequence space

$$l(M_k) = \Big\{ x = (x_k) : \sum_k M_k\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0 \Big\}$$

and equip this space with the norm

$$\|x\| = \inf\Big\{ \rho > 0 : \sum_k M_k\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

The space $l(M_k)$ is a Banach space and is called a modular sequence space. The space $l(M_k)$ also generalizes the concept of modulared sequence space earlier by Nakano [7], who considered the space $l(M_k)$ when $M_k(x) = x^{\alpha_k}$, where $1 \leq \alpha_k < \infty$ for $k \geq 1$.

In the later stage different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [8], Esi and Et [3], Esi [2] and many others.

In this article we introduce the following sequence spaces.

Let $\Omega = (M_k)$ be a sequence of Orlicz functions. Then

$$\widehat{l}_\infty(\Omega, q) = \Big\{ x = (x_k) : \sup_{m,n} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) < \infty,\ \text{for some } \rho > 0 \Big\},$$

$$m(\Omega, \phi, q) =$$

$$\Big\{ x = (x_k) : \sup_{m,n} \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) < \infty,\ \text{for some } \rho > 0 \Big\},$$

where

$$t_{mn}(x) = \frac{x_n + x_{n+1} + \cdots + x_{n+m}}{m+1}.$$

Let $x = (x_k)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of $x = (x_k)$ i.e.,

$$S(X) = \{(x_{\pi(k)}) : \pi(k) \text{ is a permutation on } \mathbb{N}\}.$$

A sequence space E is said to be symmetric if $S(X) \subset E$ for all $x \in E$.

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $x = (x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{B}$.

A sequence space E is said to be monotone, if it contains the canonical pre-images of its step spaces.

Lemma. *A sequence space E is monotone if E is solid.*

3. Main results

In this section we prove some results involving the sequence spaces $m(\Omega, \phi, q)$ and $\widehat{l}_\infty(\Omega, q)$.

Theorem 1. *$m(\Omega, \phi, q)$ and $\widehat{l}_\infty(\Omega, q)$ are linear spaces over the complex field* $\mathbf{C}$.

Proof. Let $x = (x_k), y = (y_k) \in m(\Omega, \phi, q)$ and $\alpha, \beta \in \mathbf{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{m,n} \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho_1}\Big)\Big) < \infty$$

and

$$\sup_{m,n} \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(y)}{\rho}\Big)\Big) < \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_k is non-decreasing convex function for all k and q is a seminorm, we have

$$\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(\alpha x + \beta y)}{\rho_3}\Big)\Big) \leq \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(\alpha x)}{\rho_3}\Big) + q\Big(\frac{t_{mn}(\beta y)}{\rho_3}\Big)\Big)$$

$$\leq \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho_1}\Big)\Big) + \sum_{m\in\sigma} M_m\Big(q\Big(\tfrac{t_{mn}(y)}{\rho_2}\Big)\Big).$$

$$\Rightarrow \sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(\alpha x+\beta y)}{\rho_3}\Big)\Big)$$

$$\leq \sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho_1}\Big)\Big)+\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(y)}{\rho_2}\Big)\Big).$$

Hence $m(\Omega,\phi,q)$ is a linear space.

The proof for the case $\widehat{l}_\infty(\Omega,q)$ is a routine work in view of the above proof. ■

THEOREM 2. *The space $m(\Omega,\phi,q)$ is a seminormed space, seminormed by*

$$g(x)=\inf\Big\{\rho>0:\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big)\leq 1\Big\}.$$

Proof. Clearly $g(x)\geq 0$ for all $x\in m(\Omega,\phi,q)$ and $g(\theta)=0$. Let $\rho_1>0$ and $\rho_2>0$ be such that

$$\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho_1}\Big)\Big)\leq 1$$

and

$$\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(y)}{\rho_2}\Big)\Big)\leq 1.$$

Let $\rho=\rho_1+\rho_2$. Then we have

$$\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x+y)}{\rho}\Big)\Big)$$

$$\leq \sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho_1}+\tfrac{t_{mn}(y)}{\rho_2}\Big)\Big)$$

$$\leq \sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma}\Big\{\frac{\rho_1}{\rho_1+\rho_2}M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho_1}\Big)\Big)+\frac{\rho_2}{\rho_1+\rho_2}M_m\Big(q\Big(\tfrac{t_{mn}(y)}{\rho_2}\Big)\Big)\Big\}$$

$$\leq \frac{\rho_1}{\rho_1+\rho_2}\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho_1}\Big)\Big)+$$

$$+\frac{\rho_2}{\rho_1+\rho_2}\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(y)}{\rho_2}\Big)\Big)\leq 1.$$

Since the ρ's are non-negative, we have

$$g(x+y)=\inf\Big\{\rho>0:\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\tfrac{t_{mn}(x+y)}{\rho}\Big)\Big)\leq 1\Big\}$$

$$\leq \inf\Big\{\rho_1>0:\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\tfrac{t_{mn}(x)}{\rho_1}\Big)\Big)\leq 1\Big\}$$

$$+\inf\Big\{\rho_2>0:\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(y)}{\rho_2}\Big)\Big)\leq 1\Big\}.$$

$$\Rightarrow g(x+y)\leq g(x)+g(y).$$

Next for $\lambda \in \mathbf{C}$, without loss of generality, let $\lambda \neq 0$, then

$$\begin{aligned} g(\lambda x) &= \inf\Big\{\rho > 0 : \sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\tfrac{t_{mn}(\lambda x)}{\rho}\Big)\Big) \leq 1\Big\} \\ &= \inf\Big\{|\lambda| r > 0 : \sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\tfrac{t_{mn}(x)}{r}\Big)\Big) \leq 1\Big\}, \text{ where } r = \tfrac{\rho}{|\lambda|} \\ &= |\lambda| \inf\Big\{r > 0 : \sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\tfrac{t_{mn}(x)}{r}\Big)\Big) \leq 1\Big\} = |\lambda| g(x). \end{aligned}$$

This completes the proof. ■

Proposition 3. *The space $\widehat{l_\infty}(\Omega, q)$ is a seminormed space, seminormed by*

$$h(x) = \inf\Big\{\rho > 0 : \sup_{m,n} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) \leq 1\Big\}.$$

The proof of Proposition 3 is a consequence of the above theorem.

Theorem 4. *$m(\Omega, \phi, q) \subset m(\Omega, \psi, q)$ if and only if $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$.*

Proof. Let $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$ and $x = (x_k) \in m(\Omega, \phi, q)$. Then

$$\sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) < \infty \quad \text{for some } \rho > 0.$$

$$\begin{aligned} &\Rightarrow \sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\psi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) \\ &\leq \Big(\sup_{s\geq 1} \frac{\phi_s}{\psi_s}\Big) \sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) < \infty. \end{aligned}$$

Then $x = (x_k) \in m(\Omega, \psi, q)$. Conversely, suppose that $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} = \infty$. Then there exists a sequence of natural numbers (s_i) such that $\lim_{i\to\infty} \frac{\phi_{s_i}}{\psi_{s_i}} = \infty$. Let $x = (x_k) \in m(\Omega, \phi, q)$, then there exists $\rho > 0$ such that

$$\sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) < \infty.$$

Now we have

$$\begin{aligned} &\sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\psi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) \\ &\geq \Big(\sup_{i\geq 1} \frac{\phi_{s_i}}{\psi_{s_i}}\Big) \sup_{m,n} \sup_{i\geq 1, \sigma\in P_{s_i}} \frac{1}{\phi_{s_i}} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) = \infty. \end{aligned}$$

Therefore $x = (x_k) \notin m(\Omega, \psi, q)$ which is a contradiction. Hence $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$. ■

The following result is a consequence of Theorem 4.

COROLLARY 5. *Let $\Omega = (M_m)$ be a sequence of Orlicz functions. Then $m(\Omega,\phi,q) = m(\Omega,\psi,q)$ if and only if $\sup_{s\geq 1}\xi_s < \infty$ and $\sup_{s\geq 1}\xi_s^{-1} < \infty$, where $\xi_s = \frac{\phi_s}{\psi_s}$ for all $s = 1,2,3,\ldots$.*

THEOREM 6. *Let $M, M_1, M_2 \in \Omega$ be Orlicz functions satisfying Δ_2-condition. Then*

(a) $m(M_1,\phi,q) \subset m(MoM_1,\phi,q)$,

(b) $m(M_1,\phi,q) \cap m(M_2,\phi,q) \subset m(M_1+M_2,\phi,q)$.

Proof. (a) Let $x = (x_k) \in m(M_1,\phi,q)$, then there exists $\rho > 0$ such that

$$\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma}M_1\Big(q\Big(\tfrac{t_{mn}(x)}{\rho}\Big)\Big) < \infty.$$

Let $0 < \varepsilon < 1$ and δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $y_{m,n} = M_1(q(\frac{t_{mn}(x)}{\rho}))$ and for any $\sigma \in P_s$, consider

$$\sum_{m\in\sigma}M(y_{m,n}) = \sum_1 M(y_{m,n}) + \sum_2 M(y_{m,n})$$

where the first summation is over $y_{m,n} \leq \delta$ and the second is over $y_{m,n} > \delta$. By the Remark, we have

$$\sum_1 M(y_{m,n}) \leq M(1)\sum_1 y_{m,n} \leq M(2)\sum_1 y_{m,n} \tag{3.1}$$

For $y_{m,n} > \delta$, we have $y_{m,n} < \frac{y_{m,n}}{\delta} \leq 1 + \frac{y_{m,n}}{\delta}$, since M is non-decreasing and convex, so

$$M(y_{m,n}) < M\Big(1+\frac{y_{m,n}}{\delta}\Big) < \frac{1}{2}M(2) + \frac{1}{2}M\Big(\frac{2y_{m,n}}{\delta}\Big).$$

Since M satisfies Δ_2-condition, so

$$M(y_{m,n}) < \frac{1}{2}K\frac{y_{m,n}}{\delta}M(2) + \frac{1}{2}K\frac{y_{m,n}}{\delta}M(2) = K\frac{y_{m,n}}{\delta}M(2).$$

Hence

$$\sum_2 M(y_{m,n}) \leq \max(1, K\delta^{-1}M(2))\sum_2 y_{m,n} \tag{3.2}$$

By (3.1) and (3.2), we have $x = (x_k) \in m(MoM_1,\phi,q)$.

(b) The proof is trivial. ■

Taking $M_1(x) = x$ in Theorem 6 (a), we have the following result.

COROLLARY 7. *Let $M \in \Omega$ be Orlicz function satisfying Δ_2-condition. Then $m(\phi,q) \subset m(M,\phi,q)$.*

From Theorem 4 and Corollary 7, we have:

Corollary 8. *Let $M \in \Omega$ be Orlicz function satisfying Δ_2-condition. Then $m(\phi, q) \subset m(M, \Psi, q)$ if and only if* $\sup_{s\geq 1} \frac{\phi_s}{\Psi_s} < \infty$.

Theorem 9. *Let $\Omega = (M_m)$ be a sequence of Orlicz functions. Then the sequence space $m(\Omega, \phi, q)$ is solid and symmetric.*

Proof. Let $x = (x_k) \in m(\Omega, \phi, q)$. Then there exists $\rho > 0$ such that

$$\sup_{m,n} \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) < \infty. \tag{3.3}$$

Let (λ_m) be a sequence of scalars with $|\lambda_m| \leq 1$ for all $m \in \mathbb{N}$. Then the result follows from (3.3) and the following inequality

$$\sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(\lambda x)}{\rho}\Big)\Big) \leq \sum_{m\in\sigma} |\lambda_m| M_m\Big(q\Big(\tfrac{t_{mn}(x)}{\rho}\Big)\Big) \quad \text{(by Remark)}$$
$$\leq \sum_{m\in\sigma} M_m\Big(q\Big(\tfrac{t_{mn}(x)}{\rho}\Big)\Big).$$

The symmetricity of the space follows from the definition of the space $m(\Omega, \phi, q)$ and symmetric sequence space. ∎

The following result follows from Theorem 9 and Lemma.

Corollary 10. *Let $\Omega = (M_m)$ be a sequence of Orlicz functions. Then the sequence space $m(\Omega, \phi, q)$ is monotone.*

The proof of the following result is a routine verification.

Proposition 11. *Let $\Omega = (M_m)$ be a sequence of Orlicz functions. Then the sequence space $\widehat{l}_\infty(\Omega, q)$ is solid and as such is monotone.*

Theorem 12. *Let $\Omega = (M_m)$ be a sequence of Orlicz functions. Then $m(\Omega, \phi, q) \subset \widehat{l}_\infty(\Omega, q)$.*

Proof. Let $x = (x_k) \in m(\Omega, \phi, q)$. Then there exists $\rho > 0$ such that

$$\sup_{m,n} \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x)}{\rho}\Big)\Big) < \infty \Rightarrow \sup_{m,n} \tfrac{1}{\phi_1} M_m\Big(q\Big(\tfrac{t_{mn}(x)}{\rho}\Big)\Big) < \infty$$

for some $\rho > 0$, (on taking cardinality of σ to be 1). This implies that $x = (x_k) \in \widehat{l}_\infty(\Omega, q)$. This completes the proof. ∎

Theorem 13. *Let $\Omega = (M_m)$ be a sequence of Orlicz functions. Then $m(\Omega, \phi, q) = \widehat{l}_\infty(\Omega, q)$ if and only if* $\sup_{s\geq 1} \frac{s}{\phi_s} < \infty$.

Proof. We have $m(\Omega, \psi, q) = \widehat{l}_\infty(\Omega, q)$ if $\psi_s = s$ for all $s \in \mathbb{N}$. By Theorem 4 and Theorem 9, it follows that $m(\Omega, \phi, q) = \widehat{l}_\infty(\Omega, q)$ if and only if $\sup_{s\geq 1} \frac{s}{\phi_s} < \infty$. This completes the proof. ∎

Since the proof of the following proposition is not hard, we give it without proof.

PROPOSITION 14. *Let $\Omega = (M_m)$ be a sequence of Orlicz functions and q_1 and q_2 be seminorms. Then*

(a) $m(\Omega, \phi, q_1) \cap m(\Omega, \phi, q_2) \subset m(\Omega, \phi, q_1 + q_2)$ *and* $\widehat{l}_\infty(\Omega, q_1) \cap \widehat{l}_\infty(\Omega, q_2) \subset \widehat{l}_\infty(\Omega, q_1 + q_2)$,

(b) *If q_1 is stronger than q_2, then* $m(\Omega, \phi, q_1) \subset m(\Omega, \phi, q_2)$ *and* $\widehat{l}_\infty(\Omega, q_1) \subset \widehat{l}_\infty(\Omega, q_2)$.

THEOREM 15. *Let $\Omega = (M_m)$ be a sequence of Orlicz functions and (X, q) be complete. Then $\{m(\Omega, \phi, q), g\}$ is also complete.*

Proof. Let (x^i) be a Cauchy sequence in $m(\Omega, \phi, q)$,where $x^i = (x^i_k) = (x^i_1, x^i_2, x^i_3, \dots) \in m(\Omega, \phi, q)$ for $i \in \mathbb{N}$. Let $r > 0$ and $x_0 > 0$ be fixed. Then for each $\frac{\varepsilon}{rx_0} > 0$, there exists a positive integer n_0 such that

$$g(x^i - x^j) < \frac{\varepsilon}{rx_0}, \text{ for all } i, j \geq n_0.$$

$$\Rightarrow \inf\Big\{\rho > 0 : \sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x^i - x^j)}{\rho}\Big)\Big) \leq 1\Big\} < \varepsilon, \quad (3.4)$$

for all $i, j \geq n_0$. We have for all for all $i, j \geq n_0$

$$\Rightarrow \sup_{m,n} \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{m\in\sigma} M_m\Big(q\Big(\frac{t_{mn}(x^i - x^j)}{g(x^i - x^j)}\Big)\Big) \leq 1, \text{ for some } \rho > 0.$$

$$\Rightarrow \sup_{m,n} \frac{1}{\phi_1} M_m\Big(q\Big(\frac{|t_{mn}(x^i - x^j)|}{g(x^i - x^j)}\Big)\Big) \leq 1$$

$$\Rightarrow M_m\Big(q\Big(\frac{t_{mn}(x^i - x^j)}{g(x^i - x^j)}\Big)\Big) \leq \phi_1, \text{ for all } i, j \geq n_0 \text{ and } m, n \in \mathbb{N}.$$

We can find $r > 0$ such that $\frac{rx_0}{2}\eta_m(\frac{x_0}{2}) \geq \phi_1$, where η_m is the kernel associated with M_m for all m, such that

$$M_m\Big(q\Big(\frac{|t_{mn}(x^i - x^j)|}{g(x^i - x^j)}\Big)\Big) \leq \frac{rx_0}{2}\eta_m(\frac{x_0}{2})$$

$$\Rightarrow q(t_{mn}(x^i - x^j)) < \frac{rx_0}{2}.\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}, \text{ for each } m, n \in \mathbb{N}.$$

In particular

$$q(t_{0n}(x^i - x^j)) = q(x^i_n - x^j_n) < \frac{rx_0}{2} \cdot \frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}, \text{ for each fixed } n.$$

Hence (x^i) is a Cauchy sequence in (X, q), which is complete. Therefore for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $q(x^i_n - x^j_n) \to 0$ as $i \to \infty$. Using the continuity of M_m for all m and q is a seminorm, so we have

$$\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma}M_m\Big(q\Big(\frac{t_{mn}(x^i-\lim_{j\to\infty}x^j)}{\rho}\Big)\Big)\leq 1$$
$$\Rightarrow \sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma}M_m\Big(q\Big(\frac{t_{mn}(x^i-x)}{\rho}\Big)\Big)\leq 1,\ \text{for some } \rho>0.$$

Now taking the infimum of such $\rho's$, by (3.4), we get

$$\inf\Big\{\rho>0:\sup_{m,n}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{m\in\sigma}M_m\Big(q\Big(\frac{t_{mn}(x^i-x)}{\rho}\Big)\Big)\leq 1\Big\}<\varepsilon,\ \text{for all } i\geq n_0.$$

Since $m(\Omega,\phi,q)$ is linear space and (x^i) and (x^i-x) are in $m(\Omega,\phi,q)$, so it follows that

$$(x)=(x-x^i)+(x^i)\in m(\Omega,\phi,q).$$

Hence $m(\Omega,\phi,q)$ is complete. ∎

REFERENCES

[**1**] S. Banach, *Theorie de l'Operations Linearies*, Chelsea, New York, 1955.

[**2**] A. Esi, *Some new sequence spaces defined by Orlicz functions*, Bull. Inst. Math. Acad. Sin. **27** (1999), 71–76.

[**3**] A. Esi, M. Et, *Some new sequence spaces defined by a sequence of Orlicz functions*, Indian J. Pure Appl. Math. **31** (2000), 967–972.

[**4**] A. Esi, *On a class of new type difference sequence spaces related to the space*, Far East J. Math. Sci. **13** (2004), 167–172.

[**5**] G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[**6**] J. Lindenstrauss, L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math. **10** (1971), 379–390.

[**7**] H. Nakano, *Modulared sequence spaces*, Proc. Japan Acad. **27** (1951), 508–512.

[**8**] S.D. Parashar, B. Choudhary, *Sequence spaces defined by Orlicz function*, Indian J. Pure Appl. Math. **25** (1994), 419–428.

[**9**] D. Rath, B.C. Tripathy, *Characterization of certain matrix operators*, J. Orissa Math. Soc. **8** (1989), 121–134.

[**10**] W.L.C. Sargent, *Some sequence spaces related to spaces*, J. London Math. Soc. **35** (1960), 161–171.

[**11**] B.C. Tripathy, M. Sen, *On a new class of sequences related to the space*, Tamkang J. Math. **33** (2002), 167–171.

[**12**] B.C. Tripathy, S. Mahanta, *On a class of sequences related to the space defined by Orlicz Functions*, Soochow J. Math. **29** (2003), 379–391.

[**13**] B.C. Tripathy, *Matrix maps on the power series convergent on the unit disc*, J. Analysis **6** (1998), 27–31.

[**14**] J.Y.T. Woo, *On modular sequence spaces*, Studia Math. **48** (1973), 271–289.

(received 24.03.2009, in revised form 31.07.2009)

Adiyaman University, Science and Art Faculty, Department of Mathematics, 02040, Adiyaman, Turkey

E-mail: aesi23@hotmail.com