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ON *so*-METRIZABLE SPACES

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Abstract. In this paper, we give some new characterizations for *so*-metrizable spaces, which answers a question posed by Z. Li and generalize some results on *so*-metrizable spaces. As some applications of the above results, some mappings theorems on *so*-metrizable spaces are obtained.

1. Introduction

so-networks (i.e. sequentially-open networks) were introduced and investigated by S Lin in [15]. Spaces with a σ-locally finite *so*-network are called *so*-metrizable spaces, which lie between metrizable spaces and *sn*-metrizable spaces. In [16], S. Lin gave the following characterization for *so*-metrizable spaces (see [16, Corollary 2.9 and Theorem 3.15]).

THEOREM 1.1. *The following are equivalent for a space X:*

(1) X is an so-metrizable space.

(2) X is an $\aleph$-space and contains no closed subspace having S_2 or S_ω as its sequential coreflection.

Note that there exist the following characterizations for metrizable spaces and *sn*-metrizable spaces respectively.

THEOREM 1.2. [21, Corollary 9] *The following are equivalent for a space X:*

(1) X is a metrizable space.

(2) X has a σ-discrete base.

(3) X has a σ-hereditarily closure-preserving base.

(4) X is a first countable space with a σ-hereditarily closure-preserving k-network.

THEOREM 1.3. [9, Lemma 2.2] *The following are equivalent for a space X:*

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(1) X is an sn-metrizable space.

(2) X has a σ-discrete sn-network.

(3) X has a σ-hereditarily closure-preserving sn-network.

(4) X is an snf-countable space with a σ-hereditarily closure-preserving k-network.

Z. Li posed the following question [13, Question 3.2].

QUESTION 1.4. Whether there exist some characterizations for *so*-metrizable spaces, which are similar to Theorem 1.2 or Theorem 1.3?

In this paper, we answer the above question affirmatively and give some mappings theorems on *so*-metrizable spaces. Throughout this paper, all spaces are assumed to be regular T_1, and all mappings are continuous and onto. $\mathbf{N}$, ω and ω_1 denote the set of all natural numbers, the first infinite ordinal and the first uncountable ordinal respectively. The sequence $\{x_n : n \in \mathbf{N}\}$ and the sequence $\{P_n : n \in \mathbf{N}\}$ of subsets are abbreviated to $\{x_n\}$ and $\{P_n\}$ respectively. Let P be a subset of a space X and $\{x_n\}$ be a sequence in X. $\{x_n\}$ converging to x is eventually in P if $\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbf{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a collection of subsets of X and $x \in X$. Then $(\mathcal{P})_x$ denotes the subcollection $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$, $\bigcup \mathcal{P}$ and $\bigcap \mathcal{P}$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the intersection $\bigcap\{P : P \in \mathcal{P}\}$ respectively.

2. Characterizations

DEFINITION 2.1. [7,11] Let X be a space.

(1) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever $\{x_n\}$ is a sequence converging to x, then $\{x_n\}$ is eventually in P.

(2) Let $P \subset X$. P is called a sequentially-open subset in X if P is a sequential neighborhood of x in X for each $x \in P$. F is called a sequentially-closed subset in X if $X - F$ is sequentially-open in X.

(3) X is called a sequential space if each sequentially-open subset in X is open in X.

(4) X is called a k-space, if $F \subset X$ is closed in X iff $F \cap C$ is closed in C for every compact subset C in X.

REMARK 2.2. The following are well known.

(1) P is a sequential neighborhood of x in X iff each sequence $\{x_n\}$ converging to x is frequently in P.

(2) The intersection of finitely many sequentially-open subsets of x in X is a sequentially-open subset of x in X.

(3) sequential spaces $\Longrightarrow$ k-spaces.

DEFINITION 2.3. [4] Let $\mathcal{P}$ a collection of subsets of a space X.

(1) $\mathcal{P}$ is called closure-preserving if $\overline{\bigcup \mathcal{P}'} = \bigcup\{\overline{P} : P \in \mathcal{P}'\}$ for each $\mathcal{P}' \subset \mathcal{P}$.

(2) $\mathcal{P}$ is called hereditarily closure-preserving if any collection $\{H(P) : P \in \mathcal{P}\}$ is closure-preserving, where every $H(P) \subset P \in \mathcal{P}$.

Definition 2.4. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$.

(1) $\mathcal{P}$ is called a network of X [3], if whenever $x \in U$ with U open in X there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.

(2) $\mathcal{P}$ is called a *cs*-network of X [19], if for every convergent sequence S converging to a point $x \in U$ with U open in X, S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is called a k-network of X [19], if for every compact subset $K \subset U$ with U open in X, there exists a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

Definition 2.5. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X. Assume that $\mathcal{P}$ satisfies the following (a) and (b) for each $x \in X$.

(a) $\mathcal{P}_x$ is a network at x in X.

(b) If $P_1, P_2 \in \mathcal{P}_x$, then there exists $P \in \mathcal{P}_x$ such that $P \subset P_1 \cap P_2$.

(1) $\mathcal{P}$ is called an *sn*-network of X [16,19], if every element of $\mathcal{P}_x$ is a sequential neighborhood of x for each $x \in X$, where $\mathcal{P}_x$ is called an *sn*-network at x.

(2) $\mathcal{P}$ is called an *so*-network of X [15,16], if every element of $\mathcal{P}_x$ is a sequentially-open subset, where $\mathcal{P}_x$ is called an *so*-network at x.

Definition 2.6. [16] Let X be a space. X is an sof-countable (resp. snf-countable) space if for each $x \in X$, there exists an *so*-network (resp. *sn*-network) $\mathcal{P}_x$ at x in X such that $\mathcal{P}_x$ is countable.

Definition 2.7. Let X be a space.

(1) X is an *so*-metrizable space [13] if X has a σ-locally finite *so*-network.

(2) X is an *sn*-metrizable space [9] if X has a σ-locally finite *sn*-network.

(3) X is an $\aleph$-space [11] if X has a σ-locally finite k-network.

Remark 2.8. For a space, base $\Longrightarrow$ *so*-network $\Longrightarrow$ *sn*-network $\Longrightarrow$ *cs*-network. An *so*-network for a sequential space is a base. So the following hold:

(1) First-countable $\Longrightarrow$ sof-countable $\Longrightarrow$ snf-countable.

(2) First-countable $\Longleftrightarrow$ sequential and sof-countable.

(3) metrizable spaces $\Longrightarrow$ *so*-metrizable spaces $\Longrightarrow$ *sn*-metrizable spaces $\Longrightarrow$ $\aleph$ spaces.

(4) metrizable spaces $\Longleftrightarrow$ k- and *so*-metrizable spaces.

The following example shows that "sequential" in Remarks 2.8(2) can not be relaxed to "k".

Example 2.9. There exists a k-, sof-countable space X such that is not first-countable.

Proof. Let X be the Stone-Čech compactification $\beta\mathbf{N}$ of $\mathbf{N}$. Then X is compact, and so it is a k-space. Since each convergent sequence in $\beta\mathbf{N}$ is trivial, $\mathcal{P} = \{\{x\} : x \in X\}$ is an *so*-network of X, so X is *sof*-countable. It is known that X is not first countable. ∎

Definition 2.10. Let $S = \{1/n : n \in \mathbf{N}\} \cup \{0\}$ be a space with the usual topology induced from $\mathbf{R}$. For each $\alpha < \omega_1$, let S_α be a copy of S. Then S_{ω_1} denotes the quotient space obtained from the topological sum $\oplus_{\alpha<\omega_1} S_\alpha$ by mapping all the nonisolated points into one point [12].

Lemma 2.11. *Let $\mathcal{P}$ be a hereditarily closure-preserving collection of sequentially-open subsets of a space X. Then $\bigcap \mathcal{P}$ is a sequentially-open subset of X.*

Proof. Let $x \in \bigcap \mathcal{P}$, and let $\{x_n\}$ be a sequence converging to x. By Remark 2.2(1), we only need to prove that $\{x_n\}$ is frequently in $\bigcap \mathcal{P}$. If $x_n = x$ for infinitely many $n \in \mathbf{N}$, then $\{x_n\}$ is frequently in $\bigcap \mathcal{P}$. If $x_n \neq x$ for all but finitely many $n \in \mathbf{N}$, we may assume $x_n \neq x$ for all $n \in \mathbf{N}$, then $\mathcal{P}$ is finite. Indeed, suppose $\mathcal{P}$ is infinite. Then there exists an infinite subcollection $\{P_k : k \in \mathbf{N}\}$ of $\mathcal{P}$, where $P_k \neq P_l$ if $k \neq l$. Since $\{x_n\}$ converges to x and P_k is sequentially-open for each $k \in \mathbf{N}$, we can construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in P_k$ for each $k \in \mathbf{N}$. Note that $\mathcal{P}$ is hereditarily closure-preserving and $\{x_{n_k}\}$ converges to x. So $x \in \overline{\{x_{n_k} : k \in \mathbf{N}\}} = \{x_{n_k} : k \in \mathbf{N}\}$. This is a contradiction. So $\mathcal{P}$ is finite. By Remark 2.2(2), $\bigcap \mathcal{P}$ is sequentially-open. ∎

Lemma 2.12. *Let X be a space and $x \in X$. If there exists a σ-hereditarily closure-preserving network at x in X such that its every element is a sequentially-open subset in X, then there exists a countable and decreasing so-network at x in X.*

Proof. Let $\mathcal{P}' = \bigcup\{\mathcal{P}_n : n \in \mathbf{N}\}$ is a network at x in X, where $\mathcal{P}_n$ is hereditarily closure-preserving for each $n \in \mathbf{N}$ and every element of $\mathcal{P}'$ is a sequentially-open subset in X. We may assume each $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $n \in \mathbf{N}$, put $P_n = \bigcap \mathcal{P}_n$, then $P_{n+1} \subset P_n$ as $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Put $\mathcal{P} = \{P_n : n \in \mathbf{N}\}$, then $\mathcal{P}$ is countable and decreasing.

Claim 1. $\mathcal{P}$ is a network at x in X.

Let $x \in U$ with U open in X. Since $\mathcal{P}'$ is a *so*-network, there exists $P \in \mathcal{P}_n$ for some $n \in \mathbf{N}$ such that $x \in P \subset U$. Thus $x \in P_n \subset P \subset U$. This proves that $\mathcal{P}$ is a network at x in X.

Claim 2. If $P_i, P_j \in \mathcal{P}$, then $P_k \subset P_i \cap P_j$ for some $P_k \in \mathcal{P}$.

It is clear because $\mathcal{P}$ is countable and decreasing.

Claim 3. P_n is a sequentially-open subset for each $n \in \mathbf{N}$.

It holds from Lemma 2.11.

By the above three claims, $\mathcal{P}$ is a countable and decreasing *so*-network at x in X. ∎

Corollary 2.13. *Let a space X have a σ-hereditarily closure-preserving so-network. Then X is an sof-countable space.*

Lemma 2.14. *sof-countable space contains no copy of S_{ω_1}.*

Proof. Note that S_{ω_1} is a sequential space, but it is not first-countable. By Remark 2.8(2), S_{ω_1} is not sof-countable. Obviously, sof-countable spaces are hereditary to all subspaces. So sof-countable space contains no copy of S_{ω_1}. ∎

Lemma 2.15. *Let X be an sof-countable space with a σ-hereditarily closure-preserving k-network. Then X has a σ-discrete so-network.*

Proof. Since X is sof-countable, X contains no copy of S_{ω_1} from Lemma 2.14. Note that a space is an $\aleph$-space iff it has a σ-hereditarily closure-preserving k-network, and contains no copy of S_{ω_1} [12, Theorem 2.6]. So X is an $\aleph$-space. By [6, Theorem 4], X has a σ-discrete cs-network $\mathcal{B}$. For each $x \in X$, let $\mathcal{P}'_x$ be a countable *so*-network at x in X. By Remark 2.2(2), we can assume that each $\mathcal{P}'_x$ is decreasing. For each $x \in X$, put $\mathcal{B}_x = \{B \in \mathcal{B} : P \subset B \ for\ some\ P \in \mathcal{P}'_x\}$. By a similar way as in the proof of [18, Lemma 7(3)], $\mathcal{B}_x$ is a network at x in X. For each $B \in \mathcal{B}_x$, choose $P_B \in \mathcal{P}'_x$ such that $P_B \subset B$. Put $\mathcal{P}_x = \{P_B : B \in \mathcal{B}_x\}$, and put $\mathcal{P} = \bigcup_{x\in X} \mathcal{P}_x$. It suffices to prove the following three claims.

Claim 1. $\mathcal{P}$ is σ-discrete: It holds because $\bigcup_{x\in X} \mathcal{B}_x$ is σ-discrete.

Claim 2. Every element of $\mathcal{P}$ is sequentially-open: It is clear.

Claim 3. For each $x \in X$, $\mathcal{P}_x$ is a network at x in X: Let $x \in U$ with U open in X. Since $\mathcal{B}_x$ is a network at x in X, $x \in B \subset U$ for some $B \in \mathcal{B}_x$. By the construction of $\mathcal{P}_x$, there exists $P_B \in \mathcal{P}_x$ such that $x \in P_B \subset B \subset U$. So $\mathcal{P}_x$ is a network at x in X. ∎

Now we give the main theorem in this section, which answers Question 1.4 affirmatively.

Theorem 2.16. *The following are equivalent for a space X:*

(1) X has a σ-discrete so-network.

(2) X is an so-metrizable space.

(3) X has a σ-hereditarily closure-preserving so-network.

(4) X is an sof-countable space with a σ-hereditarily closure-preserving k-network.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3): Obvious.

(3) $\Longrightarrow$ (4): By Corollary 2.13, X is sof-countable. Note that every σ-hereditarily closure-preserving *so*-network of a space is a k-network [20, Proposition 1.2(2)]. So X has a σ-hereditarily closure-preserving k-network.

(4) $\Longrightarrow$ (1): It holds by Lemma 2.15. ∎

3. Invariance and inverse invariance under mappings.

Definition 3.1. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a closed (resp. an open) mapping [5] if $f(B)$ is closed (resp. open) in Y for every closed (resp. open) subset B in X.

(2) f is called an *sn*-open mapping [10] if there exists an *sn*-network $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y such that for each $y \in Y$ and each $x \in f^{-1}(y)$, whenever U is a neighborhood of x, then $P \subset f(U)$ for some $P \in \mathcal{P}_y$.

(3) f is called a perfect mapping [5] if f is closed and $f^{-1}(y)$ is a compact subset of X for each $y \in Y$.

Remark 3.2. (1) open mappings $\Longrightarrow$ *sn*-open mappings.

(2) It is easy to obtain a simple characterization for *sn*-open mappings: A mapping $f : X \longrightarrow Y$ is *sn*-open iff $f(B)$ is a sequentially-open subset in Y for every open subset B in X. (So more precisely, *sn*-open mappings should be called sequentially-open mappings).

Definition 3.3. A space X is said to have a G_δ-diagonal [11] if $\{(x, x) : x \in X\}$ is a G_δ-set in $X \times X$.

Definition 3.4. Let X be a space. Put $\sigma = \{P \subset X : P \textit{ is sequentially open in } X\}$, and endow X with the topology σ. The space (X, σ) is called sequential coreflection of X [16], and denoted by σX.

Definition 3.5. (1) Let $L_0 = \{a_n : n \in \mathbf{N}\}$ be a sequence converging to ∞, where $\infty \notin L_0$. For each $n \in \mathbf{N}$, let L_n be a sequence converging to b_n, where $b_n \notin L_n$. Put $T_0 = L_0 \cup \{\infty\}$ and $T_n = L_n \cup \{b_n\}$ for each $n \in \mathbf{N}$. Let M be the topological sum of $\{T_n : n \geq 0\}$. Then S_2 denotes the quotient space obtained from the topological sum M by identifying a_n with b_n for each $n \in \mathbf{N}$ [1].

(2) Let $S = \{1/n : n \in \mathbf{N}\} \cup \{0\}$ be a space with the usual topology induced from $\mathbf{R}$. For each $\alpha < \omega$, let S_α be a copy of S. Then S_ω denotes the quotient space obtained from the topological sum $\oplus_{\alpha<\omega} S_\alpha$ by mapping all the nonisolated points into one point [2].

It is easy to see that a closed image of a *so*-metrizable space need not be *so*-metrizable. Now we give a sufficient and necessary condition such that closed images of *so*-metrizable spaces are *so*-metrizable spaces.

Lemma 3.6. *Let $f : X \longrightarrow Y$ be a closed mapping, and let X have a σ-hereditarily closure-preserving k-network. Then Y is so-metrizable iff Y is sof-countable.*

Proof. Necessity is obvious. We only need to prove sufficiency. Let Y be *sof*-countable. Note that closed mappings preserve σ-hereditarily closure-preserving k-networks. So Y has a σ-hereditarily closure-preserving k-network. By theorem 2.16, Y is *so*-metrizable. ∎

We immediately obtain the following result by the above lemma.

THEOREM 3.7. *A closed image of an so-metrizable space is so-metrizable iff it is sof-countable.*

Perfect mappings preserve metrizable spaces. However, we do not know even whether finite-to-one, closed mappings preserve *so*-metrizable spaces. As an applications to Theorem 3.7, we give some partial answers to this question.

LEMMA 3.8. *Let $f : X \longrightarrow Y$ be an sn-open, closed mapping and each point in X be a G_δ-set. If P is a sequentially-open subset in X, then $f(P)$ is a sequentially-open subset in Y.*

Proof. Let P be a sequentially-open subset in X and $y \in f(P)$. Let $\{y_k\}$ be a sequence converging to y. It suffices to prove that $\{y_k\}$ is frequently in $f(P)$. Without loss of generality, we assume that $y_i \neq y_j$ for all $i \neq j$ and $y_k \neq y$ for all k. Pick $x \in P$ such that $f(x) = y$, then $\{x\}$ is a G_δ-set in X. Let $\{W_n : n \in \mathbf{N}\}$ be a sequence of open neighborhoods of x such that $\overline{W_{n+1}} \subseteq W_n$ for each $n \in \mathbf{N}$ and $\bigcap_{n\in\mathbf{N}} W_n = \{x\}$. For each $n \in \mathbf{N}$, $f(W_n)$ is a sequentially-open subset of Y by Remark 3.2(2). So $\{y_k\}$ is eventually in $f(W_n)$. Thus there exists $k_n \in \mathbf{N}$ such that $y_{k_n} \in f(W_n)$. Pick $x_n \in W_n \bigcap f^{-1}(y_{k_n})$. By this method, we construct a sequence $\{x_n\}$ such that $x_n \in W_n$ and $f(x_n) = y_{k_n}$ for each $n \in \mathbf{N}$. Here, we can assume that $\{f(x_n)\} = \{y_{k_n}\}$ is a subsequence of $\{y_k\}$. Now we prove that $\{x_n\}$ converges to x.

If $\{x_n\}$ does not converge to x, then there exists a neighborhood U of x such that $\{x_n\}$ is not eventually in U. So there exists a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \notin U$ for each $i \in \mathbf{N}$. Put $L = \{x_{n_i} : i \in \mathbf{N}\}$, then L is an infinite subset of X and x is not a cluster point of L. On the other hand, $\overline{f(L)} = f(\overline{L})$ since f is closed. Thus $y \in \overline{f(L)}$ and $y \notin f(L)$, so L has a cluster point $z \neq x$. Because $\{x\} = \bigcap_{n\in\mathbf{N}} W_n = \bigcap_{n\in\mathbf{N}} \overline{W_n}$, $z \in X - \overline{W_n}$ for some $n \in \mathbf{N}$. Note that $X - \overline{W_n}$ is a neighborhood and only contains finitely many points of L. This contradicts that z is a cluster point of L. Thus we prove that $\{x_n\}$ converges to x.

Since P is a sequentially-open subset in X and $x \in P$, $\{x_n\}$ is eventually in P, and so $\{f(x_n)\} = \{y_{k_n}\}$ is eventually in $f(P)$. This shows that $\{y_n\}$ is frequently in $f(P)$. ■

THEOREM 3.9. *Let $f : X \longrightarrow Y$ be an sn-open, closed mapping. If X is so-metrizable, then Y is so-metrizable.*

Proof. Let X be *so*-metrizable. By theorem 3.7, we need to prove that Y is *sof*-countable, Let $\mathcal{P}$ be a σ-hereditarily closure-preserving *so*-network of X. Put $\mathcal{F} = \{f(P) : P \in \mathcal{P}\}$, then $\mathcal{F}$ is σ-hereditarily closure-preserving because closed mappings preserve σ-hereditarily closure-preserving collections. Let $y \in Y$, put $\mathcal{F}_y = \{f(P) : P \in \mathcal{P}, x \in f^{-1}(y) \bigcap P\}$, then $\mathcal{F}_y \subset \mathcal{F}$ is σ-hereditarily closure-preserving. Since X is *so*-metrizable, each point in X is a G_δ-set. By Lemma 3.8, every element of $\mathcal{F}_y$ is a sequentially-open subset in Y. It suffices to prove that

$\mathcal{F}_y$ is a network at y in Y from Lemma 2.12. Let $y \in U$ with U open in Y. Pick $x \in f^{-1}(y)$, then $x \in f^{-1}(U)$. Since $\mathcal{P}$ is a network of X, there exists $P \in \mathcal{P}$ such that $x \in P \subset f^{-1}(U)$. Thus $y \in f(P) \subset U$ and $f(P) \in \mathcal{F}_y$. This proves that $\mathcal{F}_y$ is a network at y in Y. ■

COROLLARY 3.10. *Let $f : X \longrightarrow Y$ be an open, closed mapping. If X is so-metrizable, then Y is so-metrizable.*

A perfect inverse image of a metrizable space is metrizable iff it has a G_δ-diagonal [11, Corollary 3.8]. Naturally, one can ask whether "metrizable" in this result can be replaced by "*so*-metrizable". The answer to this question is affirmative.

LEMMA 3.11. *Let $f : X \longrightarrow Y$ be a closed mapping, where X has a G_δ-diagonal. If B is a sequentially-closed subset of X, then $f(B)$ is a sequentially-closed subset of Y.*

Proof. Let B be a sequentially-closed subset of X. If $f(B)$ is not a sequentially-closed subset in Y, there exists $y \notin f(B)$ and a sequence $\{y_n\}$ in $f(B)$ such that $\{y_n\}$ converges to y. We can assume that $y_n \neq y_m$ if $n \neq m$. Put $K = \{y_n : n \in \mathbf{N}\}$ and pick $x_n \in f^{-1}(y_n) \cap B$ for each $n \in \mathbf{N}$, then $\{x_n\}$ is a sequence in $f^{-1}(K)$. By [18, Lemma 2(b)], there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $x \in X$. Note that $x \in f^{-1}(y)$ and $y \notin f(B)$, so $x \in X - B$. Since $X - B$ is sequentially-open in X, $\{x_{n_k}\}$ is eventually in $X - B$. This contradicts that $x_n \notin X - B$ for each $n \in \mathbf{N}$. ■

LEMMA 3.12. *If X is an $\aleph$-space that contains no closed subspace having an $\aleph$-, non-metrizable space as its sequential coreflection, then X is so-metrizable.*

Proof. Let X be an $\aleph$-space that contains no closed subspace having an $\aleph$-, non-metrizable space as its sequential coreflection. S_2 and S_ω are $\aleph$-, non-metrizable spaces [17, Example 1.8.6 and Example 1.8.7], so X contains no closed subspace having S_2 or S_ω as its sequential coreflections. By Theorem 1.1, X is *so*-metrizable.

THEOREM 3.13. *Let $f : X \longrightarrow Y$ be a perfect mapping and Y be so-metrizable. Then X is so-metrizable iff X has a G_δ-diagonal.*

Proof. Necessity is obvious. We only need to prove sufficiency.

Let X have a G_δ-diagonal. By Remark 2.8(3) and [14, Theorem 3.4], X is an $\aleph$-space. By Lemma 3.12, it suffices to prove that X contains no closed subspace having an $\aleph$-, non-metrizable space as its sequential coreflection. If not, then there exists a closed subspace S of X such that σS is homeomorphic to an $\aleph$-, non-metrizable spac T. Put $g : \sigma S \longrightarrow \sigma f(S)$, where $g = f|_{\sigma S}$ is the restriction of f on σS.

(a) g is a closed mapping: Let F is a closed subset of σS. Then F is a sequentially-closed subset of S. It is clear that S has a G_δ-diagonal and $f|_S$:

$S \longrightarrow f(S)$ is a closed mapping. By Lemma 3.11, $f|_S(F)$ is a sequentially-closed subset of $f(S)$, so $g(S) = f|_S(S)$ is a closed subset of $\sigma f(S)$.

(b) g is a compact mapping: Let $y \in \sigma f(S)$. Then $f^{-1}(y)$ is a compact subset of X. Note that $f^{-1}(y)$ has a G_δ-diagonal. So $f^{-1}(y)$ is compact metrizable from [17, Theorem 1.4.10]. Thus the topology on $f^{-1}(y) \cap S$ as a subspace of σS is equivalent to the topology on $f^{-1}(y) \cap S$ as a subspace of X. Consequently, $g^{-1}(y) = f^{-1}(y) \cap S$ is compact.

By the above (a) and (b), g is a perfect mapping. Note that $\sigma S = T$ is an $\aleph$-space and perfect mappings preserve $\aleph$-spaces [14, Theorem 2.2]. So $\sigma f(S)$ is an $\aleph$-space. It is easy to see that $f(S)$, as a subspace of Y, is sof-countable. By [16, Corollary 2.8], $\sigma f(S)$ is first countable. Thus $\sigma f(S)$ is metrizable from Theorem 1.2, so σS is a perfect pre-image of a metrizable space. By [11, Corollary 3.8], σS is metrizable. This contradicts that $\sigma S = T$ is not metrizable. ∎

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