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MULTIPLICITIES OF COMPACT LIE GROUP REPRESENTATIONS VIA BEREZIN QUANTIZATION

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Abstract. Let G be a compact Lie group and π be a unitary representation of G on a reproducing kernel Hilbert space. We study some applications of Berezin quantization to the description of the irreducible decomposition of π.

1. Introduction

A general theory of quantization on Kähler manifolds was developed by F.A. Berezin in [9], [10] and [11]. Berezin quantization has various applications to the representation theory of Lie groups and Lie algebras. Let us mention some of them : constructions of realizations of semisimple Lie algebras by holomorphic differential operators [4], [8]; constructions of generalized Fourier transforms for compact Lie groups [2], [23]; contractions of unitary irreducible representations of $SU(n)$ to unitary irreducible representations of an Heisenberg group [13], [14].

Here we are concerned with applications of Berezin quantization to the irreducible decomposition of unitary representations of compact Lie groups. In [1], Berezin quantization was used to study the restriction of a unitary irreducible representation of a compact Lie group to a closed subgroup. Similarly, a new criterion for Gel'fand pairs was obtained in [12] by means of Berezin quantization. Furthermore, a method for weight multiplicity computation in representations of semisimple compact Lie groups was introduced in [3]. This method is based on a straightforward application of the Berezin quantization theory on flag manifolds.

The purpose of the present paper is to provide a general setting in which the preceding applications fit naturally. To this aim, we consider a compact Lie group G and a unitary representation of G on a Hilbert space $\mathcal{H}$. We assume that $\mathcal{H}$ is a reproducing kernel Hilbert space whose elements are functions on a manifold M. This allows us to introduce the Berezin quantization on M. In Section 2, we review some properties of the Berezin calculus on M which is a bijection from

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the class of all bounded operators on $\mathcal{H}$ onto a class of functions on M. We then study the problem of decomposing π into its irreducible components in the context of Berezin quantization (Section 3). In particular, we give some integral formulas for the multiplicity of a unitary irreducible representation in π. Then, we recover the results of [3] relative to the computation of weight multiplicities in the unitary irreducible representations of a compact semisimple Lie group (Section 5). We also study some examples in the case where $\mathcal{H}$ is infinite-dimensional (Section 6).

2. Generalities on Berezin quantization

In this section, we review some general facts on Berezin quantization [9], [10].

Let M be a locally compact second-countable Hausdorff space endowed with a Radon measure $\tilde{\mu}$. Let $\mathcal{H}$ be a reproducing kernel Hilbert space of square integrable functions on M with respect to $\tilde{\mu}$, that is, $\mathcal{H}$ is a Hilbert with respect to the L^2-norm and, for each $x \in M$, the evaluation map $\mathcal{H} \ni f \mapsto f(x)$ is continuous. Then, for each $x \in M$, there exists a unique function $e_x \in \mathcal{H}$ such that

$$f(x) = \langle f, e_x \rangle = \int_M f(y)\overline{e_x(y)}\, d\tilde{\mu}(y) \tag{2.1}$$

for every $f \in \mathcal{H}$. The function $k(x,y) := \overline{e_x(y)} = \overline{\langle e_x, e_y \rangle} = \langle e_y, e_x \rangle$ is called the reproducing kernel of $\mathcal{H}$.

Let G be a Lie group acting on M. We consider a cocycle $\alpha : G \times M \to \mathbf{C}^*$, the cocycle condition being

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 \cdot x)\alpha(g_2, x) \tag{2.2}$$

for all $g_1, g_2 \in G$ and $x \in M$. There is an action π of G on the space of functions on M, according to the formula

$$(\pi(g)f)(x) = \alpha(g^{-1}, x)\, f(g^{-1} \cdot x).$$

We assume that $\pi(g)(f) \in \mathcal{H}$ for each $g \in G$ and $f \in \mathcal{H}$. Then π induces a representation of G on $\mathcal{H}$. The following proposition can be proved easily, see [5] for instance.

Proposition 2.1. (1) *The representation π is unitary if and only if we have*

$$d\tilde{\mu}(g \cdot x) = |\alpha(g,x)|^2 d\tilde{\mu}(x) \qquad g \in G,\, x \in M \tag{2.3}$$

(2) *If the representation π is unitary then we have*

(i) $\pi(g)e_x = \overline{\alpha(g,x)}e_{g\cdot x} \qquad g \in G,\, x \in M,$ (2.4)

(ii) $k(g \cdot x, g \cdot y) = \alpha(g,x)^{-1}\overline{\alpha(g,y)}^{-1}k(x,y) \qquad g \in G,\, x, y \in M.$ (2.5)

(iii) *The measure $d\mu(x) := k(x,x)d\tilde{\mu}(x)$ is G-invariant.*

In the rest of this section and in the following section, we shall assume that Condition (2.3) is fulfilled.

Consider now a bounded operator A on $\mathcal{H}$. The Berezin (covariant) symbol of A is the function defined on M by

$$s(A)(x) = \frac{\langle A\, e_x\, ,\, e_x\rangle}{\langle e_x\, ,\, e_x\rangle} \tag{2.6}$$

and the double Berezin symbol of A is the function defined by

$$S(A)(x,y) = \frac{\langle A\, e_y\, ,\, e_x\rangle}{\langle e_y\, ,\, e_x\rangle} \tag{2.7}$$

for x, $y \in M$ such that $\langle e_x, e_y\rangle \neq 0$ (see [9]). The operator A can be recovered from its symbol as follows:

$$\begin{aligned} A\, f(x) &= \langle A\, f\, ,\, e_x\rangle = \langle f\, ,\, A^*\, e_x\rangle = \int_M f(y)\overline{A^*\, e_x(y)}\, d\tilde{\mu}(y) \\ &= \int_M f(y)\overline{\langle A^*\, e_x, e_y\rangle}\, d\tilde{\mu}(y) = \int_M f(y)\, S(A)(x,y)\langle e_y, e_x\rangle\, d\tilde{\mu}(y). \end{aligned}$$

In particular, the map $A \mapsto S(A)$ is thus injective. The following properties of the Berezin symbols will be needed later.

Proposition 2.2. (1) *If A is a trace-class operator on $\mathcal{H}$ then*

$$\mathrm{Tr}(A) = \int_M s(A)(x)\, d\mu(x). \tag{2.8}$$

In particular, we have that $\dim \mathcal{H} < +\infty$ if and only if the integral $\int_M d\mu(x)$ exists and, in this case, $\dim \mathcal{H} = \int_M d\mu(x)$.

(2) *Let A be a bounded operator on $\mathcal{H}$. Then we have*

$$S(\pi(g^{-1})A\pi(g))(x,y) = S(A)(g\cdot x, g\cdot y) \qquad g \in G,\ x,\ y \in M. \tag{2.9}$$

Proof. For Part (1), see [9]. Part (2) easily follows from (2.4). ∎

3. Multiplicities and Berezin quantization

We retain the notation from Section 2. From now, we assume that the group G is compact. Let $\pi = \sum_\sigma m(\sigma)\sigma$ be the decomposition of π into G-irreducible components. Let us assume moreover that the multiplicities $m(\sigma)$ are finite. We denote by V_σ the σ-isotypic component and by $P_\sigma\colon \mathcal{H} \to V_\sigma \subset \mathcal{H}$ the projection operator on V_σ. Let χ_σ be the character of σ and $d(\sigma)$ be the dimension of σ. We denote by dg the normalized Haar measure on G. Finally, for $g \in G$, we set $E(g) = s(\pi(g))$ and $L(g) = S(\pi(g))$.

Proposition 3.1. 1) *The double Berezin symbol of P_σ is given by*

$$S(P_\sigma)(x,y) = d(\sigma)\int_G L(g)(x,y)\chi_\sigma(g^{-1})\, dg. \tag{3.1}$$

2) *We have*

$$m(\sigma) = \int_M d\mu(x) \left(\int_G E(g)(x)\chi_\sigma(g^{-1})\, dg \right). \tag{3.2}$$

Proof. 1) From elementary representation theory, we have $P_\sigma = d(\sigma) \times \int_G \pi(g)\chi_\sigma(g^{-1})\, dg$. Equation (3.1) then follows. 2) Applying Equation (2.8) to the operator P_σ and using 1), we obtain the desired result. ■

Note that the integral $\int_M \int_G E(g)(x)\chi_\sigma(g^{-1})\, dg d\mu(x)$ does not exist in general, as is shown by the following example.

Example 3.1. We take G to be the torus $\mathbf{T} = \{e^{i\theta} \mid \theta \in \mathbf{R}\}$ and $\mathcal{H}$ to be the Hilbert space of all holomorphic functions on $\mathbf{C}$ such that $\|f\|^2 := \frac{1}{2\pi}\int_{\mathbf{C}} |f(z)|^2 e^{-|z|^2/2}\, dx\, dy < +\infty$. We consider the unitary representation π of $\mathbf{T}$ on $\mathcal{H}$ defined by $(\pi(t)f)(z) = f(t^{-1}z)$. Then $\mathcal{H}$ has reproducing kernel $k(w,z) = e_z(w) = e^{\overline{z}w/2}$ and we have $L(t)(w,z) = e^{(t^{-1}-1)\overline{z}w/2}$.

For $p \in \mathbf{Z}$, let χ_p be the character of $\mathbf{T}$ defined by $\chi_p(t) = t^{-p}$. For $p \geq 0$, we have

$$\int_{\mathbf{T}} E(t)(z)\chi_p(t^{-1})dt = e^{-z\overline{z}/2}\int_{\mathbf{T}} e^{t^{-1}z\overline{z}/2}t^p\, dt = \frac{1}{p!}\left(\frac{z\overline{z}}{2}\right)^p e^{-z\overline{z}/2}.$$

This implies that

$$m(\chi_p) = \frac{1}{2\pi}\int_{\mathbf{C}} \frac{1}{p!}\left(\frac{z\overline{z}}{2}\right)^p e^{-z\overline{z}/2}\, dx\, dy = 1$$

for $p \geq 0$ and $m(\chi_p) = 0$ for $p < 0$, as expected. On the other hand, the integral

$$\int_{\mathbf{C}}\int_{\mathbf{T}} |E(t)(z)\chi_p(t^{-1})|\, dt\, dx\, dy = \frac{1}{2\pi}\int_{\mathbf{C}}\int_0^{2\pi} e^{(\cos\theta - 1)|z|^2/2}\, d\theta\, dx\, dy$$

does not exist, as we see by taking polar coordinates.

Proposition 3.2. *If $\mathcal{H}$ is finite-dimensional, then*

1) *the integral $\int_M \int_G E(g)(x)\chi_\sigma(g^{-1})\, dg\, d\mu(x)$ always exists,*

2) *the function $F(g) = \int_M E(g)(x)\, d\mu(x)$ is a class function, that is, $F(hgh^{-1}) = F(g)$ for each g, $h \in G$.*

Proof. 1) Clearly, we have $|E(g)(x)| \leq 1$ for each $g \in G$ and $x \in M$. The result then follows from Proposition 2.2 (1). 2) Applying Proposition 2.2 (2) to the operator $\pi(g)$ $(g \in G)$ we have

$$E(hgh^{-1})(x) = s(\pi(h)\pi(g)\pi(h)^{-1})(x) = s(\pi(g))(h^{-1}\cdot x) = E(g)(h^{-1}\cdot x)$$

for g, $h \in G$ and $x \in M$. The measure $d\mu$ being G-invariant, we then obtain

$$F(hgh^{-1}) = \int_M E(hgh^{-1})(x)\, d\mu(x) = \int_M E(g)(h^{-1}\cdot x)\, d\mu(x) = \int_M E(g)(x)\, d\mu(x).$$

Hence $F(hgh^{-1}) = F(g)$. ■

Using the Weyl Integration Formula [21], we immediately obtain the following result.

Corollary 3.3. *Suppose that G is a connected semisimple compact Lie group and that $\mathcal{H}$ is finite-dimensional. Let T be a maximal torus of G, W be the Weyl group associated with T and Δ the root system of G relative to T. Then we have*

$$m(\sigma)=\int_T\int_M E(t)(x)\chi_\sigma(t)^{-1}D(t)\,dt\,d\mu(x) \tag{3.3}$$

where dt denotes the normalized Haar measure on T and $D(t):=\prod_{\alpha\in\Delta}(1-t^\alpha)$.

Suppose now that $\mathcal{H}$ is infinite-dimensional. In many examples $\mathcal{H}$ can be easily decomposed as a direct sum of orthogonal G-invariant subspaces which are finite-dimensional.

Proposition 3.4. *Assume that $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H}=\oplus_{n\geq 0}\mathcal{H}_n$ where, for each integer $n\geq 0$, $\mathcal{H}_n$ is a finite-dimensional G-invariant subspace of $\mathcal{H}$. Let P_n be the projection operator on $\mathcal{H}_n$. Denote by $m^n(\sigma)$ the multiplicity of $\sigma\in\widehat{G}$ in the restriction of π to $\mathcal{H}_n$.*

1) *The space $\mathcal{H}_n$ has reproducing kernel $k_n(x,y):=\langle P_ne_y,e_x\rangle=S(P_n)(x,y)\langle e_y,e_x\rangle$.*
2) *We have $m^n(\sigma)=\int_M\int_G s(\pi(g)P_n)(x)\chi_\sigma(g)^{-1}\,d\mu(x)\,dg$.*
3) *We have $m(\sigma)=\sum_{n\geq 0}m^n(\sigma)$.*

Proof. 1) For $f\in\mathcal{H}_n$ and $x\in M$, we can write $f(x)=\langle f,e_x\rangle=\langle P_nf,e_x\rangle=\langle f,P_ne_x\rangle$. Then $k_n(x,y)=\langle P_ne_y,P_ne_x\rangle=\langle P_ne_y,e_x\rangle=S(P_n)(x,y)\langle e_y,e_x\rangle$. 2) This is a consequence of 1) and Proposition 3.1 2). 3) Immediate. ∎

Under the hypothesis of the previous proposition, it is possible to express $m(\sigma)$ as a limit of a double integral by generalizing the method introduced in [12].

Proposition 3.5. *Under the same hypothesis as in Proposition 3.4, for $r\in]0,1[$ we introduce the operator A_r on $\mathcal{H}$ defined by $A_r|_{\mathcal{H}_n}=r^nId_{\mathcal{H}_n}$ for each $n\geq 0$. Set $E_r(g)=s(A_r\pi(g))$. If the series $\sum_{n\geq 0}r^n\dim\mathcal{H}_n$ converges for $r\in]0,1[$ then the integral $m_r(\sigma):=\int_M\int_G E_r(g)\chi_\sigma(g)^{-1}\,d\mu(x)dg$ exists and we have $m_r(\sigma)=\sum_{n\geq 0}r^nm^n(\sigma)$ for each $r\in]0,1[$. Here $m^n(\sigma)$ denotes the multiplicity of σ in $\mathcal{H}_n$. Moreover, we have $m(\sigma)=\lim_{r\to 1}m_r(\sigma)$.*

Proof. Set $e_x^n=P_ne_x$. Then $\mathcal{H}_n$ has reproducing kernel $\langle e_y^n,e_x^n\rangle$. Note that

$$\langle A_r\pi(g)e_x,e_x\rangle=\sum_{n\geq 0}r^n\langle\pi(g)e_x^n,e_x^n\rangle.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{n\geq 0}r^n|\langle\pi(g)e_x^n,e_x^n\rangle|\leq\sum_{n\geq 0}r^n\|e_x^n\|^2.$$

We also have $\int_M \sum_{n\geq 0} r^n \|e_x^n\|^2 d\tilde{\mu}(x) = \mathrm{Tr}(A_r) = \sum_{n\geq 0} r^n \dim \mathcal{H}_n < +\infty$. Then the Lebesgue dominated convergence theorem shows that the integral

$$m_r(\sigma) = \int_M \int_G \langle A_r \pi(g) e_x, e_x \rangle \chi_\sigma(g)^{-1}\, d\tilde{\mu}(x) dg$$

exists and is equal to $\sum_{n\geq 0} r^n m^n(\sigma)$ where

$$m^n(\sigma) = \int_M \int_G \langle \pi(g) e_x^n, e_x^n \rangle \chi_\sigma(g)^{-1}\, d\tilde{\mu}(x)\, dg$$

is precisely the multiplicity of σ in $\mathcal{H}_n$. ■

In the rest in this section, by generalizing a result of [3] we introduce a method for computing the multiplicities which is simpler than the use of the preceding integral formulas. We assume that $\mathcal{H}$ is finite-dimensional and we set $n = \dim \mathcal{H}$. Fix $\sigma \in \widehat{G}$. Let $(s_i)_{1\leq i\leq n}$ be a basis of $\mathcal{H}$ and let $(\psi_k)_{1\leq k\leq m}$ be an orthonormal basis of V_σ. We can decompose the ψ_k $(1 \leq k \leq m)$ in the basis (s_i). We write $\psi_k = \sum_{l=1}^n a_{lk} s_l$ and we denote by A the $n \times m$-matrix (a_{lk}). We also introduce the $n \times n$-matrix $B = AA^* = (b_{kj})$.

Proposition 3.6. 1) *The reproducing kernel k_σ of V_σ is given by*

$$k_\sigma(x,y) = S(P_\sigma)(x,y)\langle e_y, e_x\rangle = \sum_{j,k=1}^n b_{lj} s_l(x) \overline{s_j(y)}.$$

2) *We have that $d(\sigma)m(\sigma) = \dim V_\sigma = \mathrm{rk}\, B$.*

Proof. 1) For $f \in V_\sigma$ we have

$$f(x) = \langle f, e_x\rangle = \langle f, P_\sigma e_x\rangle = \int_M f(y)\overline{P_\sigma e_x(y)}\, d\tilde{\mu}(x).$$

Then

$$k_\sigma(x,y) = \overline{P_\sigma e_x(y)} = \overline{\langle P_\sigma e_x, e_y\rangle} = \langle e_y, P_\sigma e_x\rangle = S(P_\sigma)(x,y)\langle e_y, e_x\rangle.$$

On the other hand, we have

$$\begin{aligned} k_\sigma(x,y) &= \sum_{k=1}^m \psi_k(x)\overline{\psi_k(y)} = \sum_{k=1}^m \left(\sum_{j=1}^n \overline{a_{jk} s_j(y)}\right)\left(\sum_{l=1}^n a_{lk} s_l(x)\right) \\ &= \sum_{1\leq j,l\leq n} \left(\sum_{k=1}^m \overline{a_{jk}} a_{lk}\right) s_l(x)\overline{s_j(y)} = \sum_{1\leq j,l\leq n} b_{lj} s_l(x)\overline{s_j(y)}. \end{aligned}$$

We have thus obtained the desired result.

2) Since $B = AA^*$, we have $\mathrm{rk}\, B = \mathrm{rk}\, A = \dim V_\sigma$. ■

Let us briefly describe how Proposition 3.6 can be used for explicit computations of multiplicities. In some cases, one can explicitly compute the function $k_\sigma(x,y) = S(P_\sigma)(x,y)\langle e_y, e_x\rangle$ and its development in the basis $s_l \otimes \overline{s_j}$. Then we obtain the matrix B and we can compute $\mathrm{rk}\, B = \dim V_\sigma$. In other words, the trick is that one can calculate $\dim V_\sigma = \mathrm{rk}\, A$ without knowing A.

4. Berezin quantization on flag manifolds

In this section and in the next section, we apply the general results of the preceding sections to weight multiplicities in a unitary irreducible representation of a compact semisimple Lie group. First, we recall Borel-Weil's method for constructing the irreducible unitary representations of a compact group as representations in the space of holomorphic sections of a certain line bundle and we introduce Berezin quantization on flag manifolds. We follow the presentation of [15] which is essentially based on [2] and [22]. Note that our presentation is slightly different from those of [3] which is based on geometric quantization. Also, we give explicit formulas for reproducing kernels and Berezin symbols of representation operators which are different from the formulas obtained in [3] by using generalized determinants for Kaehler potentials.

Let G be a connected simply-connected semisimple compact Lie group. Let T be a maximal torus of G. The manifold $M := G/T$ is called a flag manifold. Let Δ be the root system of G relative to T. We choose a Weyl chamber P of T relative to G. Let Δ^+ the positive roots of Δ relative to P.

Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T, respectively. We denote by $\mathfrak{g}^c$ and $\mathfrak{t}^c$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Let G^c and T^c be the connected complex Lie groups whose Lie algebras are $\mathfrak{g}^c$ and $\mathfrak{t}^c$, respectively. Let $\mathfrak{g}^c = \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root spaces decomposition of $\mathfrak{g}^c$. We set $\mathfrak{n}^+ = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}$. Then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie algebras satisfying $[\mathfrak{t}^c, \mathfrak{n}^\pm] \subset \mathfrak{n}^\pm$. We also have $\mathfrak{g}^c = \mathfrak{t}^c \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$. We denote by N^+ and N^- the analytic subgroups of G^c with Lie algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively. A complex structure on M is then defined by the diffeomorphism $M = G/T \simeq G^c/T^cN^-$ [22], 6.2.11. This complex structure depends on the choice of P. We denote by $\tau : G^c \to M \simeq G^c/T^cN^-$ the natural projection.

Let β be the Killing form on $\mathfrak{g}^c$, that is, $\beta(X,Y) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ for $X, Y \in \mathfrak{g}^c$. For each $\alpha \in \Delta$, we denote by H_α the element of $i\mathfrak{t}$ satisfying $\beta(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{t}^c$.

Let χ_0 be a character of T. Then $\lambda := d\chi_0|_{\mathfrak{t}}$ is integral i.e. $2\lambda(H_\alpha)/\alpha(H_\alpha) \in \mathbf{Z}$ for each $\alpha \in \Delta^+$. Conversely, each weight $\lambda \in \mathfrak{t}^*$ which is integral defines a unique character χ_0 on T such that $\lambda = d\chi_0|_{\mathfrak{t}}$.

Now we fix a character χ_0 on T. Let $\lambda := d\chi_0|_{\mathfrak{t}}$. Denote by χ the unique extension of χ_0 to T^cN^-. The line bundle $L_\lambda := G \times_{\chi_0} \mathbf{C}$ can be identified to $G^c \times_\chi \mathbf{C}$ by means of the map $[g, z]_0 \to [g, z]$ where $[g, z]_0$ $(g \in G, z \in \mathbf{C})$ denotes the equivalence class $\{(gh, \chi_0(h^{-1})z) \,:\, h \in T\} \in L_\lambda$ and $[g, z]$ $(g \in G^c, z \in \mathbf{C})$ denotes the equivalence class $\{(gh, \chi(h^{-1})z) \,:\, h \in T^cN^-\} \in G \times_\chi \mathbf{C}$. Thus L_λ has a natural structure of holomorphic line bundle. Recall that G^c acts on L_λ by left translations: $g\,[g', z] := [gg', z]$. A G-invariant Hermitian structure on L_λ is given by $\langle [g, z], [g, z'] \rangle = z\bar{z}'$ where $g \in G$ and $z, z' \in \mathbf{C}$.

The space $\mathcal{H}^0_\lambda$ of holomorphic sections of L_λ is endowed with the G-invariant

Hermitian scalar product defined by

$$\langle s\,,\,s'\rangle_{\mathcal{H}^0_\lambda}=\int_M \langle s(x)\,,\,s'(x)\rangle\,d\nu(x)$$

where $d\nu(x)$ is a G-invariant measure on M.

Since M is compact, $\mathcal{H}^0_\lambda$ is finite-dimensional [9], [16]. Moreover, $\mathcal{H}^0_\lambda$ carries a unitary representation π_0 of G:

$$(\pi_0(g)\,s)(x)=g\,s(g^{-1}\cdot x).$$

Suppose that λ is dominant (i.e. $2\lambda(H_\alpha)/\alpha(H_\alpha)$ is a nonnegative integer for each $\alpha\in\Delta^+$). Then, by the Borel-Weil Theorem, we have that π_0 is the irreducible (finite-dimensional) representation of G with highest weight λ.

Now we introduce an alternative realization of π_0 which is more convenient for explicit computations. Recall that (1) each g in a dense open subset of G^c has a unique Gauss decomposition $g=n^+h\,n^-$ where $n^+\in N^+$, $h\in T^c$ and $n^-\in N^-$ and (2) the map $\sigma: Z\to\tau(\exp Z)$ is a holomorphic diffeomorphism from $\mathfrak{n}^+$ onto a dense open subset of M (see [18], Chap. VIII). Then the natural action of G^c on $M\simeq G^c/T^cN^-$ induces an action (defined almost everywhere) of G^c on $\mathfrak{n}^+$. We denote by $g\cdot Z$ the action of $g\in G^c$ on $Z\in\mathfrak{n}^+$. Using again the diffeomorphism $G/T\simeq G^c/T^cN^-$, we see that for each $Z\in\mathfrak{n}^+$ there exists an element $g_Z\in G$ for which $\tau(g_Z)=\tau(\exp Z)$ or, equivalently, $g_Z\cdot 0=Z$.

We associate with each $s\in\mathcal{H}^0_\lambda$ the holomorphic function f_s on $\mathfrak{n}^+$ defined by: $s(\sigma(Z))=[\exp Z, f_s(Z)]$. For $s,\ s'\in\mathcal{H}^0_\lambda$, we have

$$\begin{aligned}\langle s(\sigma(Z)), s'(\sigma(Z))\rangle &= \langle[\exp Z, f_s(Z)]\,,\,[\exp Z, f_{s'}(Z)]\rangle\\ &=\langle[g_Z(g_Z^{-1}\exp Z), f_s(Z)]\,,\,[g_Z(g_Z^{-1}\exp Z), f_{s'}(Z)]\rangle\\ &=\langle[g_Z,\chi(g_Z^{-1}\exp Z)f_s(Z)]\,,\,[g_Z,\chi(g_Z^{-1}\exp Z)f_{s'}(Z)]\rangle\\ &=|\chi(g_Z^{-1}\exp Z)|^2\,f_s(Z)\overline{f_{s'}(Z)}.\end{aligned}$$

This implies that

$$\langle s\,,\,s'\rangle_{\mathcal{H}^0_\lambda}=\int_{\mathfrak{n}^+} f_s(Z)\overline{f_{s'}(Z)}\,|\chi(g_Z^{-1}\exp Z)|^2\,d\mu(Z)$$

where $\mu:=\sigma_*(\nu)$ is a G-invariant measure on $\mathfrak{n}^+$. We can always normalize the measure ν so that $k(0,0)=1$.

This leads us to introduce the Hilbert space $\mathcal{H}_\lambda$ of holomorphic functions f on $\mathfrak{n}^+$ such that

$$\|f\|^2_{\mathcal{H}_\lambda}:=\int_{\mathfrak{n}^+}|f(Z)|^2\,|\chi(g_Z^{-1}\exp Z)|^2\,d\mu(Z)<+\infty.$$

Moreover, for $s\in\mathcal{H}^0_\lambda$, $g\in G$ and $Z\in\mathfrak{n}^+$ we have

$$\begin{aligned}(\pi_0(g)s)(\sigma(Z))&=g\,s(g^{-1}\sigma(Z))=g\,s(\sigma(g^{-1}\cdot Z))=[g\exp(g^{-1}\cdot Z), f_s(g^{-1}\cdot Z)]\\ &=[\exp(Z),\chi(\exp(-Z)g\exp(g^{-1}\cdot Z))f_s(g^{-1}\cdot Z)].\end{aligned}$$

Hence we can conclude that the equality

$$(\pi_\lambda(g)f)(Z) = \chi(\exp(-Z)g\exp(g^{-1}\cdot Z))f(g^{-1}\cdot Z) \tag{4.1}$$

defines a unitary representation π_λ of G on $\mathcal{H}_\lambda$ which is unitarily equivalent to π_0, the intertwining operator between π_λ and π_0 being given by $s \to f_s$.

Now, we apply the general considerations of the preceding section to the Hilbert space $\mathcal{H}_\lambda$ together with the representation π_λ. We retain the notation from Section 2. The cocycle α associated with π_λ is given by

$$\alpha(g^{-1}, Z) = \chi(\exp(-Z)g\exp(g^{-1}\cdot Z)). \tag{4.2}$$

The reproducing kernel $k(W,Z)$ satisfies

$$k(Z,Z) = |\chi(g_Z^{-1}\exp Z)|^{-2}. \tag{4.3}$$

We shall deduce from (4.3) a simple expression for $k(Z,W)$ and thus for the functions e_Z ($Z \in \mathfrak{n}^+$). Following [20], we introduce the projections $\kappa : N^+T^cN^- \to T^c$ and $\zeta : N^+T^cN^- \to N^+$. Then, for $g \in G^c$ and $Z \in \mathfrak{n}^+$ we have $g\cdot Z = \log\zeta(g\exp Z)$.

We set $(X+iY)^* = -X+iY$ for X, $Y \in \mathfrak{g}$ and we denote by $g \to g^*$ the involutive automorphism of G^c which is obtained by exponentiating $X+iY \to (X+iY)^*$ to G^c.

Proposition 4.1. *We have*

1) $\alpha(g^{-1}, Z) = \chi(\kappa(g^{-1}\exp Z))^{-1}$ *for* $g \in G^c$, $Z \in \mathfrak{n}^+$.

2) $k(Z,Z) = \chi(\kappa(\exp Z^*\exp Z))^{-1}$ *for* $Z \in \mathfrak{n}^+$.

3) $k(W,Z) = e_Z(W) = \chi(\kappa(\exp Z^*\exp W))^{-1}$ *for* Z, $W \in \mathfrak{n}^+$.

Proof. 1) We can write $g^{-1}\exp Z = \exp(g^{-1}\cdot Z)h\,n$ where $h \in T^c$, $n \in N^-$. Then $\exp(-Z)g\exp(g^{-1}\cdot Z) = (h\,n)^{-1}$. Applying χ, we thus obtain $\chi(\exp(-Z)g\exp(g^{-1}\cdot Z)) = \chi(h)^{-1} = \chi(\kappa(g^{-1}\exp Z))^{-1}$.

2) We can write $g_Z = \exp(Z)h\,n$ where $h \in T^c$, $n \in N^-$. Since $g_Z \in G$, we have $g_Z^* = g_Z^{-1}$. Then $(\exp Z)^*\exp Z = h^{*-1}n^{*-1}n^{-1}h^{-1} = (h^{*-1}n^{*-1}h^*)(h^{*-1}h^{-1}) \times (h\,n^{-1}h^{-1})$. But $h\,n^{-1}h^{-1} \in N^-$ since $[\mathfrak{h}^c, \mathfrak{n}^-] \subset \mathfrak{n}^-$. Similarly, $h^{*-1}n^{*-1}h^* \in N^+$. We thus obtain $\kappa(\exp Z^*\exp Z) = h^{*-1}h^{-1}$. Hence, applying χ, we get

$$\chi(\kappa(\exp Z^*\exp Z)) = \overline{\chi(h)^{-1}}\chi(h)^{-1} = |\chi(g_Z^{-1}\exp Z)|^2.$$

3) Since χ is trivial on N^- we have

$$\begin{aligned}\chi(\kappa(\exp Z^*\exp W)) &= \chi(\zeta(\exp Z^*\exp W)\exp Z^*\exp W)\\ &= \chi(\exp(\exp Z^*\cdot W)\exp Z^*\exp W)\end{aligned}$$

and $\exp Z^*\cdot W = \sigma^{-1}(\tau(\exp Z^*\exp W))$. Then the function $\chi(\kappa(\exp Z^*\exp W))^{-1}$ is holomorphic in W and anti-holomorphic in Z. On the other hand, the function

$$k(W,Z) = \langle e_Z, e_W\rangle = e_Z(W) = \overline{e_W(Z)}$$

is also holomorphic in W and anti-holomorphic in Z. Since these two functions coincide for $W = Z$ we then obtain 3). ∎

In fact, one can give an explicit formula for the G-invariant measure $d\mu$.

PROPOSITION 4.2. [15], [20] *Let $d\mu_L$ be a fixed Lebesgue measure on $\mathfrak{n}^+$. Set $\Lambda := \sum_{\alpha\in\Delta^+} \alpha$. Let χ_Λ be the corresponding character of T^c. Then the G-invariant measure $d\mu$ on $\mathfrak{n}^+$ is $d\mu(Z) = C\,\chi_\Lambda(\kappa(\exp Z^* \exp Z))\,d\mu_L(Z)$ where the constant C is given by $C\int_{\mathfrak{n}^+} \chi_\Lambda(\kappa(\exp Z^* \exp Z))\,d\mu_L(Z) = \dim \mathcal{H}_\lambda$.*

Also, we are in position to calculate the so-called star exponential, that is, the Berezin symbol of $\pi(g)$ ($g \in G$). The star exponential plays a prominent role in the construction of the generalized Fourier transform in [2] and [23].

PROPOSITION 4.3. *Let $g \in G$. The Berezin symbol of $\pi_\lambda(g)$ is then given by*

$$L(g)(W,Z) = S(\pi_\lambda(g))(W,Z) = \chi\big(\kappa(\exp Z^* g^{-1} \exp W)^{-1}\kappa(\exp Z^* \exp W)\big). \tag{4.4}$$

Proof. We have

$$\begin{aligned} S(\pi_\lambda(g))(W,Z) &= \frac{\langle \pi_\lambda(g)e_Z\,,\, e_W\rangle}{\langle e_Z\,,\, e_W\rangle} = \frac{(\pi_\lambda(g)e_Z)(W)}{e_z(W)} \\ &= \chi(\kappa(g^{-1}\exp W))^{-1}\, e_Z(g^{-1}\cdot W)\, e_Z(W)^{-1}. \end{aligned}$$

Using Proposition 4.1 3), we get

$$S(\pi_\lambda(g))(W,Z) = \chi\big(\kappa(g^{-1}\exp W)^{-1}\,\kappa(\exp Z^* \exp(g^{-1}\cdot W))^{-1}\,\kappa(\exp Z^* \exp W)\big). \tag{4.5}$$

Now, let $h = \kappa(g^{-1}\exp W)$. We can write $g^{-1}\exp W = \exp(g^{-1}\cdot W)h\,y$ where $y \in N^-$. Then $\exp Z^* \exp(g^{-1}\cdot W) = \exp Z^* g^{-1} \exp W\, y^{-1}h^{-1}$. Thus $\kappa(\exp Z^* \exp(g^{-1}\cdot W)) = \kappa(\exp Z^* g^{-1}\exp W)\,h^{-1}$. From this and (4.5) we deduce (4.4). ∎

5. Weight multiplicities

We retain the notation from the previous section. We reformulate the results of Section 3 in the setting of Section 4 in order to recover the main results of [3] which yield a method for weight multiplicity computations in representations of semisimple compact Lie groups.

PROPOSITION 5.1. [3] *Let χ_σ be the character of T corresponding to the weight σ and P_σ the projection operator of $\mathcal{H}_\lambda$ onto the σ-isotypic component.*

1) *For $t \in T$ and Z, $W \in \mathfrak{n}^+$, we have*

$$L(t)(W,Z) = \chi_\sigma(t)\frac{\langle e_{t\cdot Z}, e_W\rangle}{\langle e_Z, e_W\rangle}.$$

2) *The double Berezin symbol of P_σ is given by*

$$S(P_\sigma)(W,Z) = \int_T L(t)(W,Z)\chi_\sigma(t)^{-1}\,dt = \int_T \chi(t)\chi_\sigma(t)^{-1}\frac{e_{t\cdot Z}(W)}{e_Z(W)}\,dt.$$

3) *The multiplicity of* σ *in* π_λ *is given by*

$$m(\sigma)=\int_T\int_{\mathfrak{n}^+}E(t)(Z)\chi_\sigma(t)^{-1}\,d\mu(Z)dt=\int_T\int_{\mathfrak{n}^+}\chi(t)\chi_\sigma(t)^{-1}\frac{\langle e_{t\cdot Z},e_W\rangle}{\langle e_Z,e_W\rangle}\,d\mu(Z)dt.$$

Proof. 1) For $t\in T$ and $Z\in\mathfrak{n}^+$, we have $\zeta(t\exp Z)=\zeta(t(\exp Z)t^{-1})=\zeta(\exp\operatorname{Ad}(t)Z)$. Then $t\cdot Z=\operatorname{Ad}(t)Z$. Consequently,

$$\alpha(t,Z)=\chi(\exp(-Z)t^{-1}\exp(t\cdot Z))=\chi(t)^{-1}.$$

Using Proposition 2.1 (2)(i), we then obtain

$$\langle e_Z,e_W\rangle L(t)(W,Z)=\langle\pi_\lambda(t)e_Z,e_W\rangle=\overline{\alpha(t,Z)}\langle e_{t\cdot Z},e_W\rangle=\chi(t)\langle e_{t\cdot Z},e_W\rangle.$$

2), 3) By 1) and Proposition 3.1. ∎

Let us denote by $\alpha_1,\alpha_2,\dots,\alpha_n$ the elements of Δ^+. Let $(E_k)_{1\le k\le n}$ be a basis for $\mathfrak{n}^+$ such that $E_k\in\mathfrak{g}_{\alpha_k}$ for $k=1,2,\dots,n$. For $Z=\sum_{k=1}^n z_kE_k$ and $p=(p_1,p_2,\dots,p_n)\in\mathbf{N}^n$, we define $Z^{(p)}=z_1^{p_1}z_2^{p_2}\cdots z_n^{p_n}$. Here $\mathbf{N}$ denotes the set of all nonnegative integers. By construction of $\mathcal{H}_\lambda$, there exists $d=(d_1,d_2,\dots,d_n)\in\mathbf{N}^n$ such that

$$\mathcal{H}_\lambda\subset\operatorname{span}\langle Z^{(p)}\,:\,p=(p_1,p_2,\dots,p_n),\,0\le p_k\le d_k,\,k=1,2,\dots,n\rangle.$$

The following proposition is analogous to Proposition 3 of [3].

Proposition 5.2. 1) *We can write* $\langle e_Z,e_W\rangle L(t)(W,Z)=\sum_\sigma\chi_\sigma(t)u_\sigma(W,Z)$ *where the sum is taken over the weights of* π_λ *and, for each weight* σ, $u_\sigma(W,Z)=\langle e_Z,e_W\rangle S(P_\sigma)(W,Z)$ *is a polynomial in the variables* $\overline{z_k}$, w_l, $(1\le k,l\le n)$.

2) *Write* $u_\sigma(W,Z)=\sum_{p,q}b^\sigma_{qp}W^{(q)}\overline{Z^{(p)}}$. *Choose an ordering on the set consisting of the elements* $p=(p_1,p_2,\dots,p_n)\in\mathbf{N}^n$ *such that* $0\le p_k\le d_k$ *for each* $1\le k\le n$ *and consider the matrix* $B_\sigma:=(b^\sigma_{qp})_{q,p}$. *Then* $m(\sigma)=\operatorname{rk}B_\sigma$.

Proof. 1) Since $\pi_\lambda(t)=\sum_\sigma\chi_\sigma(t)P_\sigma$, we have

$$\langle\pi_\lambda(t)e_Z,e_W\rangle=\sum_\sigma\chi_\sigma(t)\langle P_\sigma e_Z,e_W\rangle.$$

Then

$$\langle e_Z,e_W\rangle S(\pi_\lambda(t))(W,Z)=\sum_\sigma\chi_\sigma(t)u_\sigma(W,Z)$$

where $u_\sigma(W,Z)=\langle e_Z,e_W\rangle S(P_\sigma)(W,Z)$. Since

$$u_\sigma(W,Z)=\langle P_\sigma e_W,e_Z\rangle=(P_\sigma e_W)(Z)=\overline{(P_\sigma e_Z)(W)},$$

u_σ is a polynomial in the variables $\overline{z_k}$, w_l, $(1\le k,l\le n)$.

2) Immediate from Proposition 3.6 2). ∎

Example 5.1. We take $G=SU(2)$, $T=\{\operatorname{diag}(e^{i\theta},e^{-i\theta})\,|\,\theta\in\mathbf{R}\}$ and

$$N^+=\left\{\begin{pmatrix}1&z\\0&1\end{pmatrix}\,|\,z\in\mathbf{C}\right\}.$$

For each integer $m \geq 0$, let χ_m be the character of T defined by $\chi_m(\mathrm{diag}(e^{i\theta}, e^{-i\theta})) = e^{im\theta}$. The corresponding representation π_m of $SU(2)$ is realized on the space complex polynomials on $\mathfrak{n}^+ \simeq \mathbf{C}$ of degree $\leq m$ endowed with the Hilbert product

$$\langle f_1, f_2\rangle_m = \int_{\mathbf{C}} f_1(z)\overline{f_2(z)}\,\frac{m+1}{\pi}(1+z\overline{z})^{-m-2}\,dx\,dy\,.$$

More precisely, we have

$$(\pi_m\,(g)\,f)(z) = (a+\overline{b}\,z)^m\,f\left(\frac{\overline{a}z-b}{\overline{b}z+a}\right), \qquad g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

We easily verify that $e_z(w) = (1+\overline{z}w)^m$ and that

$$L(g)(w,z) = (a+\overline{a}\overline{z}w+\overline{b}w-b\overline{z})^m(1+\overline{z}w)^{-m}$$

for $g \in SU(2)$ as above. For $t = \mathrm{diag}(e^{i\theta}, e^{-i\theta}) \in T$, we then obtain

$$\langle e_z, e_w\rangle L(t)(w,z) = (e^{i\theta}+e^{-i\theta}\overline{z}w)^m = \sum_{k=0}^{m} \tbinom{m}{k}(\overline{z}w)^k(e^{i\theta})^{m-2k}.$$

Thus the projection operator on the $(m-2k)$-isotypic component has Berezin symbol $u_{m-2k}(w,z) = (1+\overline{z}w)^{-m}\binom{m}{k}(\overline{z}w)^k$ for $k = 0,1,2,\ldots,m$.

Example 5.2. We take $G = SU(3)$,

$$T = \{\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \quad | \quad \theta_1+\theta_2+\theta_3 = 0, \quad \theta_k \in \mathbf{R}\}$$

and

$$N^+ = \left\{ \begin{pmatrix} 1 & z_3 & z_2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \quad | \quad z_k \in \mathbf{C} \right\} \subset G^c = SL(3,\mathbf{C}).$$

We denote by $\omega_{(k,l)}$ the weight defined by $\omega_{(k,l)}(\mathrm{diag}(a,b,-a-b) = (k+l)a+lb$ and by $\chi_{(k,l)}$ the corresponding character of T. Here we consider the representation π_λ where $\lambda = \omega_{(1,1)}$. For $t \in T$ as above, we can easily compute $L(t)(W,Z)$ by writing explicitly the Gauss decomposition for $SL(3,\mathbf{C})$. Then we obtain

$$\begin{aligned}\langle e_Z, e_W\rangle L(t)(W,Z) &= (\overline{z_1}w_1)(\overline{z_3}w_3) + 2\overline{u_2(Z)}u_2(W) + e^{i(2\theta_1+\theta_2)} \\ &+ \overline{z_1}w_1e^{i(\theta_1-\theta_2)} + \overline{z_3}w_3e^{i(\theta_1+2\theta_2)} + \overline{z_3}w_3\overline{u_1(Z)}u_1(W)e^{i(-\theta_1+\theta_2)} \\ &+ \overline{z_1}w_1\overline{u_2(Z)}u_2(W)e^{i(-\theta_1-2\theta_2)} + \overline{u_1(Z)}u_1(W)\overline{u_2(Z)}u_2(W)e^{i(-2\theta_1-\theta_2)}.\end{aligned}$$

where $u_1(W) = \frac{1}{2}w_1w_3 + w_2$ and $u_2(W) = \frac{1}{2}w_1w_3 - w_2$. Then, by Proposition 5.2, the weights of π_λ are $\omega_{(1,1)}$, $\omega_{(-1,2)}$, $\omega_{(2,-1)}$, $\omega_{(1,-2)}$, $\omega_{(-2,1)}$, $\omega_{(-1,-1)}$ with multiplicity 1 and $\omega_{(0,0)}$ with multiplicity 2. Similar examples can be found in [3].

6. Some more examples: Gel'fand pairs

The action of the unitary group $U(n)$ on the $(2n+1)$-dimensional Heisenberg group H_n defined by $k\cdot(z,t) = (kz,t)$ ($k \in U(n)$, $z \in \mathbf{C}^n$, $t \in \mathbf{R}$) yields a Gel'fand pair. That is, the convolution algebra $L^1_{U(n)}(H_n)$ of $U(n)$-invariant L^1-functions on H_n is commutative [6],[7]. Also, the action of many compact subgroups G of $U(n)$ yields Gel'fand pairs (G, H_n). In fact, (G, H_n) is a Gel'fand pair if and only if the natural action of G on the space $\mathcal{P}(\mathbf{C}^n)$ of the holomorphic polynomials on $\mathbf{C}^n$ is multiplicity-free [6]. Let us introduce the Fock space, that is, the Hilbert space $\mathcal{H}$ consisting of entire functions $f : \mathbf{C}^n \to \mathbf{C}$ such that

$$\|f\|^2 := \frac{1}{(2\pi)^n} \int_{\mathbf{C}^n} |f(z)|^2 \, e^{-|z|^2/2} \, dz < +\infty$$

where $dz = dx_1 dy_1 \dots dx_n dy_n$. Let π be the unitary representation of G on $\mathcal{H}$ defined by $\pi(g)f(z) = f(g^{-1}z)$. Since $\mathcal{P}(\mathbf{C}^n)$ is dense in $\mathcal{H}$ it is clear that (G, H_n) is a Gel'fand pair if and only if π is multiplicity-free. So, we can apply the results of Section 3 to the study of such Gel'fand pairs.

It is well-known that the reproducing kernel of $\mathcal{H}$ is given by $k(w, z) = e_z(w) = e^{z^* w/2}$ (see [17] for instance). For each integer $N \geq 0$, we denote by $\mathcal{H}_N$ the space of holomorphic polynomials on $\mathbf{C}^n$ of degree N. Then $\mathcal{H} = \bigoplus_{N \geq 0} \mathcal{H}_N$ is an orthogonal decomposition of $\mathcal{H}$ into G-invariant finite-dimensional subspaces. Let P_N be the projection operator on $\mathcal{H}_N$. By Proposition 3.4 1), the reproducing kernel of $\mathcal{H}_N$ is $k_N(w, z) := \langle P_N e_z, e_w \rangle$. Since $e_z(w) = \sum_{N \geq 0} \frac{1}{N!} (\frac{z^* w}{2})^N$, we immediately obtain $k_N(w, z) = \frac{1}{N!} (\frac{z^* w}{2})^N$.

Example 6.1. Let $G = \mathbf{T}^n$ the maximal torus of $U(n)$ consisting of all matrices $t = \mathrm{diag}(t_1, t_2, \dots, t_n)$ where $t_k \in \mathbf{C}$, $|t_k| = 1$. We can directly see that $(\mathbf{T}^n, H_n)$ is a Gel'fand pair. Here, we will recover this fact by applying Proposition 3.4 and Proposition 3.5. For $p = (p_1, p_2, \dots, p_n) \in \mathbf{Z}^n$, we denote by χ_p the character of $\mathbf{T}^n$ defined by $\chi_p(t) = t_1^{p_1} t_2^{p_2} \dots t_n^{p_n}$. Also, we shall use the standard notation $z^p = z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}$, $p! = p_1! p_2! \dots p_n!$ and $|p| = p_1 + p_2 + \cdots + p_n$ for $p = (p_1, p_2, \dots, p_n) \in \mathbf{N}^n$ and $z = (z_1, z_2, \dots, z_n) \in \mathbf{C}^n$.

1) First method: by Proposition 3.4. The multiplicity of χ_p in $\pi_{\mathcal{H}_N}$ is given by the integral

$$m^N(\chi_p) = \frac{1}{(2\pi)^n} \int_{\mathbf{C}^n} \int_{\mathbf{T}^n} \langle \pi(t) P_N e_z, e_z \rangle \chi_p(t)^{-1} \, e^{-|z|^2/2} \, dz \, dt.$$

We have

$$\langle \pi(t) P_N e_z, e_z \rangle = (\pi(t) P_N e_z)(z) = (P_N e_z)(t^{-1} z) = \frac{1}{2^N N!} (z^*(t^{-1} z))^N.$$

This gives

$$\langle \pi(t) P_N e_z, e_z \rangle = \frac{1}{2^N} \sum_{k \in \mathbf{N}^n, |k| = N} \frac{1}{k!} \chi_k(t)^{-1} |z_1|^{2k_1} |z_2|^{2k_2} \dots |z_n|^{2k_n}$$

for $z \in \mathbf{C}^n$ and $t \in \mathbf{T}^n$. Then, by using the fact that $\left((1/\sqrt{2^{|p|} p!}) \, z^p\right)_{|p| = N}$ is an orthonormal basis for $\mathcal{H}_N$, we obtain

$$m^N(\chi_p) = \int_{\mathbf{T}^n} \chi_p(t)^{-1} \sum_{|k| = N} \chi_k(t)^{-1} \, dt.$$

Hence we can conclude that for each p satisfying $p_1, p_2, \ldots, p_n \leq 0$ and $p_1 + p_2 + \cdots + p_n = -N$ we have $m^N(\chi_p) = 1$ and in the other cases we have $m^N(\chi_p) = 0$. This proves that π is multiplicity-free.

2) Second method: by Proposition 3.5. Since we have $\dim \mathcal{H}_N = \binom{n+N-1}{n-1}$, we see that the series $\sum_{N\geq 0}(\dim \mathcal{H}_N) r^N$ converges for $r < 1$ and we have

$$m_r(\chi_p) = \frac{1}{(2\pi)^n} \int_{\mathbf{C}^n} \int_{\mathbf{T}^n} \langle A_r \pi(t) e_z, e_z \rangle \chi_p(t)^{-1} \, e^{-|z|^2/2} \, dz \, dt.$$

But

$$(\pi(t)e_z)(w) = e_z(t^{-1}w) = e^{z^*(t^{-1}w)/2} = \sum_{N\geq 0} \frac{1}{N!} \left(\frac{z^*(t^{-1}z)}{2} \right)^N$$

implies that

$$(A_r \pi(t) e_z)(z) = \sum_{N\geq 0} \frac{1}{N!} \left(\frac{z^*(t^{-1}z)}{2} \right)^N \, r^N.$$

Hence

$$m_r(\chi_p) = \frac{1}{(2\pi)^n} \int_{\mathbf{C}^n} \int_{\mathbf{T}^n} e^{rz^*(t^{-1}z)/2} \chi_p(t)^{-1} \, e^{-|z|^2/2} \, dz \, dt.$$

Therefore, by transforming the integral to polar coordinates we obtain

$$m_r(\chi_p) = \int_{\mathbf{T}^n} \prod_{k=1}^{n} \frac{1}{1 - r t_k^{-1}} \, t_k^{-p_k} \, dt.$$

Finally, we find that $m_r(\chi_p) = r^{-(p_1+p_2+\cdots+p_n)}$ if $p_1, p_2, \ldots, p_n \leq 0$ and $m_r(\chi_p) = 0$ otherwise. Since for each p we have $\lim_{r\to 1} m_r(\chi_p) \leq 1$, we can conclude that π is multiplicity-free.

Example 6.2. [12] Here we shall prove that $(SU(2), H_2)$ is a Gel'fand pair. Let $T \subset SU(2)$ be the torus consisting of matrices $t = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$ $(\theta \in \mathbf{R})$. For each integer $p \geq 0$, let σ_p be the $(p+1)$-dimensional unitary irreducible representation of $SU(2)$. The character χ_{σ_p} of σ_p is given by $\chi_{\sigma_p}(t) = \frac{\sin(p+1)\theta}{\sin\theta}$ for each t as above. In the notation of Proposition 3.5 we also have $D(t) = 4\sin^2\theta$. Then the multiplicity of σ_p in π is given by

$$\begin{aligned} m_r(\sigma_p) &= \frac{1}{2\pi} \int_0^{2\pi} 4\sin((p+1)\theta)\sin\theta \, \frac{1}{(1-re^{i\theta})(1-re^{-i\theta})} \, d\theta \\ &= \frac{1}{2\pi} \int_{|z|=1} \left(-z^{p+2} - \frac{1}{z^{p+2}} + z^p + \frac{1}{z^p} \right) \frac{1}{i(1-rz)(z-r)} \, dz. \end{aligned}$$

By using the Cauchy residue Theorem, we easily find that $m_r(\sigma_p) = r^p$. Then we can conclude that $(SU(2), H_2)$ is a Gel'fand pair. Similarly it could be verified by a long computation that $(SU(3), H_3)$ is a Gel'fand pair [12]. However, it seems to be difficult to prove by the same method that, more generally, $(SU(n), H_n)$ is a Gel'fand pair. In fact, there are many simple ways to verify that $(SU(n), H_n)$ is a Gel'fand pair. For instance, one can prove that the restriction of π to $\mathcal{H}_N$ is a irreducible representation of $SU(n)$; one can also use the algebraic criterion of [19]. So, we see that the use of Corollary 3.3 for explicit computations of multiplicities is limited to simple examples.

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60 YEARS OF "MATEMATIČKI VESNIK"

In 1948 the Managing Board of the Society of Mathematicians and Physicists of Serbia decided to start publishing a scientific journal named VESNIK DRUŠTVA MATEMATIČARA I FIZIČARA NR SRBIJE ("Bulletin of the Society of Mathematicians and Physicists of Serbia"). *Jovan Karamata*, a well-known Serbian mathematician, was the first Editor-in-chief, the Editorial Board consisted of *Pavle Savić*, *Dragoljub K. Jovanović*, *Miloš Radojčić* and *Dobrivoje Mihajlović*, while *Ivan Atanasijević* was the Technical Editor.

The first issue of the journal was published in the beginning of 1949. It consisted of four columns: *Scientific articles*, *Problems and exercises*, *Critics and bibliography* and *Meetings of the Society.* Eight articles (in Serbian, with abstracts in French and Russian) were published in this issue. In the column *Meetings of the Society* in this and subsequent issues (up to 1963), reports on all the important activities of the Society and, later, of the Union of Societies of Mathematicians, Physicists and Astronomers of Yugoslavia can be found.

In the subsequent years, Vesnik continued to be published in single or double-issues. The name of the journal changed to MATEMATIČKI VESNIK ("Mathematical Bulletin") in 1964, and in the period 1964–1976 it was published jointly with the Mathematical Institute from Belgrade. Starting from 1977 it has again been published by the Mathematical Society of Serbia alone.

Editors-in-chief of "Matematički Vesnik" in the past 60 years were: *Jovan Karamata*, *Dragoljub Marković*, *Zlatko Mamuzić*, *Dušan Adnađević*, *Zoran Kadelburg*, *Mila Mršević* and *Ljubiša Kočinac*, and the secretaries were: *Ivan Atanasijević*, *Milorad Bertolino*, *Dušan Adnađević*, *Vladimir Mićić*, *Zoran Kadelburg*, *Pavle Mladenović*, *Aleksandar Lipkovski*, *Darko Milinković*, *Vladimir Grujić* and *Miroslav Ristić*.

The Editorial Board was refreshed several times. Starting with 1996, some foreign mathematicians were included in the Board, in an effort to raise the quality of articles. The list of the present Editorial Board can be found in each issue of the journal.

We can conclude that MATEMATIČKI VESNIK has played a very important role in the development of mathematical sciences in Yugoslavia. Some of the most eminent mathematicians published their articles in it, and, on the other hand, a lot of our mathematicians had an opportunity to publish their first articles in this journal. All the articles from Vesnik have been regularly reviewed in the main reviewing journals—*Mathematical Reviews*, *Zentralblatt für Mathematik und ihre Grenzgebiete* and *Реферативный журнал.* Finally, starting with 1996, Matematički Vesnik is published electronically, too, as a part of the ELibEMS (Electronic Library of the European Mathematical Society) and it can be obtained through Internet on `http://www.emis.de/journals/MV/` or `http://www.dms.org.rs`.

This is the last issue of the jubilar, 60th volume of Matematički Vesnik; we are sure that this jubilee will not be the last one.

Editorial Board