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SIGNED DEGREE SETS IN SIGNED 3-PARTITE GRAPHS

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Abstract. If each edge of a 3-partite graph is assigned a positive or a negative sign then it is called a signed 3-partite graph. Also, signed degree of a vertex x in a signed 3-partite graph is the number of positive edges incident with x less than the number of negative edges incident with x. The set of distinct signed degrees of the vertices of a signed 3-partite graph is called its signed degree set. In this paper, we prove that every set of n integers is the signed degree set of some connected signed 3-partite graph.

1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graph is given by Harary [3]. Let G be a signed graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The signed degree of $v_i \in V$ is $sdeg(v_i) = d_i = d_i^+ - d_i^-$, where $d_i^+(d_i^-)$ is the number of positive (negative) edges incident with v_i. A signed degree sequence $\sigma = [d_1, d_2, \ldots, d_n]$ of a signed graph G is formed by listing the vertex signed degrees in non-increasing order. An integral sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \ldots, d_n]$ is standard sequence if σ is non-increasing, $\sum_{i=1}^{n} d_i$ is even, $d_1 > 0$, each $|d_i| < n$, and $|d_1| \geq |d_n|$.

Chartrand et al. [1] obtained the necessary and sufficient conditions for an integral sequence to be s-graphical, which is similar to Hakimi's result for degree sequences in graphs [2]. Another necessary and sufficient conditions for an integral sequence to be the signed degree sequence of a signed graph is given by Yan et al. [8].

The set of distinct signed degrees of the vertices of a signed graph is called its signed degree set. Pirzada et al. [6] proved that every set of positive(negative) integers is the signed degree set of some connected graph and determined the smallest possible order for such a signed graph.

A signed 3-partite graph is a 3-partite graph in which each edge is assigned a positive or a negative sign. Let $G(U, V, W)$ be a signed 3-partite graph with

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$U = \{u_1, u_2, \ldots, u_p\}$, $V = \{v_1, v_2, \ldots, v_q\}$ and $W = \{w_1, w_2, \ldots, w_r\}$. Then, signed degree of u_i is sdeg $(u_i) = d_i = d_i^+ - d_i^-$, where $d_i^+(d_i^-)$ is the number of positive(negative) edges incident with u_i, signed degree of v_j is sdeg $(v_j) = e_j = e_j^+ - e_j^-$, where $e_j^+(e_j^-)$ is the number of positive(negative) edges incident with v_j, and signed degree of w_k is sdeg $(w_k) = f_k = f_k^+ - f_k^-$, where $f_k^+(f_k^-)$ is the number of positive(negative) edges incident with w_k. Then, the sequences $\alpha = [d_1, d_2, \ldots, d_p], \beta = [e_1, e_2, \ldots, e_q]$ and $\gamma = [f_1, f_2, \ldots, f_r]$ are called the signed degree sequences of $G(U, V, W)$. Also, the sequences of integers α, β and γ are said to be standard sequences if α is non-zero and non-increasing, $\sum_{i=1}^{p} d_i + \sum_{j=1}^{q} e_j + \sum_{k=1}^{r} f_k$ is even, $d_1 > 0$, each $|d_i| \leq q + r$, each $|e_j| \leq r + p$, each $|f_k| \leq p + q$, $|d_1| \geq |e_j|$ and $|d_1| \geq |f_k|$ for each j and k.

The following result is given by Pirzada et al. [5].

Theorem 1.1. *Let $\alpha = [d_1, d_2, \ldots, d_p], \beta = [e_1, e_2, \ldots, e_q]$ and $\gamma = [f_1, f_2, \ldots, f_r]$ be standard sequences. Then, α, β and γ are the signed degree sequences of a signed 3-partite graph if and only if there exists integers g and h with $d_1 = g - h$ and $0 \leq h \leq \frac{q+r-d_1}{2}$ such that α', β' and γ' are the signed degree sequences of a signed 3-partite graph, where α' is obtained from α by deleting d_1 and β' and γ' are obtained from β and γ by reducing g greatest entries of β and γ by 1 each and adding h least entries of β and γ by 1 each.*

The characterization of signed degree sequences in signed bipartite graphs can be found in [7]. That every set of integers is the signed degree set of some connected signed bipartite graph is proved in [4].

For any two sets X and Y, we denote by $X \oplus Y$ to mean that each vertex of X is joined to every vertex of Y by a positive edge.

2. Main Results

A signed 3-partite graph $G(U, V, W)$ is said to be connected if each vertex of one partite set is connected to every vertex of other partite sets. The set S of distinct signed degrees of the vertices of a signed 3-partite graph $G(U, V, W)$ is called its signed degree set.

First, we obtain the following result which shows that every set of positive integers is a signed degree set of some connected signed 3-partite graph.

Theorem 2.1. *Let $d_1, d_2, \ldots, d_n$ be positive integers. Then, there exists a connected signed 3-partite graph with signed degree set $S = \{s_i | i = 1, 2, \ldots, n\}$, where $s_i = \sum_{j=1}^{i} d_j$.*

Proof. If $n = 1$, $d_1 = 1$ then a signed 3-partite graph $G(U, V, W)$ with $U = \{u_1, u_2\}$, $V = \{v_1, v_2, v_3\}$, $W = \{w_1\}$ and in which $u_1v_1, u_2v_2, u_2v_3, v_1w_1$ are positive edges and u_2v_1 is negative edge has signed degree set $S = \{d_1\}$.

Also, if $n = 1$, $d_1 > 0$, then a signed 3-partite graph $G(U, V, W)$ with $|U| = 1, |V| = d_1 - 1, |W| = d_1, U \oplus W$ and $V \oplus W$ has signed degree set $S = \{d_1\}$.

Now, assume that $n \geq 2$. Construct a signed 3-partite graph $G(U, V, W)$ as follows.

Let $U = X_1 \cup X_2 \cup \cdots \cup X_{n-1} \cup X_n$, $V = Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1}$, $W = Z_1 \cup Z_2 \cup Z_2' \cup \cdots \cup Z_{n-1} \cup Z_{n-1}' \cup Z_n \cup Z_n'$, with $X_i \cap X_j = \phi$, $Y_i \cap Y_j = \phi$, $Z_i \cap Z_j = \phi$, $Z_i \cap Z_j' = \phi$, $Z_i' \cap Z_i' = \phi$ $(i \neq j)$, $|X_i| = |Z_i| = d_i$ for all i, $1 \leq i \leq n$, $|Y_i| = \left|Z_{i+1}'\right| = d_1 + d_2 + \cdots + d_i$ for all i, $1 \leq i \leq n-1$, $X_i \oplus Z_j$ whenever $i \geq j$, $Y_i \oplus Z_{i+1}$ for all i, $1 \leq i \leq n-1$, $Y_i \oplus Z_{i+1}'$ for all i, $1 \leq i \leq n-1$. Then, the signed degrees of the vertices of $G(U, V, W)$ are as follows.

For $1 \leq i \leq n$, $sdeg(x_i) = \sum_{j=1}^{i} |Z_j| = \sum_{j=1}^{i} d_j = d_1 + d_2 + \cdots + d_i$, for all $x_i \in X_i$; for $1 \leq i \leq n-1$, $sdeg(y_i) = |Z_{i+1}| + \left|Z_{i+1}'\right| = d_{i+1} + d_1 + d_2 + \cdots + d_i = d_1 + d_2 + \cdots + d_i + d_{i+1}$ for all $y_i \in Y_i$; for $1 \leq i \leq n$, $sdeg(z_i) = (\sum_{j=i1}^{n} |X_j|) + |Y_{i-1}| = \sum_{j=1}^{n} d_j + d_1 + d_2 + \cdots + d_{i-1} = d_1 + d_2 + \cdots + d_n$ for all $z_i \in Z_i$; and for $2 \leq i \leq n$, $\mathrm{sdeg}(z_i') = |Y_{i-1}| = d_1 + d_2 + \cdots + d_{i-1}$, for all $z_i' \in Z_i'$.

Therefore, signed degree set of $G(U, V, W)$ is S. Clearly, all the signed 3-partite graphs constructed above are conected. Hence, the result. ∎

The next result follows from Theorem 2.1 by interchanging positive edges with negative edges.

Corolarry 2.1. *Every set of negative integers is a signed degree set of some connected signed 3-partite graph.*

Now we have the following result.

Theorem 2.2 *Every set of integers is a signed degree set of some connected signed 3-partite graph.*

Proof. If S is a set of integers, then we have the following cases.

(i) S is a set of positive(negative) integers. Then, by Theorem 2.1(Corollary 2.1), the result follows.

(ii) $S = \{0\}$. Then, a signed 3-partite graph $G(U, V, W)$ with $U = \{u_1\}$, $V = \{v_1, v_2\}$, $W = \{w_1\}$ and in which u_1v_1, v_2w_1 are positive edges and u_1v_2, v_1w_1 are negative edges has signed degree set S.

(iii) S is a set of non-negative (non-positive) integers. Let $S = S_1 \cup \{0\}$, where S_1 is a set of negative (positive) integers. Then, by Theorem 2.1(Corollary 2.1), there is a connected signed 3-partite graph $G_1(U_1, V_1, W_1)$ with signed degree set S_1. Construct a new signed 3-partite graph $G(U, V, W)$ as follows.

Let $U = U_1 \cup \{u\}$, $V = V_1 \cup \{v\}$, $W = W_1 \cup \{w\}$, with $U_1 \cap \{u\} = \phi, V_1 \cap \{v\} = \phi, W_1 \cap \{w\} = \phi$ and let uv, v_1w be positive edges and uv_1, vw be negative edges, where $v_1 \in V_1$. Then, $G(U, V, W)$ has signed degree set S. Also, note that addition of such edges do not effect the signed degrees of the vertices of $G_1(U_1, V_1, W_1)$, and the vertices u, v, w have signed degrees zero each.

(iv) S is a set of non-zero integers. Let $S = S_1 \cup S_2$, where S_1 and S_2 are sets of positive and negative integers respectively. Then, by Theorem 2.1 and Corollary

2.1, there are connected signed 3-partite graphs $G_1(U_1, V_1, W_1)$ and $G_2(U_2, V_2, W_2)$ with S_1 and S_2 respectively.

Construct a new signed 3-partite graph $G(U, V, W)$ as follows.

Let $U = U_1 \cup U_2, V = V_1 \cup V_2, W = W_1 \cup W_2$ with $U_1 \cap U_2 = \phi, V_1 \cap V_2 = \phi, W_1 \cap W_2 = \phi$ and let u_1v_2, u_2w_1, v_1w_2 be positive edges and u_1w_2, u_2v_1, w_1v_2 be negative edges where $u_i \in U_i, v_i \in V_i, w_i \in W$. Then, signed degree set of $G(U, V, W)$ is S. Clearly addition of such edges do not effect the signed degrees of the vertices of $G_1(U_1, V_1, W_1)$ and $G_2(U_2, V_2, W_2)$.

(v) S is a set of all integers. Let $S = S_1 \cup S_2 \cup \{0\}$, where S_1 and S_2 are sets of positive and negative integers respectively. Then, by Theorem 2.1 and Corollary 2.1, there exist connected signed 3-partite graphs $G_1(U_1, V_1, W_1)$ and $G_2(U_2, V_2, W_2)$ with signed degree sets S_1 and S_2 respectively. Construct a new signed 3-partite graph $G(U, V, W)$ as follows.

$U = U_1 \cup U_2 \cup \{u\}, V = V_1 \cup V_2, W = W_1 \cup W_2$ with $U_1 \cap U_2 = \phi, U_i \cap \{u\} = \phi$, $V_1 \cap V_2 = \phi, W_1 \cap W_2 = \phi$ and let uv_1, u_2w_1 be positive edges, where $u_2 \in U_2, v_1 \in V_1, w_1 \in W_1$. Then, signed degree set of $G(U, V, W)$ is S. We note that addition of such edges do not effect the signed degrees of the vertices of $G_1(U_1, V_1, W_1)$ and $G_2(U_2, V_2, W_2)$, and the signed degree of u is zero. Clearly, by construction, all the above signed 3-partite graphs are connected. This proves the result. ■

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