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SOME REMARKS ABOUT BOUNDED DERIVATIONS ON THE HILBERT SPACE OF SQUARE SUMMABLE MATRICES

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Abstract. It is known that not every Banach algebra has non-trivial bounded derivations. For instance, consider large families of weighted semisimple Banach algebras. In particular, we will be concerned with derivations within the concrete frame of the non-abelian, non-unitary, involutive Banach algebra $l^2(\mathbf{N}^2)$. The theoretical interest in this algebra is based on the well-known fact that it is isomorphic to the class of Hilbert-Schmidt operators acting between two given separable Hilbert spaces (cf. [8]). In this article, we characterize and determine the explicit structure of *all* bounded derivations on $l^2(\mathbf{N}^2)$.

1. Preliminaries

The study of bounded and unbounded derivations on Banach algebras raised the attention of several researchers (cf. [3], [9], [11], etc.). Among other authors, for mapping properties of derivations, determination of their ranges and other theoretic results, the reader can see [4], [5], [6], [7], [10]. Though the existence of non-trivial derivations on a Banach algebra is known, it is usually difficult to describe their general structure. Our matter in this article is to develop the structure of bounded derivations on spaces of complex matrices. We shall consider the space $l^2\left(\mathbf{N}^2\right)$ of infinite matrices $a = \left(a_{i,j}\right)_{i,j\in\mathbf{N}}$ with complex entries such that the extended real number $\|a\|_2^2 = \sum_{i,j\in\mathbf{N}} \left|a_{i,j}\right|^2$ is finite. Endowed with the usual inner product $\langle a,b\rangle = \sum_{i,j\in\mathbf{N}} a_{i,j}\,\overline{b_{i,j}}$, $a,b \in l^2\left(\mathbf{N}^2\right)$, $\left(l^2\left(\mathbf{N}^2\right), \|\circ\|_2\right)$ becomes a Hilbert space. Moreover, if we define

$$a\cdot b = \left\{\sum_{k=1}^{\infty} a_{i,k}\, b_{k,j}\right\}_{i,j\in\mathbf{N}},\ \ a,b\in l^2\left(\mathbf{N}^2\right),$$

then $\left(l^2\left(\mathbf{N}^2\right), \|\circ\|_2, \cdot\right)$ is a non-abelian complex Banach algebra without unit, endowed with an involution $z \mapsto z^*$, where $z^*_{k,h} = \overline{z_{h,k}}$ for each $k,h \in \mathbf{N}$. Moreover, as we have already pointed out, this algebra is isometrically isomorphic to the Banach

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algebra of Hilbert-Schmidt operators acting between two given separable Hilbert spaces (cf. [8]). More generally, if $\mathfrak{U}$ is a Banach algebra we will write $\mathcal{B}(\mathfrak{U})$ (resp. $\mathcal{D}(\mathfrak{U})$) to denote the class of bounded linear operators (resp. continuous derivations) on $\mathfrak{U}$. As usual, if $a \in \mathfrak{U}$ then L_a and R_a will denote the bounded linear operators on $\mathfrak{U}$ of left and right multiplication by a, respectively. Of course, by a *derivation* we mean a linear map $\Delta\colon \mathfrak{U} \to \mathfrak{U}$ so that the Leibnitz rule $\Delta(a \cdot b) = \Delta(a) \cdot b + a \cdot \Delta(b)$ is satisfied for all $a, b \in \mathfrak{U}$. For instance, if $a \in \mathfrak{U}$ then $\Delta_a = L_a - R_a$ defines a derivation on $\mathfrak{U}$. As usual such derivations are called *inner* or *Lie derivations.* A non-inner derivation is said to be an *outer derivation.* In particular, all inner derivations are clearly bounded. If $\mathfrak{U}$ is a $*$-algebra, a derivation Δ is called a $*$ derivation if $\Delta(x)^* = \Delta(x^*)$ for each $x \in \mathfrak{U}$. The set $\mathcal{D}(\mathfrak{U})$ becomes an algebra if for elements Δ_1, Δ_2 we define their *Lie product* as $[\Delta_1, \Delta_2] = \Delta_1 \circ \Delta_2 - \Delta_2 \circ \Delta_1$. Indeed, if $a, b \in \mathfrak{U}$ then $[\Delta_a, \Delta_b] = \Delta_{a \cdot b - b \cdot a}$.

Example 1. Since $\langle z \cdot w, t\rangle = \langle z, t \cdot w^*\rangle = \langle w, z^* \cdot t\rangle$ if $z, w, t \in l^2\left(\mathbf{N}^2\right)$ then $\Delta_a^* = \Delta_{a^*}$ and the class of inner derivations becomes a self-adjoint subalgebra of $\mathcal{D}\left(l^2\left(\mathbf{N}^2\right)\right)$.

Example 2. The map $a \mapsto \Delta_a$ is a non-isometric monomorphism between $l^2\left(\mathbf{N}^2\right)$ and $\mathcal{D}\left(l^2\left(\mathbf{N}^2\right)\right)$. For, $(e_{i,j} \cdot a)_{k,h} = \delta_k^i\, a_{j,h}$ and $(a \cdot e_{i,j})_{k,h} = \delta_h^j\, a_{k,i}$, where $i, j, k, h \in \mathbf{N}$, δ_k^i is Kronecker's symbol and $e_{i,j} = \left(\delta_k^i\, \delta_h^j\right)_{k,h \in \mathbf{N}}$. So, if $\Delta_a = 0$ we deduce that a is a scalar multiple of the identity matrix and since $a \in l^2\left(\mathbf{N}^2\right)$ it must be the zero matrix. On the other hand, if $i \neq j$ in $\mathbf{N}$ and $x \in l^2\left(\mathbf{N}^2\right)$ we write

$$\begin{aligned}\left\|\Delta_{e_{i,j}}(x)\right\|_2^2 &= \left\|\sum_{h=1}^{\infty}\left(x_{i,h}\, e_{j,h} - x_{h,j}\, e_{h,i}\right)\right\|_2^2 = \|x\|_2^2 - 2\ \mathrm{Re}\left(x_{i,i}\, \overline{x_{j,j}}\right) \\ &\leq \|x\|_2^2 + 2\ |x_{i,i}\, x_{j,j}| \leq \|x\|_2^2 + |x_{i,i}|^2 + |x_{j,j}|^2 \leq 2\ \|x\|_2^2,\end{aligned}$$

i.e. $\left\|\Delta_{e_{i,j}}\right\| \leq \sqrt{2}$. Indeed, $\left\|\Delta_{e_{i,j}}\left((e_{i,i} - e_{j,j})/\sqrt{2}\right)\right\|_2 = \sqrt{2}$ and so

$$1 = \|e_{i,j}\|_2 < \left\|\Delta_{e_{i,j}}\right\| = \sqrt{2}.$$

Remark 3. The linear operator $\Delta(z) = \sum_{k,l \in \mathbf{N}} (k - l)\, z_{k,l} e_{k,l}$, defined on the subalgebra

$$D(\Delta) = \left\{z \in l^2\left(\mathbf{N}^2\right) : \sum_{k,l \in \mathbf{N}} (k - l)^2\ |z_{k,l}|^2 < \infty\right\},$$

is an unbounded derivation on $l^2\left(\mathbf{N}^2\right)$. Since $\sum_{n=1}^{\infty} e_{n,1}/n \in l^2\left(\mathbf{N}^2\right) - D(\Delta)$ then $D(\Delta) \subsetneq l^2(\mathbf{N}^2)$. Indeed, it is dense because $e_{i,j} \in D(\Delta)$ if $i, j \in l^2(\mathbf{N}^2)$. It is known that every derivation on a C^*-algebra is continuous (cf. [1], Ch. 4, 4.6.65, 301–302). If $k, l \in l^2(\mathbf{N}^2)$ then

$$\left\|(e_{k,l} + e_{l,k})^*\,(e_{k,l} + e_{l,k})\right\|_2 = \sqrt{2} < 2 = \|(e_{k,l} + e_{l,k})\|_2^2$$

and so $l^2\left(\mathbf{N}^2\right)$ is not a C^*-algebra.

EXAMPLE 4. In a von Neumann algebra every derivation is bounded and inner ([2], Ch. 8, 8.7.55, 582). An example of a bounded outer derivation on $l^2\left(\mathbf{N}^2\right)$ is

$$\Delta(z)=\sum_{i\in\mathbf{N}-\{1\},\, j\in\mathbf{N}}\left(z_{i-1,j}-z_{i,j+1}\right)e_{i,j}-\sum_{j=1}^{\infty}z_{1,j+1}\,e_{1,j},\; z\in l^2\left(\mathbf{N}^2\right).$$

REMARK 5. All eigenvectors corresponding to nonzero eigenvalues of an inner derivation are nilpotent. For, let z_0 be an eigenvector corresponding to a nonzero eigenvalue ζ of a derivation Δ_a. Since

$$a\cdot z_0^2-z_0\cdot a\cdot z_0=\zeta\, z_0^2=z_0\cdot a\cdot z_0-z_0^2\cdot a$$

we have

$$z_0\cdot a\cdot z_0=a\cdot z_0^2-\zeta\, z_0^2=\zeta\, z_0^2+z_0^2\cdot a$$

and $\Delta_a\left(z_0^2\right)=2\zeta\, z_0^2$. Then $\Delta_a\left(z^n\right)=n\zeta\, z_0^n$, $n\in\mathbf{N}$, as follows by an iterated application of the Leibniz rule, i.e. $n\zeta\in\sigma(\Delta)$ whenever $z_0^n\neq 0$. The claim follows by the boundedness of the spectrum of Δ.

2. On the structure of general derivations

DEFINITION 6. An infinite complex matrix $\omega=\{\omega_{i,j}\}_{i,j\in\mathbf{N}}$ is said to be *nearly-inner* if the formal operator $L_\omega-R_\omega$ belongs to $\mathcal{B}\left(l^2\left(\mathbf{N}^2\right)\right)$. We will denote by $\mathfrak{Q}$ the class of all nearly-inner matrices.

REMARK 7. Observe that $\mathfrak{Q}\supsetneq l^2\left(\mathbf{N}^2\right)$. For instance, the identity matrix is nearly-inner though not square-summable. Further, if we write

$$\omega(m)=\begin{cases}\sum_{n=1}^{\infty}e_{n,m+n}, & \text{if } m\geq 0,\\ \sum_{n=1}^{\infty}e_{-m+n,n}, & \text{if } m<0,\end{cases}$$

then $\{\omega(m)\}_{m\in\mathbf{Z}}\subseteq\mathfrak{Q}-l^2\left(\mathbf{N}^2\right)$.

REMARK 8. If $\alpha\in\mathfrak{Q}$, the following extended number

$$\eta=\sup_{k,h\in\mathbf{N}}\sum_{l=1}^{\infty}\left(\left|\alpha_{k,l}\right|^2+\left|\alpha_{l,h}\right|^2\right)$$

is finite. For, if $k,h\in\mathbf{N}$ then

$$\sum_{l=1}^{\infty}\left(\left|\alpha_{k,l}\right|^2+\left|\alpha_{l,h}\right|^2\right)=\left\|\left(L_\alpha-R_\alpha\right)\left(e_{h,k}\right)\right\|^2\leq\left\|L_\alpha-R_\alpha\right\|^2.$$

PROPOSITION 9. *Necessary and sufficient conditions in order that the formal operator $\delta_\beta(z)=\sum_{i,j\in\mathbf{N}}\beta_{i,j}\,z_{i,j}\,e_{i,j}$ defines a bounded derivation on $l^2\left(\mathbf{N}^2\right)$ are that $\sup_{i,j\in\mathbf{N}}\left|\beta_{i,j}\right|<\infty$ and that for any $i,j,k\in\mathbf{N}$ the following identities $\beta_{i,k}+\beta_{k,j}=\beta_{i,j}$ hold.*

Remark 10. The proof of the above proposition is straightforward. In consequence, the coefficients $\beta_{i,j}$'s verify $\beta_{i,i} = 0$ and so $\beta_{i,j} + \beta_{j,i} = 0$. Thus $\beta_{i,j} = \beta_{i,1} + \beta_{1,j} = -\beta_{1,i} + \beta_{1,j}$, i.e. the first row determines the whole β. So any bounded sequence $\xi = (\xi_n)_{n\in\mathbf{N}}$ gives rise to a matrix $\beta = \beta(\xi)$. For instance, if $\xi = (0,1,0,0,\dots)$ then

$$\beta_{i,j} = \begin{cases} 0, & \text{if } i = j = 2 \text{ or } 2 \notin \{i,j\}, \\ -1, & \text{if } i = 2 \text{ and } j \neq 2, \\ 1, & \text{if } i \neq 2 \text{ and } j = 2. \end{cases}$$

and $\delta_\beta(z) = \sum_{i\in\mathbf{N}-\{2\}} z_{i,2}\, e_{i,2} - \sum_{j\in\mathbf{N}-\{2\}} z_{2,j}\, e_{2,j}$.

Theorem 11. *A bounded linear endomorphism Δ of $l^2(\mathbf{N}^2)$ is a derivation if and only if there are matrices $\alpha = \{\alpha_{i,j}\}_{i,j\in\mathbf{N}}$ and $\beta = \{\beta_{i,j}\}_{i,j\in\mathbf{N}}$ of complex numbers uniquely determined so that:*

(i) *for any $i \in \mathbf{N}$, $\alpha_{i,i} = 0$;*

(ii) $\sup_{i,j\in\mathbf{N}} |\beta_{i,j}| < \infty$;

(iii) $\alpha \in \mathfrak{Q}$;

(iv) *for any $i,j,k \in \mathbf{N}$ the identities $\beta_{i,j} + \beta_{j,k} = \beta_{i,k}$ hold.*

Moreover,

$$\Delta(z) = \sum_{k,l\in\mathbf{N}} \left(\sum_{j=1}^{\infty} (z_{j,l}\,\alpha_{k,j} - \alpha_{j,l}\, z_{k,j}) + z_{k,l}\,\beta_{k,l} \right) e_{k,l} \quad \text{if } z \in l^2(\mathbf{N}^2) \tag{1}$$

and $\Delta = \Delta_\alpha$ if $\alpha \in l^2(\mathbf{N}^2)$ and $\beta = 0$.

Proof. Let $\Delta \in \mathcal{D}(l^2(\mathbf{N}^2))$ and for $i,j \in \mathbf{N}$ let us denote

$$A_i = \Delta(e_{i,i}) = \{a^i_{j,k}\}_{j,k\in\mathbf{N}} \quad \text{and} \quad B_{i,j} = \Delta(e_{i,j}) = \left(b^{k,h}_{i,j}\right)_{k,h\in\mathbf{N}}.$$

Since $e^2_{i,i} = e_{i,i}$ we have

$$A_i = A_i \cdot e_{i,i} + e_{i,i} \cdot A_i = \sum_{h=1}^{\infty} (a^i_{i,h}\, e_{i,h} + a^i_{h,i}\, e_{h,i}). \tag{2}$$

Whence, for any $k \in \mathbf{N}$ is $a^i_{j,k} = 0$ if $i \notin \{j,k\}$ or $i = j = k$. Since

$$e_{i,j} = e_{i,i} \cdot e_{i,j} = e_{i,j} \cdot e_{j,j}$$

we obtain

$$\begin{aligned} \sum_{h=1}^{\infty} (a^i_{h,i}\, e_{h,j} + b^{i,h}_{i,j}\, e_{i,h}) &= A_i \cdot e_{i,j} + e_{i,i} \cdot B_{i,j} \\ &= B_{i,j} = B_{i,j} \cdot e_{j,j} + e_{i,j} \cdot A_j = \sum_{h=1}^{\infty} (b^{h,j}_{i,j}\, e_{h,j} + a^j_{j,h}\, e_{i,h}). \end{aligned} \tag{3}$$

If $i,j,k,h \in \mathbf{N}$ from (3) is

$$b_{i,j}^{k,h} = a_{k,i}^{i}\,\delta_j^h + b_{i,j}^{i,h}\,\delta_i^k = b_{i,j}^{k,j}\,\delta_j^h + a_{j,h}^{j}\,\delta_i^k. \tag{4}$$

So, by (4) it is $b_{i,j}^{k,k} = 0$ if $k \notin \{i,j\}$. Indeed,

$$\begin{aligned} &\text{if } \ i = k \text{ and } j \neq h \Rightarrow a_{j,h}^{j} = b_{i,j}^{i,h},\\ &\text{if } \ i \neq k \text{ and } j = h \Rightarrow a_{k,i}^{i} = b_{i,j}^{k,j},\\ &\text{if } \ i \neq k \text{ and } j \neq h \Rightarrow b_{i,j}^{k,h} = 0 \end{aligned}$$

and

$$\Delta\left(e_{i,j}\right) = \sum_{h=1}^{\infty}\left(a_{h,i}^{i}\,e_{h,j} + a_{j,h}^{j}\,e_{i,h}\right) + b_{i,j}^{i,j}\,e_{i,j}. \tag{5}$$

In particular, (5) coincides with (2) if $i = j$. Now, if $j \notin \{i,k\}$ then $e_{i,j}\cdot e_{k,j} = 0$ and

$$0 = \Delta\left(0\right) = \Delta\left(e_{i,j}\right)\cdot e_{k,j} + e_{i,j}\cdot\Delta\left(e_{k,j}\right) = \left(a_{j,k}^{j} + a_{j,k}^{k}\right)e_{i,j},$$

i.e. $a_{j,k}^{j} = -a_{j,k}^{k}$. Hence, if $z \in l^2\left(\mathbf{N}^2\right)$ we obtain that

$$\Delta\left(z\right) = \sum_{i,j\in\mathbf{N}} z_{i,j}\left(\sum_{h=1}^{\infty}\left(a_{h,i}^{i}\,e_{h,j} - a_{j,h}^{h}\,e_{i,h}\right) + b_{i,j}^{i,j}\,e_{i,j}\right). \tag{6}$$

From now on we will write $\alpha_{i,j} = a_{i,j}^{j}$ and $\beta_{i,j} = b_{i,j}^{i,j}$ for all indices i and j. In particular we have already obtained (i). Since

$$\begin{aligned} \|\Delta(e_{i,j})\|_2^2 &= \sum_{h=1}\left(\left|a_{h,i}^{i}\right|^2 + \left|a_{j,h}^{j}\right|^2\right) + \left|b_{i,j}^{i,j}\right|^2\\ &= \sum_{h=1}\left(\left|\alpha_{h,i}\right|^2 + \left|\alpha_{j,h}\right|^2\right) + \left|\beta_{i,j}\right|^2 \leq \|\Delta\|^2 \end{aligned}$$

then (ii) follows. Now, if $z \in l^2\left(\mathbf{N}^2\right)$ the linear form $z \mapsto \langle\Delta\left(z\right), e_{k,l}\rangle$ is clearly bounded if $k,l \in \mathbf{N}$ and by (6) it is

$$\langle\Delta\left(z\right), e_{k,l}\rangle = \sum_{i,j\in\mathbf{N}}\left(\alpha_{k,i}\,\delta_l^j - \alpha_{j,l}\,\delta_k^i\right)z_{i,j} + \beta_{k,l}\,z_{k,l}. \tag{7}$$

Since

$$\begin{aligned} \sum_{i,j\in\mathbf{N}}\left|\left(\alpha_{k,i}\,\delta_l^j - \alpha_{j,l}\,\delta_k^i\right)z_{i,j}\right| &\leq \sum_{i=1}^{\infty}\left|\alpha_{k,i}\,z_{i,l}\right| + \sum_{j=1}^{\infty}\left|\alpha_{j,l}\,z_{k,j}\right|\\ &\leq \left(\sum_{i=1}^{\infty}\left|\alpha_{k,i}\right|^2\sum_{i=1}^{\infty}\left|z_{i,l}\right|^2\right)^{1/2} + \left(\sum_{j=1}^{\infty}\left|\alpha_{j,l}\right|^2\sum_{j=1}^{\infty}\left|z_{k,j}\right|^2\right)^{1/2}\\ &\leq 2\,\|\Delta\|\,\|z\|_2 < \infty \end{aligned}$$

by (7) it follows that

$$\begin{aligned} \Delta\left(z\right) &= \sum_{k,l\in\mathbf{N}} e_{k,l}\left(\sum_{i,j\in\mathbf{N}}\left(\alpha_{k,i}\,\delta_l^j - \alpha_{j,l}\,\delta_k^i\right)z_{i,j} + \beta_{k,l}\,z_{k,l}\right)\\ &= \sum_{k,l\in\mathbf{N}} e_{k,l}\left(\sum_{j=1}^{\infty}\left(z_{j,l}\,\alpha_{k,j} - \alpha_{j,l}\,z_{k,j}\right) + \beta_{k,l}\,z_{k,l}\right). \end{aligned} \tag{8}$$

So, if we denote

$$\delta_\alpha = L_\alpha - R_\alpha \quad \text{and} \quad \delta_\beta(z) = \sum_{k,l\in\mathbf{N}} e_{k,l}\,\beta_{k,l}\,z_{k,l} \tag{9}$$

by (ii) it is clear that $\delta_\beta \in \mathcal{B}\left(l^2\left(\mathbf{N}^2\right)\right)$. By (8) it is $\Delta\left(z\right) = \delta_\alpha(z) + \delta_\beta(z)$ for each z, so we infer that $\delta_\alpha \in \mathcal{B}\left(l^2\left(\mathbf{N}^2\right)\right)$ and (iii) holds. In order to see that δ_α is a derivation let us consider $z, w \in l^2\left(\mathbf{N}^2\right)$ so that only a finite number of entries w are non-zero. With the notation of Remark 8, if $k, h \in \mathbf{N}$ we get

$$\begin{aligned}\sum_{l,j\in\mathbf{N}} \left|\alpha_{j,l}\,z_{k,j}\,w_{l,h}\right| &\leq \|z\|_2 \sum_{l=1}^{\infty} \left|w_{l,h}\right| \left(\sum_{j=1}^{\infty} \left|\alpha_{j,l}\right|^2\right)^{1/2} \\ &\leq \eta\,\|z\|_2\,\|w\|_2\,\left[\sharp\left\{l\in\mathbf{N} : w_{l,h}\neq 0\right\}\right]^{1/2} < \infty,\end{aligned} \tag{10}$$

$$\begin{aligned}\sum_{l,j\in\mathbf{N}} \left|\alpha_{l,h}\,z_{k,j}\,w_{j,l}\right| &= \sum_{j=1}^{\infty} \left|z_{k,j}\right| \sum_{l=1}^{\infty} \left|\alpha_{l,h}\,w_{j,l}\right| \\ &\leq \eta \sum_{j=1}^{\infty} \left|z_{k,j}\right| \left(\sum_{l=1}^{\infty} \left|w_{j,l}\right|^2\right)^{1/2} \leq \eta\,\|z\|_2\,\|w\|_2 < \infty,\end{aligned} \tag{11}$$

$$\begin{aligned}\sum_{l,j\in\mathbf{N}} \left|\alpha_{k,j}\,z_{j,l}\,w_{l,h}\right| &= \sum_{j=1}^{\infty} \left|\alpha_{k,j}\right| \sum_{l=1}^{\infty} \left|z_{j,l}\,w_{l,h}\right| \\ &\leq \sum_{j=1}^{\infty} \left|\alpha_{k,j}\right| \left(\sum_{l=1}^{\infty} \left|z_{j,l}\right|^2\right)^{1/2} \left(\sum_{l=1}^{\infty} \left|w_{l,h}\right|^2\right)^{1/2} \\ &\leq \left[\sum_{j=1}^{\infty} \left|\alpha_{k,j}\right| \left(\sum_{l=1}^{\infty} \left|z_{j,l}\right|^2\right)^{1/2}\right] \|w\|_2 \leq \eta\,\|z\|_2\,\|w\|_2 < \infty.\end{aligned} \tag{12}$$

So, by (10), (11) and (12) we can write

$$\begin{aligned}&\left\langle \delta_\alpha\left(z\right)\cdot w + z\cdot\delta_\alpha(w),\, e_{k,h}\right\rangle \\ &\quad= \sum_{l=1}^{\infty} w_{l,h} \sum_{j=1}^{\infty} \left(z_{j,l}\,\alpha_{k,j} - \alpha_{j,l}\,z_{k,j}\right) + \sum_{j=1}^{\infty} z_{k,j} \sum_{l=1}^{\infty} \left(w_{l,h}\,\alpha_{j,l} - \alpha_{l,h}\,w_{j,l}\right) \\ &= \sum_{l=1}^{\infty} w_{l,h} \sum_{j=1}^{\infty} \left(z_{j,l}\,\alpha_{k,j} - \alpha_{j,l}\,z_{k,j}\right) + \sum_{l=1}^{\infty} w_{l,h} \sum_{j=1}^{\infty} z_{k,j}\,\alpha_{j,l} - \sum_{j=1}^{\infty} z_{k,j} \sum_{l=1}^{\infty} \alpha_{l,h}\,w_{j,l} \\ &\quad= \sum_{j=1}^{\infty} \left(\alpha_{k,j} \sum_{l=1}^{\infty} w_{l,h}\,z_{j,l} - \alpha_{j,h} \sum_{l=1}^{\infty} z_{k,l}\,w_{l,j}\right) = \left\langle \delta_\alpha\left(z\cdot w\right), e_{k,h}\right\rangle\end{aligned}$$

Since k, h are arbitrary, Leibnitz rule follows for z and w. In the general case we write $w = \lim_{\sigma\in S} w_\sigma$, where $\left\{w_\sigma\right\}_{\sigma\in S}$ is a net in $l^2\left(\mathbf{N}^2\right)$ whose elements have only a finite number of non zero entries. Then

$$\delta_\alpha\left(z\cdot w\right) = \lim_{\sigma\in S} \delta_\alpha\left(z\cdot w_\sigma\right) = \lim_{\sigma\in S} \left(\delta_\alpha\left(z\right)\cdot w_\sigma + z\cdot\delta_\alpha(w_\sigma)\right) = \delta_\alpha\left(z\right)\cdot w + z\cdot\delta_\alpha(w).$$

As a consequence δ_β becomes a bounded derivation and (iv) follows from Proposition 9.

On the other hand, let $\alpha = \{\alpha_{i,j}\}_{i,j\in\mathbf{N}}$ and $\beta = \{\beta_{i,j}\}_{i,j\in\mathbf{N}}$ be matrices which satisfy (i), (ii) (iii) and (iv). With the notation of (9), $\delta_\alpha \in \mathcal{B}\left(l^2\left(\mathbf{N}^2\right)\right)$ by (iii) and $\delta_\beta \in \mathcal{D}\left(l^2\left(\mathbf{N}^2\right)\right)$ by (ii), (iv) and Proposition 9. Remark 8 is still applicable as well as the above argument to show that $\delta_\alpha \in \mathcal{D}\left(l^2\left(\mathbf{N}^2\right)\right)$. So, if $\Delta = \delta_\alpha + \delta_\beta$ then $\Delta \in \mathcal{D}\left(l^2\left(\mathbf{N}^2\right)\right)$ and the theorem is proved. ∎

Remark 12. With the notation of Theorem 11, the proof of the following assertions is straightforward:

1. Δ is a $*$-derivation if and only if $\alpha_{k,h} = -\overline{\alpha_{h,k}}$ and $\beta_{k,h} = \overline{\beta_{h,k}}$ for all $k, h \in \mathbf{N}$.
2. Δ is self-adjoint if and only if $\alpha_{k,h} = \overline{\alpha_{h,k}}$ for all $k, h \in \mathbf{N}$ and β is a real matrix.

Open Question 13. In this manuscript we introduce the *nearly-innerness concept*. It is easy to give examples of such matrices and in Remark 10 we have observed a necessary condition on them. Is it possible to characterize nearly-inner matrices in terms of growth conditions of their entries? If it is possible, what can be said about the norm of a derivation induced by a nearly-inner matrix according to Theorem 11?

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