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RENORMALIZING ITERATED REPELLING GERMS OF C^2

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Abstract. We find bounded-degree renormalizing polynomial families for iterated repelling germs of $(C^2, 0)$. These families consist in contracting mappings and yield germs whose differentials have rank two.

1. Foreword

In this paper we deal with renormalizing families of iterates of holomorphic mappings with a repelling fixed point at 0. The process of renormalization could be described as composing on the right the elements of a holomorphic family $\mathcal{F}$ with a family of mappings of a fixed type (depending on the nature of the problem: linear, affine, polynomial ones, etc) and then extracting a normally convergent subsequence from the new family. This is a useful tool in complex analysis: in one variable, for instance, it allows, as shown in [5], Chap. 8, to get a 'linearizing' change of coordinates (known as Königs coordinates) in the neighbourhood of a repelling (or attractive) fixed point p of a holomorphic endomorphism φ of $\mathbb{P}^1$: at a repelling fixed point, this coordinates could be in fact achieved by composing on the right the iterates φ^n with multiplication by $[\varphi'(p)]^{-n}$ and letting n diverge.

Moreover, by a slightly different point of view, one could consider 'Zalcman's renormalization lemma' (see [7], [1], p. 9), which amounts to getting a normal family from a nonnormal one by means of composition on the right with a family of contracting *affine* functions: this yields an *entire* limit function and allows quite direct proofs of both great Picard's and Montel's theorems (see [1], pp. 10/11): this topics will not be discussed in this paper.

The situation is different in higher dimension: the following example, adapted from [6] (9.2) shows, for instance, the existence of nonnormal families of iterates of an automorphisms of $\mathbb{C}^2$, with a repelling fixed point in 0, which are by no means linearly renormalizable.

Let $F \in Aut(\mathbb{C}^2)$ be defined by $F(z,w) = (\alpha z, \beta w + z^2)$, with $|\alpha| > 1$, $|\beta| > 1$ and $|\beta| \geq |\alpha|^2$; F admits a repelling fixed point in 0, hence $\{F^{o\,k}\}$ cannot be normal at 0.

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Now $F^{o\,k}(z,w)=(\alpha^k z,\beta^k w+\beta^{k-1}[1+c+\cdots c^{k-1}]z^2)$, where $c=\alpha^2/\beta$; then $F^{o\,k}o[F_*(0)]^{-k}(z,w)=(z,w+\beta^{-1}\big[c^{-k}+c^{-k+1}+\cdots+c^{-1}\big]z^2)$; since $c\le 1$, the coefficient of z^2 diverges as $k\to\infty$.

Thus, composing with 'division' by differentials in 0 allows in general no kind of renormalization; however, see [3] for a deep analysis of renormalization by differentials in connection with 'Lattes examples' in $\mathbb{P}^N$.

By contrast, we shall show that the family of the iterates of a repelling germ of $(\mathbb{C}^2,0)$ admits a renormalizing family consisting of *polynomial* mappings fixing 0, with *uniformly bounded degrees* and converging unformly on compacta to 0, in such a way that a holomorphic germ $\boldsymbol{H}$ tangent to the identity is yielded after renormalization. This fact implies, in particular, that the dimension of the image of any representative of $\boldsymbol{H}$ will be maximal.

2. Some definitions and lemmata

2.1. Elementary mappings

We recall that $G=(g_1,g_2)$ is called an *elementary mapping* of $\mathbb{C}^2$ if $g_1(z)=c_1z_1$, $g_2(z)=c_2z_2+h(z_1)$, where the c_k's are complex constants and h is a holomorphic function of z_1; G is an automorphism if and only if each c_k is nonzero.

Lemma 1. *If $H\colon\mathbb{C}^2\to\mathbb{C}^2$ is defined by setting $H(x,y)=(\alpha x,\beta y+h(x))$, where $h(x)=\sum_{l=0}^{\infty}\eta_l x^l$ and α, β are complex constants, then, for $n\ge 0$, $H^{\circ n}(x,y)=\Big(\alpha^n x,\Big[\sum_{k=0}^{n-1}\beta^k h(\alpha^{n-1-k}x)\Big]+\beta^n y\Big)$; moreover, if $\alpha\ne 0$, $\beta\ne 0$, then H is invertible and there holds $H^{-n}(x,y)=(\alpha^{-n}x,[-\sum_{k=1}^{n}\beta^{-k}h(\alpha^{-n-1+k}x)]+\beta^{-n}y)$.*

Proof: by induction on n. ■

Let now $p_N(x)=\sum_{l=0}^{N}\eta_l x^l$ be the N-th degree truncation of the development of h and

$$H_N^{-n}(x,y)=(\alpha^{-n}x,[-\sum_{k=1}^{n}\beta^{-k}p_N(\alpha^{-n-1+k}x)]+\beta^{-n}y)$$

the corresponding truncation of H^{-n}; note that, if $|\alpha|>1$ and $|\beta|>1$, then H_N^{-n} is a polynomial contracting mapping for every $n\ge 0$ and $\{H^{-n}\}\to 0$ uniformly on compacta in $\mathbb{C}^2$.

Theorem 2. *If $|\alpha|>1$, $|\beta|>1$ and $|\beta|<|\alpha|^N$, then $\{H^{\circ n}\circ H_N^{-n}\}$ converges uniformly on compacta to a lower triangular automorphism of the form $H(x,y)=(x,h(x)+y)$ for a suitable entire function ψ.*

Proof. Trivially $\left(H^{\circ n}\circ H_N^{-n}\right)_1(x,y)\equiv x$ and

$$\begin{aligned}\left(H^{\circ n}\circ H_N^{-n}\right)_2(x,y)&=\sum_{k=0}^{n-1}\beta^k h(\alpha^{-1-k}x)-\sum_{k=1}^{n}\beta^{n-k}p_N(\alpha{-}n-1+kx)+y\\&=\sum_{k=0}^{n-1}\beta^k h(\alpha^{-1-k}x)-\sum_{k=0}^{n-1}\beta^k p_N(\alpha{-}1-kx)+y\end{aligned}$$

$$= \sum_{k=0}^{n-1} \beta^k R_N(\alpha{-}1 - kx) + y := \psi_n(x) + y,$$

where R_N is the N-th remainder in the development of h.

Now $h_n(x) = \sum_{k=0}^{n-1} \beta^k \sum_{l=N}^{\infty} \eta_l (\alpha^{k+1})^{-l} x^l$; since, for $N \geq l$, we have

$$|\beta^k \alpha^{-(k+1)l}| \leq |\beta\alpha^{-N}|^k,$$

we can let n diverge and exchange the order of summation, getting, uniformly on compact sets:

$$\begin{aligned} h(x) := \lim_{n\to\infty} \psi_n(x) &= \sum_{l=N}^{\infty} (\sum_{k=0}^{\infty} \beta^k (\alpha^{k+1})^{-l}) \eta_l x^l \\ &= [\sum_{k=0}^{\infty} (\beta\alpha^{-l})^k] \alpha^{-l} \eta_l x^l = \sum_{l=N}^{\infty} (\alpha^l - \beta)^{-1} \eta_l x^l; \end{aligned} \tag{1}$$

by comparison with $h = \sum_{l=0}^{\infty} \eta_l x^l$, we see that the last series in (1) represents an entire function ψ: this ends the proof. ■

2.2. A lemma by Rosay and Rudin

We recall the lemma from the Appendix of [6], specialized to $\mathbb{C}^2$.

LEMMA 3. *Let V be a neighbourhood of* 0 *in $\mathbb{C}^2$, $F : V \to \mathbb{C}^2$ a holomorphic mapping with $F(0) = 0$ and $F_*(0)$ lower triangular ; suppose that all eigenvalues λ_i of $F_*|_0 := A$ satisfy $|\lambda_i| < 1$. Then there exist:* (i) *an elementary polynomial automorphism G of $\mathbb{C}^2$ such that $G(0) = 0$ and $G_*|_0 = A$ (thus $c_i = \lambda_i$ for every i),* (ii) *polynomial applications $T_m \colon \mathbb{C}^2 \to \mathbb{C}^2$, with $T_m(0) = 0$, $T_{m*}|_0 =$* **id** *such that $G^{-1} \circ T_m \circ F - T_m = O(|z|^m)$, $(m = 2, 3, \dots)$.* ■

2.3. A lemma on attractive germs

The following lemma shows that a holomorphic mapping has a contracting behaviour near an attractive fixed point.

LEMMA 4. *Let V be a neighbourhood of* 0 *in $\mathbb{C}^N$, $F : V \to \mathbb{C}^N$ a holomorphic mapping admitting an attractive fixed point at* 0*: then there exists $A < 1$ and a neighbourhood $\mathbb{R} \subset V$ of* 0 *such that $F^n(\mathbb{R}) \subset A^n \mathbb{R}$.*

Proof. By Schur's lemma, we may suppose, without loss of generality, that

$$F_*(0) = \begin{pmatrix} \lambda_1 & \dots & \dots & \dots & 0 \\ a_{21} & \lambda_2 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{N1} & \dots & \dots & a_{N\,N-1} & \lambda_N \end{pmatrix},$$

where the λ_k's (with $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_N|$) are the eigenvalues of $F_*(0)$ and the a_{jk}'s complex constants.

Set

$$E_\varepsilon = \begin{pmatrix} \varepsilon^N & 0 & \dots & 0 \\ 0 & \varepsilon^{N-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \varepsilon \end{pmatrix}:$$

if ε is small enough, there exists $A < 1$ such that

$$\|E_\varepsilon^{-1} F_*(0) E_\varepsilon\| (= \|E_\varepsilon^{-1} F_* E_\varepsilon(0)\|) < A;$$

thus there exists $\varrho > 0$ such that

$$E_\varepsilon^{-1} \circ F \circ E_\varepsilon(B(0,\varrho)) \subset B(0,\varrho),$$

hence, if $p \in B(0,\varrho)$, $\|E_\varepsilon^{-1} F_*^n E_\varepsilon(p)\| < A^n$ and $\|E_\varepsilon^{-1} F^n E_\varepsilon(p)\| < A^n \|p\| \leq A^n \varrho$, i.e. $E_\varepsilon^{-1} \circ F^n \circ E_\varepsilon(B(0,\varrho)) \subset B(0, A^n \varrho)$.

This fact will eventually imply

$$F^n(E_\varepsilon(B(0,\varrho))) \subset E_\varepsilon(B(0, A^n\varrho)) = A^n E_\varepsilon(B(0,\varrho));$$

we conclude by setting $\mathbb{R} = E_\varepsilon(B(0,\varrho))$. ■

3. The main result

LEMMA 5. *Let f be a holomorphic mapping in a neighbourhood of $0 \in \mathbb{C}^2$, with $f(0) = 0$ and $f_*(0)$ attractive. There exist: a neighbourhood $\mathbb{R}$ of 0, a biholomorphic mapping $\Psi : \mathbb{R} \to \mathbb{C}^2$ with $\psi(0) = 0$ and $\psi_*(0) = \mathbf{id}$ and an elementary polynomial automorphism G of $\mathbb{C}^2$ such that $G^n \circ \Psi = \Psi \circ f^n$ for each $n \geq 0$.*

Proof: By Lemma 4 there exists a neighbourhood $\mathbb{R}$ of 0 such that $f^{\circ n}(\mathbb{R}) \subset A^n \mathbb{R}$ for a suitable real constant $A < 1$ and every $n > 0$.

We may suppose that $f_*(0)$ is lower triangular (necessarily attractive) at 0: Lemma 3 gives us a lower triangular polynomial automorphism $G : \mathbb{C}^2 \to \mathbb{C}^2$ such that $G(0) = 0$ and $G_*(0) = f_*(0)$; by Lemma 1, there exists a complex constant γ such that $|G^{-k}(w) - G^{-k}(w') \leq \gamma^k |w - w'|$ for each $(w, w') \in [\mathbb{D}\,(0, 1/2)]$. Take an integer m such that $A^m \leq 1/\gamma$: Lemma 3 yields a polynomial application $T := T_m$ correspondingly.

Let us proceed exactly like in the proof if the Theorem in the appendix of [6] (see from line 29 of page 84 up to line 4 of page 85), whose notations we have kept; this shows that the limit $\lim_{k\to\infty} G^{-k} \circ T \circ f^k$ exists uniformly on each compact set of $\mathbb{R}$ so it canonically defines a holomorphic application $\Psi : \mathbb{R} \to \mathbb{C}^2$ such that $\Psi(0) = 0$, $\Psi_*(0) = \mathbf{id}$ and $G^n \circ \Psi = \Psi \circ f^n$. ■

THEOREM 6. *Let $\boldsymbol{h}$ be a repelling holomorphic germ of $(\mathbb{C}^2, 0)$, with $\boldsymbol{h}_*$ admitting the eigenvalues α, β such that $\beta \geq \alpha > 1$ and $|\beta| < |\alpha|^N$; then there exists a sequence $\{Q_n\}$ of polynomial mappings which*

- *are contracting in a neighbourhood of 0;*
- *converge uniformly on compacta to 0 in $\mathbb{C}^2$;*

- *have uniformly bounded degrees*

such that $\{\boldsymbol{h}\circ Q_n\}$ *converges to a holomorphic germ* $\boldsymbol{H}$ *of* $(\mathbb{C}^2, 0)$, *with* $\boldsymbol{H}_*(0) = \mathbf{id}$.

Proof. Let $(\mathcal{U}, h)$ be a representative of $\boldsymbol{h}$ with inverse $(\mathcal{V}, f)$; maybe shrinking $\mathcal{V}$ we may suppose, by Lemma 4, that there exists $A \in \mathbb{R}$ such that $f^n \subset A^n\mathcal{V}$ (i.e. $\mathcal{V} = \mathbb{R}$ with respect to the notation of Lemma 4.

By lemma 5 there exist: a biholomorphic mapping $\psi : \mathcal{V} \to \mathbb{C}^2$ tangent to the identity in 0 and an elementary polynomial automorphism G of $\mathbb{C}^2$ such that $f^n(z) = \psi^{-1}G^{\circ n}\psi(z)$, for each $n > 0$ and $z \in \mathcal{V}$: thus $h^n(z) = \psi^{-1}G^{-n}\psi(z)$, for each $n > 0$ and $z \in f^n(\mathcal{V})$.

Set $H := G^{-1}$: we have $H_*(0) = G_*^{-1}(0) = f_*^{-1}(0) = h_*(0)$ hence $\boldsymbol{H}$ will be defined by setting $H(x, y) = (\alpha x, \beta y + h(x))$, with h entire and $h(0) = 0$.

For each $M \geq 0$, let Θ_M be a polynomial mapping such that

$$\psi \circ \Theta_M = \mathbf{id} + O(|z|^M).$$

By Theorem 2, $\{H^{\circ n}\}$ admits a family of contracting polynomial mappings $\{P_n\}$ converging uniformly on compacta to 0, with $P_n(0) = 0$ and uniformly bounded degrees such that $\{H^{\circ n}\circ P_n\}$ converges uniformly on compacta to an entire mapping S on $\mathbb{C}^2$ (note that $P_n := H_N^{-n}$ with respect to the notation of Theorem 2).

Since $|\alpha| \leq |\beta|$ and $h(0) = 0$, we have:

$$|\sum_{k=1}^{n} \beta^{-k} h(\alpha^{-n-1+k}u)| = O(n|\alpha|^{-n}|u|);$$

thus $P_n(z) = O\left((n+1)|\alpha|^{-n}|z|\right)$, where $z = (u, v)$; moreover

$$\psi \circ \Theta_M \circ P_n = P_n + O\left([(n+1)|\alpha|^{-n}|z|]^M\right).$$

Now $P_n^{-1} \circ f^n$ is tangent to the identity at 0 for all n, hence we may assume that there exists a neighbourhood $\mathcal{A}$ of 0 such that $P_n^{-1} \circ f^n(\mathcal{V}) \supset \mathcal{A}$.

This eventually implies that, if $z \in \mathcal{A}$,

$$\begin{aligned}
\psi \circ h^n \circ \Theta_M \circ P_n &= H^{-n} \circ \psi \circ \Theta_M \circ P_n \\
&= H^{-n} \circ \psi \circ \Theta_M \circ \left(P_n + O\left(\left[(n+1)|\alpha|^{-n}|z|\right]^M\right)\right) \\
&= S + O\left((n+1)|\beta|^n \left[(n+1)|\alpha|^{-n}|z|\right]^M\right) \\
&= S + O\left((n+1)^{M+1}|\alpha|^{n(N-M)}|z|^M\right).
\end{aligned}$$

By choosing $M > N$ and passing to germs at 0, we see that $\boldsymbol{\Psi} \circ \boldsymbol{h}^n \circ \Theta_M \circ P_n$ converges to S, hence $\boldsymbol{h}^n \circ \Theta_M \circ P_n$ converges to $\boldsymbol{\Psi}^{-1} \circ S$.

Now the $\Theta_M \circ P_n$'s are polynomial mappings, contracting in a neighbourhood of 0, with uniformly bounded degrees, since so are the P_n's; they converge uniformly on compacta to 0 by the corresponding property of the P_n's; moreover, $\boldsymbol{\Psi}^{-1}$ is tangent to the identity at 0 and $rk(S_*(0)) = 2$ by Theorem 2: setting $\boldsymbol{H} = \boldsymbol{\Psi}^{-1} \circ S$ ends the proof. ■

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