

r -FUZZY STRONGLY PREOPEN SETS IN FUZZY TOPOLOGICAL SPACES

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Abstract. We introduce r -fuzzy strongly preopen and r -fuzzy strongly preclosed sets in fuzzy topological spaces in the sense of the definition of Šostak [8] and investigate some of their properties. Fuzzy strongly precontinuous, fuzzy strongly preopen and fuzzy strongly preclosed mappings between fuzzy topological spaces are defined. Their properties and the relationship between these mappings and other mappings introduced previously are investigated.

1. Introduction and preliminaries

A. P. Šostak [8] introduced a fuzzy topology as an extension of Chang's fuzzy topology [2]. It has been developed in many directions [3,4,7]. B. Krsteska [1] introduced fuzzy strongly preopen and fuzzy strongly preclosed sets in the Chang's fuzzy topology.

In this paper we define r -fuzzy strongly preopen and r -fuzzy strongly preclosed sets in a fuzzy topological space in view of the definition of Šostak [8] and investigate some of their properties. We show that fuzzy strong precontinuity implies fuzzy precontinuity, but the converse is not true. We obtain some properties of fuzzy strongly precontinuous mappings.

Throughout this paper, let X be a non-empty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha$, for all $x \in X$. All other notations and definitions are standard in the fuzzy set theory.

DEFINITION 1.1. [8] A function $\tau: I^X \rightarrow I$ is called a fuzzy topology on X if it satisfies the following conditions:

- (O1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,
- (O2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,
- (O3) $\tau(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset I^X$.

The pair (X, τ) is called a fuzzy topological space (fts, for short).

AMS Subject Classification: 54 A 40

Keywords and phrases: r -fuzzy strongly preopen, r -fuzzy strongly preclosed, fuzzy strong precontinuity, fuzzy strongly preopen mapping, fuzzy strongly preclosed mapping.

REMARK 1.2. Let (X, τ) be an fts. Then, for each $\alpha \in I$, $\tau_\alpha = \{\mu \in I^X : \tau(\mu) \geq \alpha\}$ is a Chang's fuzzy topology on X .

THEOREM 1.1. [7] Let (X, τ) be an fts. For each $r \in I_0$, $\lambda \in I^X$, define an operator $C_\tau: I^X \times I_0 \rightarrow I^X$ as follows

$$C_\tau(\lambda, r) = \bigwedge \{ \mu \in I^X : \lambda \leq \mu, \tau(\bar{1} - \mu) \geq r \}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_0$ it satisfies the following conditions:

- (1) $C_\tau(\bar{0}, r) = \bar{0}$.
- (2) $\lambda \leq C_\tau(\lambda, r)$.
- (3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.
- (4) $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$ if $r \leq s$.
- (5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

THEOREM 1.2. [6] Let (X, τ) be an fts. For each $r \in I_0$, $\lambda \in I^X$, define an operator $I_\tau: I^X \times I_0 \rightarrow I^X$ as follows

$$I_\tau(\lambda, r) = \bigvee \{ \mu \in I^X : \lambda \geq \mu, \tau(\mu) \geq r \}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_0$ it satisfies the following conditions:

- (1) $I_\tau(\bar{1} - \lambda, r) = \bar{1} - C_\tau(\lambda, r)$ and $C_\tau(\bar{1} - \lambda, r) = \bar{1} - I_\tau(\lambda, r)$.
- (2) $I_\tau(\bar{1}, r) = \bar{1}$.
- (3) $I_\tau(\lambda, r) \leq \lambda$.
- (4) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$.
- (5) $I_\tau(\lambda, r) \geq I_\tau(\lambda, s)$ if $r \leq s$.
- (6) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

DEFINITION 1.2. [6] Let (X, τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$:

(1) λ is called an r -fuzzy semi-open (r -fso, for short) set if there exists $\nu \in I^X$ with $\tau(\nu) \geq r$ such that $\nu \leq \lambda \leq C_\tau(\nu, r)$. Equivalently, $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$.

(2) λ is called an r -fuzzy semi-closed (r -fss, for short) set if there exists $\nu \in I^X$ with $\tau(\bar{1} - \nu) \geq r$ such that $I_\tau(\nu, r) \leq \lambda \leq \nu$. Equivalently, $I_\tau(C_\tau(\lambda, r), r) \leq \lambda$.

DEFINITION 1.3. [6] Let (X, τ) and (Y, τ^*) be fts's. A mapping $f: X \rightarrow Y$ is said to be:

- (1) fuzzy continuous iff $\tau^*(\mu) \leq \tau(f^{-1}(\mu))$ for each $\mu \in I^Y$.
- (2) fuzzy open iff $\tau^*(f(\mu)) \geq \tau(\mu)$ for each $\mu \in I^X$.
- (3) fuzzy closed iff $\tau^*(\bar{1} - f(\mu)) \geq \tau(\bar{1} - \mu)$ for each $\mu \in I^X$.

DEFINITION 1.4. [6] Let (X, τ) and (Y, τ^*) be fts's. A mapping $f: X \rightarrow Y$ is said to be:

- (1) fuzzy semi-continuous iff $f^{-1}(\mu)$ is r -fso for each $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\mu) \geq r$.
- (2) fuzzy semi-open iff $f(\mu)$ is r -fso for each $\mu \in I^X$, $r \in I_0$ with $\tau(\mu) \geq r$.
- (3) fuzzy semi-closed iff $f(\mu)$ is r -fsc for each $\mu \in I^X$, $r \in I_0$ with $\tau(\bar{1} - \mu) \geq r$.

2. r -fuzzy strongly preopen and r -fuzzy strongly preclosed sets

DEFINITION 2.1. [6] Let (X, τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$:

- (1) λ is called r -fuzzy preopen (r -fpo, for short) iff

$$\lambda \leq I_\tau(C_\tau(\lambda, r), r).$$

- (2) λ is called r -fuzzy preclosed (r -fpc, for short) iff

$$C_\tau(I_\tau(\lambda, r), r) \leq \lambda.$$

- (3) λ is called r -fuzzy strongly semi-open (r -fsso, for short) iff

$$\lambda \leq I_\tau(C_\tau(I_\tau(\lambda, r), r), r).$$

- (4) λ is called r -fuzzy strongly semi-closed (r -fssc, for short) iff

$$C_\tau(I_\tau(C_\tau(\lambda, r), r), r) \leq \lambda.$$

DEFINITION 2.2. [6] Let (X, τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$:

- (1) the r -fuzzy preinterior of λ , denoted by $PI_\tau(\lambda, r)$, is defined by

$$PI_\tau(\lambda, r) = \bigvee \{ \nu \in I^X : \nu \leq \lambda, \nu \text{ is } r\text{-fpo} \},$$

- (2) the r -fuzzy preclosure of λ , denoted by $PC_\tau(\lambda, r)$, is defined by

$$PC_\tau(\lambda, r) = \bigwedge \{ \nu \in I^X : \nu \geq \lambda, \nu \text{ is } r\text{-fpc} \},$$

- (3) the r -fuzzy strongly semi-interior of λ , denoted by $SSI_\tau(\lambda, r)$, is defined by

$$SSI_\tau(\lambda, r) = \bigvee \{ \nu \in I^X : \nu \leq \lambda, \nu \text{ is } r\text{-fsso} \},$$

- (4) the r -fuzzy strongly semi-closure of λ , denoted by $SSC_\tau(\lambda, r)$, is defined by

$$SSC_\tau(\lambda, r) = \bigwedge \{ \nu \in I^X : \nu \geq \lambda, \nu \text{ is } r\text{-fssc} \}.$$

THEOREM 2.1. *Let (X, τ) be an fts, $\lambda \in I^X$ and $r \in I_0$. Then*

- (1) $\lambda \vee C_\tau(I_\tau(\lambda, r), r) \leq PC_\tau(\lambda, r)$,
- (2) $PI_\tau(\lambda, r) \leq \lambda \wedge I_\tau(C_\tau(\lambda, r), r)$,
- (3) $I_\tau(PC_\tau(\lambda, r), r) \leq I_\tau(C_\tau(\lambda, r), r)$,
- (4) $I_\tau(C_\tau(I_\tau(\lambda, r), r), r) \leq I_\tau(PC_\tau(\lambda, r), r)$,
- (5) $PC_\tau(\bar{1} - \lambda, r) = \bar{1} - PI_\tau(\lambda, r)$ and $PI_\tau(\bar{1} - \lambda, r) = \bar{1} - PC_\tau(\lambda, r)$.

Proof. (1) Since $PC_\tau(\lambda, r)$ is an r -fpc set, we have

$$C_\tau(I_\tau(\lambda, r), r) \leq C_\tau(I_\tau(PC_\tau(\lambda, r), r), r) \leq PC_\tau(\lambda, r).$$

Thus, $\lambda \vee C_\tau(I_\tau(\lambda, r), r) \leq PC_\tau(\lambda, r)$.

- (2) It can be shown as (1).
- (3) It follows from the relation $PC_\tau(\lambda, r) \leq C_\tau(\lambda, r)$.
- (4) From (1) we have

$$I_\tau(PC_\tau(\lambda, r), r) \geq I_\tau(\lambda \vee C_\tau(I_\tau(\lambda, r), r), r) \geq I_\tau(C_\tau(I_\tau(\lambda, r), r), r).$$

- (5) Straightforward. ■

DEFINITION 2.3. Let (X, τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$:

- (1) λ is called r -fuzzy strongly preopen (r -fspo, for short) iff

$$\lambda \leq I_\tau(PC_\tau(\lambda, r), r).$$

- (2) λ is called r -fuzzy strongly preclosed (r -fspc, for short) iff

$$C_\tau(PI_\tau(\lambda, r), r) \leq \lambda.$$

- (3) The r -fuzzy strong preinterior of λ , denoted by $SPI_\tau(\lambda, r)$, is defined by

$$SPI_\tau(\lambda, r) = \bigvee \{ \nu \in I^X : \nu \leq \lambda, \nu \text{ is } r\text{-fspo} \}.$$

- (4) The r -fuzzy strong preclosure of λ , denoted by $SPC_\tau(\lambda, r)$, is defined by

$$SPC_\tau(\lambda, r) = \bigwedge \{ \nu \in I^X : \nu \geq \lambda, \nu \text{ is } r\text{-fspc} \}.$$

THEOREM 2.2. *Let (X, τ) be an fts, $\lambda \in I^X$ and $r \in I_0$. Then*

- (1) if $\tau(\lambda) \geq r$, then λ is an r -fspo set;
- (2) if λ is r -fspo, then λ is r -fspo;
- (3) if λ is r -fspo, then λ is r -fspo.

Proof. It follows from Theorem 2.1. ■

The following examples show that the converses in the above theorem are not true in general.

EXAMPLE 2.1. Let $X = \{a, b, c\}$. Define fuzzy sets $\lambda_1, \lambda_2, \mu \in I^X$ as follows:

$$\begin{aligned} \lambda_1(a) &= 0.3, & \lambda_1(b) &= 0.2, & \lambda_1(c) &= 0.7, \\ \lambda_2(a) &= 0.8, & \lambda_2(b) &= 0.8, & \lambda_2(c) &= 0.4, \\ \mu(a) &= 0.8, & \mu(b) &= 0.7, & \mu(c) &= 0.6. \end{aligned}$$

Define a fuzzy topology $\tau: I^X \rightarrow I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{3}, & \text{if } \lambda = \lambda_1, \\ \frac{1}{2}, & \text{if } \lambda = \lambda_2, \\ \frac{2}{3}, & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ \frac{2}{3}, & \text{if } \lambda = \lambda_1 \wedge \lambda_2, \\ 0, & \text{otherwise.} \end{cases}$$

For the fts (X, τ) with $0 < r \leq \frac{1}{3}$, μ is an r -fspo set from

$$\mu \leq I_\tau(PC_\tau(\mu, r), r) = \bar{1}.$$

For $0 < r \leq \frac{1}{3}$, μ is neither r -fssso nor $\tau(\mu) \geq r$ from

$$\mu \not\leq I_\tau(C_\tau(I_\tau(\mu, r), r), r) = \lambda_1 \wedge \lambda_2.$$

EXAMPLE 2.2. Let X be a non-empty set. We define fuzzy topology $\tau: I^X \rightarrow I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.2}, \\ \frac{1}{3}, & \text{if } \lambda = \overline{0.4}, \\ \frac{1}{2}, & \text{if } \lambda = \overline{0.5}, \\ \frac{3}{4}, & \text{if } \lambda = \overline{0.6}, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < r \leq \frac{1}{3}$, $\overline{0.7}$ is an r -fpo set but it is not r -fspo, from the following:

$$\begin{aligned} \overline{0.7} &\leq I_\tau(C_\tau(\overline{0.7}, r), r) = \bar{1}, \\ \overline{0.7} &> I_\tau(PC_\tau(\overline{0.7}, r), r) = I_\tau(\overline{0.7}, r) = \overline{0.4}. \end{aligned}$$

For $\frac{1}{3} < r \leq \frac{2}{3}$, $\overline{0.3}$ is r -fso but it is not r -fspo, from the following:

$$\begin{aligned} \overline{0.3} &\leq C_\tau(I_\tau(\overline{0.3}, r), r) = \overline{0.4}, \\ \overline{0.3} &> I_\tau(PC_\tau(\overline{0.3}, r), r) = I_\tau(\overline{0.4}, r) = \overline{0.2}. \end{aligned}$$

REMARK 2.1. From Example 2.1, for $0 < r \leq \frac{1}{3}$, μ is an r -fspo set but not an r -fspo set because

$$\mu \not\leq C_\tau(I_\tau(\mu, r), r) = \bar{1} - \lambda_1 \wedge \lambda_2.$$

Also, in Example 2.2, $\overline{0.3}$ is an r -fspo but not r -fspo set for $\frac{1}{3} < r \leq \frac{2}{3}$. From the above discussion there is no relation between the concepts of r -fspo sets and r -fspo sets.

THEOREM 2.3. Let (X, τ) be an fts and $r \in I_0$.

(1) Any union of r -fspo sets is an r -fspo set.

(2) Any intersection of r -fspo sets is an r -fspo set.

Proof. (1) Let $\{\lambda_\alpha : \alpha \in \Gamma\}$ be a family of r -fspo sets. For each $\alpha \in \Gamma$, $\lambda_\alpha \leq I_\tau(PC_\tau(\lambda_\alpha, r), r)$. Hence we have

$$\bigvee_{\alpha \in \Gamma} \lambda_\alpha \leq \bigvee_{\alpha \in \Gamma} I_\tau(PC_\tau(\lambda_\alpha, r), r) \leq I_\tau\left(PC_\tau\left(\bigvee_{\alpha \in \Gamma} \lambda_\alpha, r\right), r\right).$$

So, $\bigvee_{\alpha \in \Gamma} \lambda_\alpha$ is an r -fspo set.

(2) It is easily proved in the same manner. ■

REMARK 2.2. The intersection of two r -fspo sets need not be an r -fspo set. And the union of two r -fspo sets need not be an r -fspo set. We will show it in the next example.

EXAMPLE 2.3. Consider the fts (X, τ) from Example 2.1. The fuzzy set ρ defined as:

$$\rho(a) = 0.4, \quad \rho(b) = 0.2, \quad \rho(c) = 0.8,$$

is a $\frac{1}{3}$ -fspo set, but $\lambda_2 \wedge \rho$ is not a $\frac{1}{3}$ -fspo set. Also, $(\bar{1} - \lambda_2) \vee (\bar{1} - \rho)$ is not a $\frac{1}{3}$ -fspo set in (X, τ) .

THEOREM 2.4. Let (X, τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$ the following statements hold:

(1) $C_\tau(\lambda, r)$ is r -fspo.

(2) λ is r -fspo iff $\lambda = SPI_\tau(\lambda, r)$.

(3) λ is r -fspo iff $\lambda = SPC_\tau(\lambda, r)$.

(4) $I_\tau(\lambda, r) \leq SPI_\tau(\lambda, r) \leq PI_\tau(\lambda, r) \leq \lambda \leq PC_\tau(\lambda, r) \leq SPC_\tau(\lambda, r) \leq C_\tau(\lambda, r)$.

(5) $SPI_\tau(\bar{1} - \lambda, r) = \bar{1} - SPC_\tau(\lambda, r)$ and $SPC_\tau(\bar{1} - \lambda, r) = \bar{1} - SPI_\tau(\lambda, r)$.

(6) $C_\tau(SPC_\tau(\lambda, r), r) = SPC_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Proof. (1), (2), (3) and (4) follow from the definitions.

(5) For $\lambda \in I^X$, $r \in I_0$ we have the following:

$$\begin{aligned} \bar{1} - SPI_\tau(\lambda, r) &= \bar{1} - \bigvee\{\nu : \nu \leq \lambda, \nu \text{ is } r\text{-fspo}\} \\ &= \bigwedge\{\bar{1} - \nu : \bar{1} - \lambda \leq \bar{1} - \nu, \bar{1} - \nu \text{ is } r\text{-fspo}\} \\ &= SPC_\tau(\bar{1} - \lambda, r). \end{aligned}$$

(6) From (1) and (3), $SPC_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$. We only show that

$$C_\tau(SPC_\tau(\lambda, r), r) = C_\tau(\lambda, r).$$

Since $\lambda \leq SPC_\tau(\lambda, r)$, $C_\tau(\lambda, r) \leq C_\tau(SPC_\tau(\lambda, r), r)$. Suppose that

$$C_\tau(\lambda, r) \not\leq C_\tau(SPC_\tau(\lambda, r), r).$$

There exist $x \in X$ and $r \in I_0$ such that

$$C_\tau(\lambda, r)(x) < C_\tau(SPC_\tau(\lambda, r), r)(x).$$

By the definition of C_τ , there exists $\rho \in I^X$ with $\lambda \leq \rho$ and $\tau(\bar{1} - \rho) \geq r$ such that

$$C_\tau(SPC_\tau(\lambda, r), r)(x) > \rho(x) \geq C_\tau(\lambda, r)(x).$$

On the other hand, since $\rho = C_\tau(\rho, r)$, $\lambda \leq \rho$ implies

$$SPC_\tau(\lambda, r) \leq SPC_\tau(\rho, r) = SPC_\tau(C_\tau(\rho, r), r) = C_\tau(\rho, r) = \rho.$$

Thus,

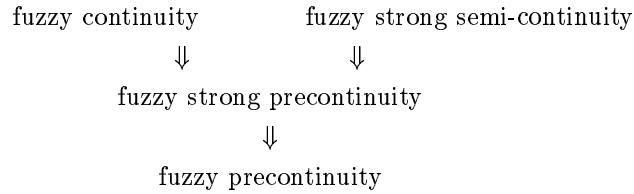
$$C_\tau(SPC_\tau(\lambda, r), r) \leq \rho.$$

It is a contradiction. Hence, $C_\tau(SPC_\tau(\lambda, r), r) \leq C_\tau(\lambda, r)$. ■

3. Fuzzy strong precontinuity

DEFINITION 3.1. Let (X, τ) and (Y, τ^*) be fts's and $f: X \rightarrow Y$ be a mapping. Then, f is called fuzzy strongly precontinuous (resp. fuzzy strongly semi-continuous, fuzzy precontinuous) if $f^{-1}(\mu)$ is an r -fspo (resp. r -fssso, r -fpo) set in X for each $\mu \in I^Y$ and $r \in I_0$ with $\tau^*(\mu) \geq r$.

The implications contained in the following diagram are true.



The following examples show that the reverse may not be true in general.

EXAMPLE 3.1. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ as follows:

$$\begin{aligned}
 \lambda_1(a) = 0.2, \quad \lambda_1(b) = 0.4, \quad \lambda_2(a) = 0.6, \quad \lambda_2(b) = 0.5; \\
 \lambda_3(a) = 0.3, \quad \lambda_3(b) = 0.4, \quad \lambda_4(a) = 0.6, \quad \lambda_4(b) = 0.7.
 \end{aligned}$$

Let $\tau, \tau^*: I^X \rightarrow I$ be fuzzy topologies on X defined as:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1 \text{ or } \overline{0.4}, \\ \frac{2}{3}, & \text{if } \lambda = \lambda_2 \text{ or } \overline{0.7}, \\ 1, & \text{otherwise.} \end{cases} \quad \tau^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{2}, & \text{if } \lambda = \lambda_3, \\ \frac{1}{2}, & \text{if } \lambda = \lambda_4, \\ 0, & \text{otherwise.} \end{cases}$$

Then the identity mapping $id_X: (X, \tau) \rightarrow (X, \tau^*)$ is fuzzy precontinuous but not fuzzy strongly precontinuous because: for $0 < r \leq \frac{1}{2}$, λ_3 is r -fpo and λ_4 is r -fpo but not r -fspo because

$$\begin{aligned}\lambda_3 &\leq I_\tau(C_\tau(\lambda_3, r), r) = I(\tau)(\bar{1} - \lambda_2, r) = \overline{0.4}, \\ \lambda_4 &\leq I_\tau(C_\tau(\lambda_4, r), r) = \bar{1}, \\ \lambda_4 &> I_\tau(PC_\tau(\lambda_4, r), r) = I_\tau(\lambda_4, r) = \lambda_2.\end{aligned}$$

EXAMPLE 3.2. We consider Example 2.1 and put

$$\tau^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{3}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

For fts (X, τ^*) the identity mapping $id_X: (X, \tau) \rightarrow (X, \tau^*)$ is fuzzy strongly precontinuous but it is neither fuzzy continuous nor fuzzy strongly semi-continuous.

THEOREM 3.1. *Let (X, τ) and (Y, τ^*) be fts's and $f: X \rightarrow Y$. The following statements are equivalent:*

- (1) *The mapping f is fuzzy strongly precontinuous.*
- (2) *$f^{-1}(\mu)$ is r -fspc in X for each $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\bar{1} - \mu) \geq r$.*
- (3) *$f(SPC_\tau(\lambda, r)) \leq C_{\tau^*}(f(\lambda), r)$, $\forall \lambda \in I^X$, $r \in I_0$.*
- (4) *$SPC_\tau(f^{-1}(\mu), r) \leq f^{-1}(C_{\tau^*}(\mu, r))$, $\forall \mu \in I^Y$, $r \in I_0$.*
- (5) *$f^{-1}(I_{\tau^*}(\mu, r)) \leq SPI_\tau(f^{-1}(\mu), r)$, $\forall \mu \in I^Y$, $r \in I_0$.*

Proof. (1) \iff (2). Clear.

(1) \implies (3). For all $\lambda \in I^X$, $r \in I_0$, since $\tau^*(\bar{1} - C_{\tau^*}(f(\lambda), r)) \geq r$ from the definition of C_{τ^*} , and Definition 1.1 (O3),

$$f^{-1}(\bar{1} - C_{\tau^*}(f(\lambda), r)) = \bar{1} - f^{-1}(C_{\tau^*}(f(\lambda), r))$$

is r -fspc. By Definition 2.3.(2), $f^{-1}(C_{\tau^*}(f(\lambda), r))$ is an r -fspc set in X . Since

$$\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(C_{\tau^*}(f(\lambda), r)),$$

we have $SPC_\tau(\lambda, r) \leq f^{-1}(C_{\tau^*}(f(\lambda), r))$. Hence

$$f(SPC_\tau(\lambda, r)) \leq f(f^{-1}(C_{\tau^*}(f(\lambda), r))) \leq C_{\tau^*}(f(\lambda), r).$$

(3) \implies (4). For $\mu \in I^Y$, $r \in I_0$, let $\lambda = f^{-1}(\mu)$. By (3)

$$f(SPC_\tau(f^{-1}(\mu), r)) \leq C_{\tau^*}(f(f^{-1}(\mu)), r) \leq C_{\tau^*}(\mu, r).$$

It implies $SPC_\tau(f^{-1}(\mu), r) \leq f^{-1}(C_{\tau^*}(\mu, r))$.

(4) \implies (5). For $\mu \in I^Y$, $r \in I_0$, by (4) we have

$$f^{-1}(C_{\tau^*}(\bar{1} - \mu, r)) \geq SPC_{\tau}(f^{-1}(\bar{1} - \mu), r) = SPC_{\tau}(\bar{1} - f^{-1}(\mu), r).$$

Then by Theorem 2.4.(5) we have

$$\begin{aligned} f^{-1}(C_{\tau^*}(\bar{1} - \mu, r)) &\geq \bar{1} - SPI_{\tau}(f^{-1}(\mu), r), \\ \bar{1} - f^{-1}(\bar{1} - I_{\tau^*}(\mu, r)) &\leq SPI_{\tau}(f^{-1}(\mu), r), \\ f^{-1}(I_{\tau^*}(\mu, r)) &\leq SPI_{\tau}(f^{-1}(\mu), r). \end{aligned}$$

(5) \implies (1). For each $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\mu) \geq r$, since $I_{\tau^*}(\mu, r) = \mu$,

$$f^{-1}(\mu) = f^{-1}(I_{\tau^*}(\mu, r)) \leq SPI_{\tau}(f^{-1}(\mu), r).$$

Hence, by definition of $SPI_{\tau}(f^{-1}(\mu), r)$, $f^{-1}(\mu) = SPI_{\tau}(f^{-1}(\mu), r)$. By Theorem 2.4.(2), $f^{-1}(\mu)$ is r -fspo. Therefore f is a fuzzy strongly precontinuous mapping. ■

THEOREM 3.2. *Let (X, τ) and (Y, τ^*) be fts's and $f: X \rightarrow Y$. The following statements are equivalent:*

- (1) *The mapping f is fuzzy strongly precontinuous.*
- (2) $C_{\tau}(PI_{\tau}(f^{-1}(\mu), r), r) \leq f^{-1}(C_{\tau^*}(\mu, r))$, $\forall \mu \in I^Y$, $r \in I_0$.
- (3) $f(C_{\tau}(PI_{\tau}(\lambda, r), r)) \leq C_{\tau^*}(f(\lambda), r)$, $\forall \lambda \in I^X$, $r \in I_0$.

Proof. The proof is standard and therefore omitted. ■

THEOREM 3.3. *Let (X, τ) and (Y, τ^*) be fts's and $f: X \rightarrow Y$. The following statements are equivalent:*

- (1) *The mapping f is fuzzy precontinuous.*
- (2) $f^{-1}(\mu)$ is r -fpc in X for each $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\bar{1} - \mu) \geq r$.
- (3) $f(PC_{\tau}(\lambda, r)) \leq C_{\tau^*}(f(\lambda), r)$, $\forall \lambda \in I^X$, $r \in I_0$.
- (4) $PC_{\tau}(f^{-1}(\mu), r) \leq f^{-1}(C_{\tau^*}(\mu, r))$, $\forall \mu \in I^Y$, $r \in I_0$.
- (5) $f^{-1}(I_{\tau^*}(\mu, r)) \leq PI_{\tau}(f^{-1}(\mu), r)$, $\forall \mu \in I^Y$, $r \in I_0$.

Proof. Similar to the proof of Theorem 3.1. ■

THEOREM 3.4. *Let (X, τ) and (Y, τ^*) be fts's and $f: X \rightarrow Y$ be a bijective mapping. The following statements are equivalent:*

- (1) *The mapping f is fuzzy strongly precontinuous.*
- (2) $I_{\tau^*}(f(\lambda), r) \leq f(SPI_{\tau}(\lambda, r))$, $\forall \lambda \in I^X$, $r \in I_0$.

Proof. (1) \implies (2). Let f be a fuzzy strongly precontinuous mapping and $\lambda \in I^X$, $r \in I_0$. Then, $f^{-1}(I_{\tau^*}(f(\lambda), r))$ is an r -fspo set in X . By Theorem 3.1.(5), and the fact that f is injective, we have

$$f^{-1}(I_{\tau^*}(f(\lambda), r)) \leq SPI_{\tau}(f^{-1}(f(\lambda)), r) = SPI_{\tau}(\lambda, r).$$

Since f is surjective, we have

$$I_{\tau^*}(f(\lambda), r) = f(f^{-1}(I_{\tau^*}(f(\lambda), r))) \leq f(SPI_{\tau}(\lambda, r)).$$

(2) \implies (1). Let $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\mu) \geq r$. Then $I_{\tau^*}(\mu, r) = \mu$ and by (2) we have

$$\mu = I_{\tau^*}(\mu, r) \leq f(SPI_{\tau}(f^{-1}(\mu), r)).$$

Since f is injective we have

$$f^{-1}(\mu) \leq f^{-1}f(SPI_{\tau}(f^{-1}(\mu), r)) = SPI_{\tau}(f^{-1}(\mu), r).$$

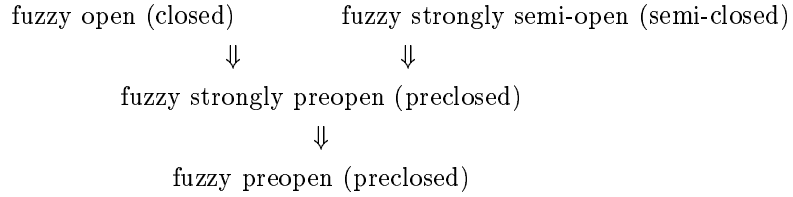
Then, by the definition of $SPI_{\tau}(f^{-1}(\mu), r)$ we have $f^{-1}(\mu) = SPI_{\tau}(f^{-1}(\mu), r)$. By Theorem 2.4.(2) we have $f^{-1}(\mu)$ is an r -fspo set in X . Thus, f is a fuzzy strongly precontinuous mapping. ■

4. Fuzzy strongly preopen and fuzzy strongly preclosed mappings

DEFINITION 4.1. Let (X, τ) and (Y, τ^*) be fts's and $f: X \rightarrow Y$ be a mapping. Then f is called:

- (1) fuzzy strongly preopen (resp. fuzzy strongly semi-open, fuzzy preopen) if $f(\lambda)$ is an r -fspo (resp. r -fssso, r -fpo) set in Y for each $\lambda \in I^X$, $r \in I_0$ with $\tau(\lambda) \geq r$.
- (2) fuzzy strongly preclosed (resp. fuzzy strongly semi-closed, fuzzy preclosed) if $f(\lambda)$ is an r -fspc (resp. r -fssc, r -fpc) set in Y for each $\lambda \in I^X$, $r \in I_0$ with $\tau(\bar{1} - \lambda) \geq r$.

The implications contained in the following diagram are true.



The following examples show that the reverse may not be true.

EXAMPLE 4.1. In Example 3.1, the identity mapping $id_X: (X, \tau^*) \rightarrow (X, \tau)$ is a fuzzy preopen mapping but not a fuzzy strongly preopen mapping.

EXAMPLE 4.2. In Example 3.2, the identity mapping $id_X: (X, \tau^*) \rightarrow (X, \tau)$ is a fuzzy strongly preopen mapping but it is neither fuzzy open nor fuzzy strongly semi-open mapping.

THEOREM 4.1. *Let $f: (X, \tau) \rightarrow (Y, \tau^*)$ be a mapping. Then the following statements are equivalent:*

- (1) f is a fuzzy strongly preopen mapping.
- (2) $f(I_\tau(\lambda, r)) \leq SPI_{\tau^*}(f(\lambda), r)$ for each $\lambda \in I^X$, $r \in I_0$.
- (3) $I_\tau(f^{-1}(\mu), r) \leq f^{-1}(SPI_{\tau^*}(\mu, r))$ for each $\mu \in I^Y$, $r \in I_0$.

Proof. (1) \implies (2). For all $\lambda \in I^X$, $r \in I_0$, since $\tau(I_\tau(\lambda, r)) \geq r$, $f(I_\tau(\lambda, r))$ is r - (τ^*, τ^*) -fspo. From Theorem 2.4.(2),

$$f(I_\tau(\lambda, r)) = SPI_{\tau^*}(f(I_\tau(\lambda, r), r)) \leq SPI_{\tau^*}(f(\lambda), r).$$

(2) \implies (1). For all $\lambda \in I^X$, $r \in I_0$, with $\tau(\lambda) \geq r$ we have $I_\tau(\lambda, r) = \lambda$. From (2), we have

$$f(\lambda) = f(I_\tau(\lambda, r)) \leq SPI_{\tau^*}(f(\lambda), r) \leq f(\lambda).$$

Then, $f(\lambda) = SPI_{\tau^*}(f(\lambda), r)$, and by Theorem 2.4.(2) we have that $f(\lambda)$ is an r -fspo set in Y . Therefore f is a fuzzy strongly preopen mapping.

(2) \implies (3). For all $\mu \in I^Y$, $r \in I_0$, by (2) we have

$$f(I_\tau(f^{-1}(\mu), r)) \leq SPI_{\tau^*}(ff^{-1}(\mu), r) \leq SPI_{\tau^*}(\mu, r).$$

It implies that $I_\tau(f^{-1}(\mu), r) \leq f^{-1}f(I_\tau(f^{-1}(\mu), r)) \leq f^{-1}(SPI_{\tau^*}(\mu, r))$.

(3) \implies (2). For all $\lambda \in I^X$, $r \in I_0$, by (3) we have

$$I_\tau(\lambda, r) \leq I_\tau(f^{-1}(f(\lambda)), r) \leq f^{-1}(SPI_{\tau^*}(f(\lambda), r)).$$

It implies that $f(I_\tau(\lambda, r)) \leq f(f^{-1}(SPI_{\tau^*}(f(\lambda), r))) \leq SPI_{\tau^*}(f(\lambda), r)$. ■

THEOREM 4.2. *Let $f: (X, \tau) \rightarrow (Y, \tau^*)$ be a mapping. Then the following statements are equivalent:*

- (1) f is a fuzzy strongly preclosed mapping.
- (2) $SPC_{\tau^*}(f(\lambda), r) \leq f(C_\tau(\lambda, r))$ for each $\lambda \in I^X$, $r \in I_0$.

Proof. Similar to the proof of Theorem 4.1. ■

THEOREM 4.3. *Let $f: (X, \tau) \rightarrow (Y, \tau^*)$ be a bijective mapping. Then the following statements are equivalent:*

- (1) f is a fuzzy strongly preclosed mapping.
- (2) $f^{-1}(SPC_{\tau^*}(\mu, r)) \leq C_\tau(f^{-1}(\mu), r)$ for each $\mu \in I^Y$, $r \in I_0$.

Proof. (1) \implies (2). For all $\mu \in I^Y$, $r \in I_0$, since $\tau(\bar{1} - C_\tau(f^{-1}(\mu), r)) \geq r$, from (1) we have $f(C_\tau(f^{-1}(\mu), r))$ is an r -fspc set in Y . Then by Theorem 2.4.(3) we have

$$f(C_\tau(f^{-1}(\mu), r)) = SPC_{\tau^*}(f(C_\tau(f^{-1}(\mu), r)), r) \geq SPC_{\tau^*}(ff^{-1}(\mu), r).$$

By the surjectivity of f we have

$$f(C_\tau(f^{-1}(\mu), r)) \geq SPC_{\tau^*}(\mu, r).$$

Also, by injectivity of f we have

$$f^{-1}(SPC_{\tau}(\mu, r)) \leq f^{-1}(f(C_{\tau}(f^{-1}(\mu), r))) = C_{\tau}(f^{-1}(\mu), r).$$

(2) \implies (1). For all $\lambda \in I^X$, $r \in I_0$ with $\tau(\bar{1} - \lambda) \geq r$, from (2) we have

$$f^{-1}(SPC_{\tau^*}(f(\lambda), r)) \leq C_{\tau}(f^{-1}(f(\lambda)), r).$$

By the injectivity of f we have

$$f^{-1}(SPC_{\tau^*}(f(\lambda), r)) \leq C_{\tau}(\lambda, r) = \lambda$$

and by surjectivity of f we have

$$SPC_{\tau^*}(f(\lambda), r) \leq f(\lambda) \leq SPC_{\tau^*}(f(\lambda), r).$$

Then $f(\lambda) = SPC_{\tau^*}(f(\lambda), r)$. Hence, $f(\lambda)$ is an r -fspc set in Y . So, f is a fuzzy strongly preclosed mapping. ■

THEOREM 4.4. *Let $f: (X, \tau) \rightarrow (Y, \tau^*)$ be a mapping. Then f is fuzzy strongly preopen iff for each $\nu \in I^Y$ and each $\lambda \in I^X$, $r \in I_0$ with $\tau(\bar{1} - \lambda) \geq r$, when $f^{-1}(\nu) \leq \lambda$, there exists an r -fspc set μ in Y such that $\nu \leq \mu$ and $f^{-1}(\mu) \leq \lambda$.*

Proof. Suppose that f is a fuzzy strongly preopen mapping, $\nu \in I^Y$ and $\lambda \in I^X$, $r \in I_0$ with $\tau(\bar{1} - \lambda) \geq r$ such that $f^{-1}(\nu) \leq \lambda$. Then, $f(\bar{1} - \lambda) \leq f(f^{-1}(\bar{1} - \nu)) \leq \bar{1} - \nu$. Since $\tau(\bar{1} - \lambda) \geq r$ and f is a fuzzy strongly preopen mapping, then $f(\bar{1} - \lambda)$ is an r -fspo set in Y . So,

$$\begin{aligned} f(\bar{1} - \lambda) &= SPI_{\tau^*}(f(\bar{1} - \lambda), r) \leq SPI_{\tau^*}(\bar{1} - \nu, r), \\ \bar{1} - \lambda &\leq f^{-1}(f(\bar{1} - \lambda)) \leq f^{-1}(SPI_{\tau^*}(\bar{1} - \nu, r)), \\ \bar{1} - f^{-1}(SPI_{\tau^*}(\bar{1} - \nu, r)) &= f^{-1}(\bar{1} - SPI_{\tau^*}(\bar{1} - \nu, r)) \leq \lambda, \\ f^{-1}(SPC_{\tau^*}(\nu, r)) &\leq \lambda. \end{aligned}$$

Let $\mu = SPC_{\tau^*}(\nu, r)$. Then, μ is an r -fspc set in Y , $\nu \leq SPC_{\tau^*}(\nu, r) = \mu$ and $f^{-1}(\mu) \leq \lambda$.

Conversely, for all $\eta \in I^X$, $r \in I_0$ with $\tau(\eta) \geq r$, take $\lambda = \bar{1} - \eta \in I^X$ and $\nu = \bar{1} - f(\eta) \in I^Y$. Then, $\tau(\bar{1} - \lambda) \geq r$ and since $\eta \leq f^{-1}f(\eta)$ we have

$$\bar{1} - f^{-1}(f(\eta)) = f^{-1}(\bar{1} - f(\eta)) \leq \bar{1} - \eta,$$

which implies that $f^{-1}(\nu) \leq \lambda$. Then there exists an r -fspc set in Y such that $\nu = \bar{1} - f(\eta) \leq \mu$ and $f^{-1}(\mu) \leq \bar{1} - \eta$. Then, $\bar{1} - \mu = f(\eta)$. But μ is an r -fspc set in Y and so $\bar{1} - \mu = f(\eta)$ is an r -fspo set in Y . Hence, f is a fuzzy strongly preopen mapping. ■

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(received 25.09.2000, in revised form 06.06.2003)

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