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AN L_p ESTIMATE FOR THE DIFFERENCE OF DERIVATIVES OF SPECTRAL EXPANSIONS ARISING BY ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

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Abstract. We prove the estimate

$$\|\sigma'_\mu(x,f) - \tilde{\sigma}'_\mu(x,f)\|_{L_p(G)} \le C\|f\|_{BV(G)} \cdot \mu^{1-1/p},$$

where $2 \le p < +\infty$, and $\sigma_\mu(x,f), \tilde{\sigma}_\mu(x,f)$ are the partial sums of spectral expansions of a function $f(x) \in BV(G)$, corresponding to arbitrary non-negative self-adjoint extensions of the operators $\mathcal{L}u = -u'' + q(x)u$, $\tilde{\mathcal{L}}u = -u'' + \tilde{q}(x)u$ $(x \in G)$ respectively; the operators are defined on an arbitrary bounded interval $G \subset \mathbb{R}$.

1. Introduction

Let $G = (a,b)$ be an arbitrary bounded interval, and let the operators

$$\mathcal{L}u = -u'' + q(x)u, \qquad \tilde{\mathcal{L}}u = -u'' + \tilde{q}(x)u \tag{1}$$

be defined on G, with potentials $q(x), \tilde{q}(x) \in L_s(G)$, $1 < s \le 2$. Denote by $L, \tilde{L}$ arbitrary non-negative self-adjoint extensions, with discrete spectrum, of the operators (1) respectively (see §17, [1]). Let $\{u_n(x)\}_{n=1}^\infty, \{\tilde{u}_n(x)\}_{n=1}^\infty$ be complete (in $L_2(G)$) and orthonormal systems of eigenfunctions of those extensions, and $\{\lambda_n\}_{n=1}^\infty, \{\tilde{\lambda}_n\}_{n=1}^\infty$ the corresponding systems of non-negative eigenvalues, enumerated in non-decreasig order. If $f(x) \in L_1(G)$ and $\mu \ge 2$, we can form the partial sums of order μ:

$$\sigma_\mu(x,f) = \sum_{\sqrt{\lambda_n}<\mu} f_n u_n(x), \qquad \tilde{\sigma}_\mu(x,f) = \sum_{\sqrt{\tilde{\lambda}_n}<\mu} \tilde{f}_n \tilde{u}_n(x),$$

where $f_n = \int_a^b f(x)u_n(x)\,dx$, $\tilde{f}_n = \int_a^b f(x)\tilde{u}_n(x)\,dx$. Let $AC(G)$ be the set of absolutely continuous functions on the closed interval $\overline{G}$. Denote by $BV(G)$ the Banach space of functions having bounded variation on $\overline{G}$, with the norm $\|f\|_{BV(G)} = \sup_{x\in\overline{G}}|f(x)| + V_a^b(f)$, where $V_a^b(f)$ stands for the total variation of $f(x)$ on $\overline{G}$.

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The problem of behavior of function $\sigma_\mu(x,f)$ (and its derivatives) on subsets of $\overline{G}$, as $\mu\to+\infty$, is the classical one. One of the most fruitful approaches to the problem is so-called "equiconvergence approach": one studies the behavior of the difference $\sigma_\mu(x,f)-S_\mu(x,f)$, as $\mu\to+\infty$, where $S_\mu(x,f)$ is the corresponding partial sum of the trigonometrical Fourier series of function f (for a review see [2]). It seems that the first results concerning the equiconvergence rate estimates, in the case of arbitrary self-adjoint Sturm-Liouville operators, were obtained by V.A. Il'in and I. Joo in [3]. They obtained the following estimate:

If $q(x),\tilde q(x)\in L_s(G)$ $(s>1)$, $f(x)\in AC(G)$, and $K\subset G$ is an arbitrary compact set, then there exists a constant $C(K,f)>0$ such that

$$\max_{x\in K}|\sigma_\mu(x,f)-\tilde\sigma_\mu(x,f)|\le C(K,f)\cdot\frac{1}{\mu},\qquad \mu\ge 2; \tag{2}$$

$C(K,f)$ does not depend on μ. The estimate is exact in order with respect to μ.

In order to "globalize" the estimate (2), I.S. Lomov has considered the L_p metric instead of the uniform one; in paper [4] he proved the following assertion: If $q(x),\tilde q(x)\in L_s(G)$ $(s>1)$, $f(x)\in BV(G)$, and $2\le p<+\infty$, then the estimate

$$\|\sigma_\mu(x,f)-\tilde\sigma_\mu(x,f)\|_{L_p(G)}\le C\|f\|_{BV(G)}\cdot\frac{1}{\mu^{1/p}},\qquad \mu\ge 3, \tag{3}$$

holds, where $C>0$ does not depend on f and μ. (Note that in earlier paper [5] Lomov obtained estimate (3) with $\mu^{-1/p}\ln\mu$ instead of $\mu^{-1/p}$.)

A local uniform estimate for the difference of the first derivatives $\sigma'_\mu(x,f)$, $\tilde\sigma'_\mu(x,f)$ was obtained by I. Joo and N. Lažetić in paper [6]. They proved: If $q(x)$ and $\tilde q(x)$ belong to $L_s(G)$ $(1<s\le 2)$, $f(x)\in AC(G)$, and $K\subset G$ is an arbitrary compact set, then the estimate

$$\max_{x\in K}|\sigma'_\mu(x,f)-\tilde\sigma'_\mu(x,f)|\le C(K,f),\qquad \mu\ge 2, \tag{4}$$

holds, where $C(K,f)>0$ is independent of μ. This estimate is exact in order with respect to the spectral parameter μ.

Recently, the estimate (4) has been extendend on the set $BV(G)$. Namely, the authors of this paper have proved ([7]) that for every function $f(x)\in BV(G)$ and every compact set $K\subset G$ the following estimate is valid:

$$\max_{x\in K}|\sigma'_\mu(x,f)-\tilde\sigma'_\mu(x,f)|\le C(K)\|f\|_{BV(G)},\qquad \mu\ge 2. \tag{5}$$

It is supposed that $q(x),\tilde q(x)\in L_s(G)(s>1)$.

In this paper we propose an L_p estimate for the difference mentioned above. That estimate "globalizes" (5), and shows how the estimate (3) is affected by the operation of differentiation (compare with estimates (8)-(9) below). Hence, our result is the following assertion.

Theorem. *Suppose* $q(x),\tilde q(x)\in L_s(G)(1<s\le 2), f(x)\in BV(G)$, $p\in[2,+\infty)$, *and* $\mu\ge 2$. *There exists a constant* $C>0$, *independent of* f *and* μ, *such that the following estimate holds*:

$$\|\sigma'_\mu(x,f)-\tilde\sigma'_\mu(x,f)\|_{L_p(G)}\le C\|f\|_{BV(G)}\cdot\mu^{1-1/p}. \tag{6}$$

2. Auxiliary results

Proof of the theorem is based on estimate (5). But we will also use a variety of known results listed below.

Let $q(x) \in L_1(G)$. Then for systems of eigenfunctions and eigenvalues of an arbitrary non-negative self-adjoint extension L of the operator $\mathcal{L}$ the following estimates are valid:

$$\sum_{|\sqrt{\lambda_n}-\mu|<1} 1 \le A, \qquad \mu > 0, \tag{7}$$

where $A > 0$ does not depend on μ (see [8] and [9]);

$$\sup_{x\in G} |u_n(x)| \le C(G), \tag{8}$$

where $C(G) > 0$ is independent of $n \in \mathbb{N}$ ([8]);

$$\sup_{x\in G} |u_n'(x)| \le C_1(G)(\sqrt{\lambda_n}+1), \tag{9}$$

with $C_1(G) > 0$ non-depending on $n \in \mathbb{N}$ ([10]).

If $f(x) \in BV(G)$, then for its Fourier coefficients f_n (with respect to the system $\{u_n(x)\}_{n=1}^{\infty}$) the estimate

$$|f_n| \le \frac{C}{\sqrt{\lambda_n}} \cdot \|f\|_{BV(G)} \tag{10}$$

holds, where $C > 0$ does not depend on $n \in \mathbb{N}$ (see [5]).

We will also use so-called "mean value formula" for the derivatives $u_n'(x)$ ([10]): If $x \in G$ and $t > 0$ are such that $x \pm t \in G$, then

$$u_n'(x+t) - u_n'(x-t) = -2\sqrt{\lambda_n}u_n(x)\sin\sqrt{\lambda_n}t + \\ + \int_{x-t}^{x+t} q(\xi)u_n(\xi)\cos\sqrt{\lambda_n}(|x-\xi|-t)\,d\xi. \tag{11}$$

(Note that a function $u_\lambda(x)$ is called an eigenfunction corresponding to an eigenvalue λ of the operator L if $u_\lambda(x), u_\lambda'(x) \in AC(G)$ and the equality

$$-u_\lambda''(x) + q(x)u_\lambda(x) = \lambda u_\lambda(x)$$

holds a.e. on G.)

Finally, recall the "second part" of the known Riesz theorem ([11]): Let $\{v_n(x)\}_{n=1}^{\infty}$ be an orthogonal system of functions defined on a bounded interval G, and such that $\sup_{x\in G}|v_n(x)| \le M$, where $M > 0$ is independent on $n \in \mathbb{N}$. If $1 < r \le 2$ and $1/r + 1/p = 1$, then for every sequence of (complex) numbers $\{g_n\}_{n=1}^{\infty}$, satisfying $\left(\sum_{n=1}^{\infty}|g_n|^r\right)^{1/r} < +\infty$, there exists a function $g(x) \in L_p(G)$ such that $g_n = \int_a^b g(y)\overline{v_n(y)}\,dy$ and

$$\|g\|_{L_p(G)} \le M^{2/r-1}\Big(\sum_{n=1}^{\infty}|g_n|^r\Big)^{1/r}. \tag{12}$$

Note that in proving the estimate (5) we have used all the results (7)–(12).

3. Proof of the theorem

The first step of the proof is the same as the one in the proof of Lemma 2 [5]. Let $K = [c, d] \subset G$ be an arbitrary fixed closed interval. Then we have

$$\|\sigma'_\mu(x,f) - \tilde{\sigma}'_\mu(x,f)\|^p_{L_p(G)} = \|(\cdot)\|^p_{L_p((a,c))} + \|(\cdot)\|^p_{L_p(K)} + \|(\cdot)\|^p_{L_p((d,b))}. \tag{13}$$

In estimating the members on the right-hand side, we will assume, with no loss of generality, that $\lambda_n \geq 1$ $(n \in \mathbb{N})$. (This assumption is based on the equation $-u''_n(x) + [q(x)+1]u_n(x) = (\lambda_n + 1)u_n(x)$.) Set $\mu_n \stackrel{\text{def}}{=} \sqrt{\lambda_n}$.

Let us consider the first member. Introducing a new variable $z = x + h$, with $h \in (0, (d-c)/2)$ fixed, we obtain

$$\|\sigma'_\mu(x,f) - \tilde{\sigma}'_\mu(x,f)\|_{L_p((a,c))} = \int_{K_1} |\sigma'_\mu(z-h,f) - \tilde{\sigma}'_\mu(z-h,f)|^p\, dz, \tag{14}$$

where $K_1 \stackrel{\text{def}}{=} [a+h, c+h] \subset G$. By the mean value formula (11), we can write

$$\begin{aligned}\sigma'_\mu(z-h,f) = \sum_{\mu_n<\mu} f_n u'_n(z+h) + \sum_{\mu_n<\mu} 2\mu_n f_n u_n(z)\sin\mu_n h -\\ - \sum_{\mu_n<\mu} f_n \int_z^{z+h} q(\xi)u_n(\xi)\cos\mu_n(\xi - z - h)\, d\xi +\\ + \sum_{\mu_n<\mu} f_n \int_{z-h}^{z} q(\xi)u_n(\xi)\cos\mu_n(z-\xi-h)\, d\xi.\end{aligned}$$

Analogous equality can be written for $\tilde{\sigma}'_\mu(z-h,f)$. Therefore, the following equality holds on K_1:

$$\begin{aligned}\sigma'_\mu(z-h,f) - \tilde{\sigma}'_\mu(z-h,f) = \sum_{\mu_n<\mu} f_n u'_n(z+h) - \sum_{\tilde{\mu}_n<\mu} \tilde{f}_n \tilde{u}'_n(z+h) +\\ + \sum_{\mu_n<\mu} 2\mu_n f_n u_n(z)\sin\mu_n h - \sum_{\tilde{\mu}_n<\mu} 2\tilde{\mu}_n \tilde{f}_n \tilde{u}'_n(z)\sin\tilde{\mu}_n h -\\ - \sum_{\mu_n<\mu} f_n \int_z^{z+h} q(\xi)u_n(\xi)\cos\mu_n(\xi - x - h)\, d\xi +\\ + \sum_{\tilde{\mu}_n<\mu} \tilde{f}_n \int_z^{z+h} \tilde{q}(\xi)\tilde{u}_n(\xi)\cos\tilde{\mu}_n(\xi - z - h)\, d\xi +\\ + \sum_{\mu_n<\mu} f_n \int_{z-h}^{z} q(\xi)u_n(\xi)\cos\mu_n(z-\xi-h)\, d\xi -\\ - \sum_{\tilde{\mu}_n<\mu} \tilde{f}_n \int_{z-h}^{z} \tilde{q}(\xi)\tilde{u}_n(\xi)\cos\tilde{\mu}_n(z-\xi-h)\, d\xi.\end{aligned}$$

That is why we have the inequality

$$\|\sigma'_\mu(z-h,f) - \tilde{\sigma}'_\mu(z-h,f)\|^p_{L_p(K_1)} \leq C_p \|\sigma'_\mu(z+h,f) - \tilde{\sigma}'_\mu(z+h,f)\|^p_{L_p(K_1)} +$$

$$+ C_p \Big\| \sum_{\mu_n<\mu} 2\mu_n f_n u_n(z) \sin \mu_n h \Big\|^p_{L_p(K_1)} +$$
$$+ C_p \Big\| \sum_{\tilde\mu_n<\mu} 2\tilde\mu_n \tilde f_m \tilde u_n(z) \sin \tilde\mu_n h \Big\|^p_{L_p(K_1)} +$$
$$+ C_p \int_{K_1} \Big| \sum_{\mu_n<\mu} f_n \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - z - h)\, d\xi \Big|^p dz +$$
$$+ C_p \int_{K_1} \Big| \sum_{\tilde\mu_n<\mu} \tilde f_n \int_z^{z+h} \tilde q(\xi) \tilde u_n(\xi) \cos \tilde\mu_n(\xi - x - h)\, d\xi \Big|^p dz +$$
$$+ C_p \int_{K_1} \Big| \sum_{\mu_n<\mu} f_n \int_{z-h}^{z} q(\xi) u_n(\xi) \cos \mu_n(x - \xi - h)\, d\xi \Big|^p dz +$$
$$+ C_p \int_{K_1} \Big| \sum_{\tilde\mu_n<\mu} \tilde f_n \int_{z-h}^{z} \tilde q(\xi) \tilde u_n(\xi) \cos \tilde\mu_n(x - \xi - h)\, d\xi \Big| dz. \tag{15}$$

Here and further, we denote by C_p not necessarily equal positive constants.

In order to estimate the first member on the right-hand side of (15), we will use the estimate (5). Having in mind that $z + h \in K_2$ if $z \in K_1$, where $K_2 \stackrel{\text{def}}{=} [a + 2h, c + 2h] \subset G$, we have the inequalities

$$\|\sigma'_\mu(z+h,f) - \tilde\sigma'_\mu(z+h,f)\|^p_{L_p(K_1)} \le (c-a) C(K_2)^p \|f\|^p_{BV(G)}$$
$$\le C_p \|f\|_{BV(G)} \cdot \mu^{(1-1/p)p}. \tag{16}$$

The next two members have the same "structure", and they will be estimated by the Riesz theorem. First we introduce a new function:

$$g(z) = \sum_{\mu_n<\mu} (2\mu_n f_n \sin \mu_n h) u_n(z), \qquad z \in G.$$

It belongs to $L_p(G) \subset L_2(G)$, and its Fourier coefficients (with respect to the system $\{u_n(z)\}_{n=1}^\infty$) are given by

$$g_n = \begin{cases} 2\mu_n f_n \sin \mu_n h & \text{if} \quad \mu_n < \mu, \\ 0 & \text{if} \quad \mu_n \ge \mu. \end{cases}$$

Let $r \in (1,2]$ be a number such that $1/p + 1/r = 1$. By estimates (7) and (10), we obtain

$$\Big(\sum_{n=1}^\infty |g_n|^r\Big)^{1/r} \le C\|f\|_{BV(G)} \Big(\sum_{\mu_n<\mu} 1\Big)^{1/r}$$
$$\le C\|f\|_{BV(G)} \Big(\sum_{k=1}^{[\mu]} \Big(\sum_{k\le\mu_n<k+1} 1\Big)\Big)^{1/r} \le 2^{1/r} C A^{1/r} \|f\|_{BV(G)} \cdot \mu^{1/r}. \tag{17}$$

Hence, we can use the second part of the Riesz theorem: from estimate (12) it follows that the inequalities

$$\|g\|_{L_p(K_1)} \le \|g\|_{L_p(G)} \le (C(G))^{2/r-1} \Big(\sum_{n=1}^\infty |g_n|^r\Big)^{1/r}$$

are valid. That is why we can conclude, by (17), that for the second member it holds:

$$\Big\|\sum_{\mu_n<\mu} 2\mu_n f_n u_n(z)\sin\mu_n h\Big\|^p_{L_p(K_1)} \le C_p\|f\|^p_{BV(G)}\cdot\mu^{(1-1/p)p}. \tag{18}$$

The same estimate holds for the third member:

$$\Big\|\sum_{\tilde\mu_n<\mu} 2\tilde\mu_n \tilde f_n \tilde u_n(z)\sin\tilde\mu_n h\Big\|^p_{L_p(K_1)} \le \tilde C_p\|f\|^p_{BV(G)}\cdot\mu^{(1-1/p)p}. \tag{19}$$

In the case of the fourth member, using estimates (7)-(8), (10), and the Hölder inequality, we obtain

$$\int_{K_1}\Big|\sum_{\mu_n<\mu} f_n\int_z^{z+h} q(\xi)u_n(\xi)\cos\mu_n(\xi-x-h)\,d\xi\Big|^p dz \le$$

$$\int_{K_1}\Big(\sum_{\mu_n<\mu}|\mu_n f_n|\Big|\frac{1}{\mu_n}\int_z^{z+h} q(\xi)u_n(\xi)\cos\mu_n(\xi-x-h)\,d\xi\Big|\Big)^p \le$$

$$\Big(\sum_{\mu_n<\mu}|\mu_n f_n|^r\Big)^{p/r}\int_{K_1}\Big(\sum_{\mu_n<\mu}\Big|\frac{1}{\mu_n}\int_z^{z+h} q(\xi)u_n(\xi)\cos\mu_n(\xi-x-h)\,d\xi\Big|^p\Big)dz \le$$

$$C_p\|f\|^p_{BV(G)}\cdot\mu^{(1-1/p)p}\|q\|^p_{L_1(G)}(c-a)\Big(\sum_{\mu_n<\mu}\frac{1}{\mu_n^p}\Big) \le C_p\|f\|^p_{BV(G)}\cdot\mu^{(1-1/p)p}.$$

Here $1/p+1/r=1$. Also we have in mind that

$$\sum_{\mu_n<\mu}\frac{1}{\mu_n^p} \le \sum_{k=1}^{\infty}\Big(\sum_{k\le\mu_n<k+1}\frac{1}{\mu_n^p}\Big) \le A\sum_{k=1}^{\infty}\frac{1}{k^p}.$$

Therefore, the following estimate holds:

$$\int_{K_1}\Big|\sum_{\mu_n<\mu} f_n\int_z^{z+h} q(\xi)u_n(\xi)\cos\mu_n(\xi-x-h)\,d\xi\Big|^p dz \le$$

$$\le C_p\|f\|^p_{BV(G)}\cdot\mu^{(1-1/p)p}. \tag{20}$$

The estimates of the same form, with possibly different constants C_p, are valid for the last three members on the right-hand side of (15). So we get, by (14)–(16) and (18)–(20), the final estimate

$$\|\sigma'_\mu(x,f)-\tilde\sigma'_\mu(x,f)\|^p_{L_p((a,c))} \le C_p\|f\|^p_{BV(G)}\cdot\mu^{(1-1/p)p}. \tag{21}$$

Using the analogous argument, one can prove the estimate

$$\|\sigma'_\mu(x,f)-\tilde\sigma'_\mu(x,f)\|^p_{L_p((d,b))} \le C_p\|f\|_{BV(G)}\cdot\mu^{(1-1/p)p}. \tag{22}$$

Finally, by estimate (5), we obtain

$$\|\sigma'_\mu(x,f)-\tilde\sigma'_\mu(x,f)\|^p_{L_p(K)} \le (b-a)C(K)^p\|f\|^p_{BV(G)}\cdot\mu^{(1-1/p)p}. \tag{23}$$

Now, the estimate (6) follows from (13) and (21)–(23). The theorem is proved. ■

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