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# An explicit separate solution for the Tzitzéica equation

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**Abstract**

In this note we present an explicit solution of the Tzitzéica equation in terms of the $P$-Weierstrass function, representing non-scattering double waves. This solution is obtained by combining a "nonclassical" method and certain separation of variables technique.

# 1 Introduction

The Tzitzéica equation (TE) originates from the work of G. Tzitzéica [1] in the study of integrable surfaces. He was the first to formulate the linear differential equations for which the compatibility conditions give us the original equation. The complete integrability has been established in the study of the two-dimensionnal Toda lattice by A. Mikhailov [2]. It is known [3] that the Klein-Gordon equations $u_{xt} = f(u)$ allow contact Auto-Bäcklund Transformation (Auto-BT) of the first order if and only if $f(u) = e^u, \sin u$ or $f(u)$ =constant. Boldin, Safin and Sharipov [4] obtained second order Auto-BT for the Tzitzéica equation. However this Auto-BT have a strongly non symmetric structure and therefore they are integrated with difficulty if the starting solution is non trivial. A new Auto-BT of the TE was proposed in [5] by the conditional symmetry approach. This Auto-BT is composed of a second order ordinary differential equation which belongs to the fifth class of Gambier and a first order partial differential equation. In this note we prove that by introducing new and separate functions of the independent variables $X_1(x^1)$ and $X_2(x^2)$, and acting on the resulting form of the TE by annihilators, we are able to find an explicit solution for such equation. This solution is obtained by combining a "nonclassical" method [6] and certain separation of variables technique first developed in [7].

# 2 The Tzitzéica equation subjected to differential constraints of first order and separation of variables

Let us consider the Tzitzéica equation

$$u_{x^1x^1} - u_{x^2x^2} = e^u - e^{-2u}. \tag{1}$$

We look for a class of real solutions of (1) invariant under only one vector field

$$Z = X_1\partial_{X_1} - X_2\partial_{X_2}, \tag{2}$$

where $X_1$ and $X_2$ are some real-valued functions of real independent variables $x^1$ and $x^2$, respectively. We assume the existence of functions $\xi$ and $\eta$ of $X_1$ and $X_2$, respectively, such that

$$\frac{dX_1}{dx^1} = \epsilon|\xi(X_1)|^{\frac{1}{2}},\ \frac{dX_2}{dx^2} = \epsilon|\eta(X_2)|^{\frac{1}{2}}, \quad \epsilon = \pm 1. \tag{3}$$

This means that the functions $X_1$ and $X_2$ are assumed to be locally piecewise monotonic functions. Making use of the fact that the function $\varphi(s)$, where $s = X_1X_2$, is the basic invariant of the vector field $Z$, we reduce equation (1) to the following differential equation

$$\varphi''(s)\left(\xi {X_2}^2 - \eta {X_1}^2\right) + \frac{\epsilon}{2}\varphi'(s)\left(|\xi|^{-\frac{1}{2}}\xi' X_2 - |\eta|^{-\frac{1}{2}}\eta' X_1\right) = e^{\varphi(s)} - e^{-2\varphi(s)}, \tag{4}$$

where, to simplify the formulae, we use the notation

$$\frac{d\varphi}{ds} = \varphi'(s), \quad \frac{dX_1}{dx^1} = \dot{X}_1, \quad \frac{dX_2}{dx^2} = \dot{X}_2, \quad \frac{d\xi}{dX_1} = \xi', \quad \frac{d\eta}{dX_2} = \eta'.$$

We operate with the generator $Z$ on equation (4)

$$\varphi''(s) + \frac{1}{2s}\left[Z^*(\ln|2\xi{X_1}^{-2} - \xi' {X_1}^{-1} + 2\eta{X_2}^{-2} - \eta' X_2^{-1}|) + 2\right]\varphi'(s) = 0, \tag{5}$$

where the adjoint generator

$$Z^* = X_1\partial_{X_1} + X_2\partial_{X_2}, \tag{6}$$

which commutes with $Z$ (i.e $[Z, Z^*] = 0$), is introduced. Equation (5) implies that the function

$$Z^*(\ln|F|) = Z^*(\ln|2\xi{X_1}^{-2} - \xi' {X_1}^{-1} + 2\eta{X_2}^{-2} - \eta' {X_2}^{-1}|),$$

is a function of the basic invariant $s$ only, i.e

$$ZZ^*\ln|F| = 0. \tag{7}$$

This means that there exist two functions $h_1$ and $h_2$ of one variable $s = X_1X_2$ and $r = X_1/X_2$, respectively, such that

$$F = h_1(s)h_2(r). \tag{8}$$

We substitute (8) into (7) and differentiate the obtained equation twice, with respect to $X_1$ and $X_2$. After the separation of variables $s$ and $r$ we get a system of Cauchy-Euler equations

$$s^2h_1'' + sh_1' - \lambda h_1 = 0, \quad r^2h_2'' + rh_2' - \lambda h_2 = 0, \tag{9}$$

where $\lambda$ is a constant of separation. The general integral of the system (9) has the form

$$h_1(s) = C_1s^{\sqrt{\lambda}} + C_2s^{-\sqrt{\lambda}}, \quad h_2(r) = D_1r^{\sqrt{\lambda}} + D_2r^{-\sqrt{\lambda}}, \tag{10}$$

where $C_i, D_i, i = 1,2$ are arbitrary constants. Note that the solutions of (10) depend on the sign of $\lambda$. This fact leads to three different situations (i.e $\lambda > 0, \lambda = 0, \lambda < 0$). It is easy to show that explicit separable solutions of TE (1) can be derived in the case when $\lambda = 0$. In this case the solutions of (9) have the form

$$h_1(s) = C_1 + C_2 \ln|s|, \quad h_2(r) = D_1 + D_2 \ln|r|. \tag{11}$$

Substitution of (11) and (8) into (7), followed by separation of variables with respect to $X_1$ and $X_2$, leads to the ODEs which can be easily integrated in terms of the inverse P-Weierstrass function

$$\begin{aligned} x^1 &= (\frac{12}{C_2D_2})^{1/3}\mathcal{P}^{-1}\left(-(\frac{C_2D_2}{12})^{1/3}(\ln|X_1| + \frac{C_1D_2 + C_2D_1}{2C_2D_2}), g_2, g_3\right), \\ x^2 &= -(\frac{12}{C_2D_2})^{1/3}\mathcal{P}^{-1}\left((\frac{C_2D_2}{12})^{1/3}(\ln|X_2| + \frac{C_1D_2 - C_2D_1}{2C_2D_2}), g_2, g_3^{'}\right), \end{aligned} \tag{12}$$

where

$$g_2 = (\frac{12}{C_2D_2})^{1/3}\left(\frac{(C_1D_2 - C_2D_1)^2}{4C_2D_2} + \mu\right),$$

$$g_3 = \frac{(C_1D_2 + C_2D_1)(\mu - C_1D_1)}{2C_2D_2} + \frac{(C_1D_2 + C_2D_1)^3}{12(C_2D_2)^2} - \xi_0,$$

and

$$g_3^{'} = \frac{\mu(C_2D_1 - C_1D_2)}{2C_2D_2} - \frac{(C_1D_2 - C_2D_1)^3}{12(C_2D_2)^2} - \eta_0, \text{ where } \mu, \xi_0, \eta_0 \in \mathbb{R}.$$

Substituting (11) and (12) into (5) and integrating the obtained equation, we have

$$\varphi(s) = \frac{E}{C_2}\ln|C_1 + C_2\ln|s|| + \varphi_0. \tag{13}$$

Next, substituting (12) and (13) into (4) we obtain

$$\frac{E}{h_1(s)^2}\left[-\frac{D_2}{6C_2}h_1(s)^3 + G - C_2(\xi_0 - \eta_0)\right] = e^{\varphi(s)} - e^{-2\varphi(s)}. \tag{14}$$

Using (13) we find $s$ as a function of $\varphi$ and next substituting this result into (14), we obtain

$$Ee^{-2\frac{C_2}{E}(\varphi-\varphi_0)}\left[-\frac{D_2}{6C_2}e^{3\frac{C_2}{E}(\varphi-\varphi_0)} + G - C_2(\xi_0 - \eta_0)\right] = e^{\varphi} - e^{-2\varphi}, \tag{15}$$

where

$$G = \frac{C_1^3 D_2}{6C_2} - \frac{C_1^2 D_1}{2} + C_1\mu.$$

The TE (1) can be obtained from (15) through the appropriate selection of constants, namely

$$C_2 = E = 2b^3,\ \varphi_0 = 0,\ D_2 = -6,\ \xi_0 = \eta_0, \tag{16}$$

and

$$C_1{}^3 + C_1^2 b^3 D_1 - 2C_1 b^3 \mu = 1. \tag{17}$$

Hence, (12) becomes

$$\begin{aligned} x^1 &= -\frac{6^{1/3}}{b}\mathcal{P}^{-1}\left(b(\ln|X_1| + \frac{6C_1 - 2b^3 D_1}{24b^3}), g_2, g_3\right), \\ x^2 &= \frac{6^{1/3}}{b}\mathcal{P}^{-1}\left(b(\ln|X_2| + \frac{6C_1 + 2b^3 D_1}{24b^3}), g_2, g_3^{'}\right). \end{aligned} \tag{18}$$

Using (13) with the choice of constants satisfying (16) and (17) we obtain the explicit rank-2 solution of TE (1), representing the superposition of two traveling waves (rank-one solutions [8] expressed in term of the $\mathcal{P}$-Weierstrass functions)

$$u(x^1, x^2) = \ln|2b^2\left[\mathcal{P}(bx^1, g_2, g_3) - \mathcal{P}(bx^2, g_2, g_3^{'})\right]|, \tag{19}$$

where

$$g_3 = A - \frac{1}{8b^6},\ g_3^{'} = A + \frac{1}{8b^6},$$

and $A, g_2, b$ are arbitrary constants. After substituting (18) into (2) we obtain that the solution (19) is invariant under the vector field

$$\begin{aligned} Z &= [4\mathcal{P}^3\left(\frac{b}{6^{1/3}}x^1\right) - g_2\mathcal{P}\left(\frac{b}{6^{1/3}}x^1\right) - g_3]^{-1/2}\partial_{x^1} \\ &- [4\mathcal{P}^3\left(\frac{b}{6^{1/3}}x^2\right) - g_2\mathcal{P}\left(\frac{b}{6^{1/3}}x^2\right) - g_3^{'}]^{-1/2}\partial_{x^2}. \end{aligned} \tag{20}$$

Note that the vector field (20) generates a "nonclassical" symmetry since it can not be represented as a linear combination of the vector fields

$$z_1 = \partial_{x^1},\ z_2 = \partial_{x^2},\ z_3 = x^2\partial_{x^1} + x^1\partial_{x^2},$$

which span the classical symmetry algebra of TE (1).

The solution (19) is a reconstruction of the result obtained in [9] by postulating the form of the solution. It is worth underlining however that our approach (like other versions of symmetry reduction method) provides us with much more information about the properties of obtained solutions than the direct method or the prolongation technique.

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