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The structure of the Haar systems on locally compact groupoids

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Abstract. We shall prove a decomposition property of a Haar system on a locally compact groupoid. Using this decomposition, we shall establish the structure of the C^*-algebra associated to a locally compact groupoid whose associated equivalence relation is a closed set.

Key words and phrases. locally compact groupoid, Haar system, C^*-algebra.

1. Introduction

We shall use the definition of a topological groupoid given by J. Renault in [8]. For a groupoid G, $G^{(0)}$ will denote its unit space and $G^{(2)}$ the set of the composable pairs. Usually, elements of G will be denoted by letters as x, y, or z, and the elements of $G^{(0)}$ by letters as u, v, or w. The inverse map is written $x \to x^{-1}$ $[: G \to G]$ and the product map is written $(x, y) \to xy$ $\left[: G^{(2)} \to G\right]$. The range and the source maps from G to $G^{(0)}$ will be denoted respectively by r and d. The fibers of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets A, $B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G_B^A = r^{-1}(A) \cap d^{-1}(B)$. The reduction of G is $G|A = G_A^A$.

We shall assume that G admits a continuous Haar system $\left\{\nu^u, u \in G^{(0)}\right\}$. Consequently, $r, d : G \to G^{(0)}$ are open maps. We shall also assume that $(r, d)(G)$ is a closed subset of $G^{(0)} \times G^{(0)}$.

Let $\pi : G^{(0)} \to G^{(0)}/G$ ($u\tilde{\ }v <=> \exists\ x \in G$ such that $r(x) = u$ and $d(x) = v$) be the canonical projection. Let $[u] = \{v : v\tilde{\ }u\}$.

We shall prove that for each orbit $[u]$, there is a probability $\eta_{\pi(u)}$ supported on $[u]$ such that:

1) $d_*(\nu^v)\ \tilde{}\ \eta_{\pi(u)}$ for all $v \in [u]$

2) $\nu^u = \int \nu_{u,v} d\eta_{\pi(u)}(v)$

3) If μ is a quasi-invariant probability, then

$$\int \nu^u d\mu(u) = \int\int\int \nu_{u,v} d\eta_{\pi(w)}(u)\, d\ \eta_{\pi(w)}(v)\, \mu(w),$$

where $\{\nu_{u,v},\ u\ \tilde{}\ v\}$ is a system of measures resulted by applying Hahn Structure Theorem on the Haar measure $\left(\int \nu^u d\mu(u), \mu\right)$.

As a consequence of this decomposition of the Haar system, we shall establish the structure of the C^*-algebra of G with respect to the Haar system $\{\nu^u, u \in G^{(0)}\}$. If G is transitive it is well known that the C^*-algebra of G is isomorphic to $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$, where H is the isotropy group G_u^u at any unit $u \in G^{(0)}$, μ is an essentially unique measure on $G^{(0)}$, $C^*(H)$ denotes the group C^*-algebra of H, and $\mathcal{K}\left(L^2(\mu)\right)$ denotes the compact operators on $L^2(\mu)$ (see [6]). In the general case we shall describe the C^*-algebra of G using the C^*-algebras of the transitive components of G, whose structures are known.

Let $C^*(G)$ be the C^*-algebra of G with respect to the Haar system $\{\nu^u, u \in G^{(0)}\}$ and $\|\|$ be the C^*-norm. For each orbit $[u]$, let $C^*(G|[u])$ be the C^*-algebra of the locally compact groupoid transitive $G|[u]$ with respect to the Haar system $\{\nu^v, v \in [u]\}$ and $\|\|_{\pi(u)}$ be the C^*-norm of this algebra.

Let $f \in C_c(G)$. We shall prove that $\|f\| = \sup\limits_{[u]} \left\|f|_{G|[u]}\right\|_{\pi(u)}$. Consequently, $C^*(G)$ is isomorphic with the completion of

$$\left\{\left(f_{\pi(u)}\right)_{\pi(u) \in G^{(0)}/G} : (\exists)\ f \in C_c(G) \text{ such that } f|_{G|[u]} = f_{\pi(u)}\right\}$$

in the norm $\left\|\left(f_{\pi(u)}\right)_{\pi(u)}\right\| = \sup \left\|f_{\pi(u)}\right\|_{\pi(u)}$.

In order to prove the equality $\|f\| = \sup\limits_{[u]} \left\|f|_{G|[u]}\right\|_{\pi(u)}$, we shall show that every representation of $C_c(G)$ induced by a representation $\left(\mu, G^{(0)} * \mathcal{H}, L\right)$ of G

can be written $L = \int L_{\pi(u)} d\mu(u)$, where $L_{\pi(u)}$ is the representation induced by $\left(\eta_{\pi(u)}, \left(G^{(0)} * \mathcal{H}\right)|_{[u]}, L|_{G|[u]}\right)$ with $\eta_{\pi(u)}$ a probability measure equivalent to $d_*(\nu^u)$.

2. The decomposition of a Haar system on a locally compact groupoid

Let G be a locally compact second countable groupoid with a continuous Haar system, i.e. a family of positive Radon measures on G, $\left\{\nu^u,\, u \in G^{(0)}\right\}$, such that

1) For all $u \in G^{(0)}$, $supp(\nu^u) = G^u$.

2) For all $f : G \to \mathbf{C}$ continuous with compact support,

$$u \to \int f(x)\, d\nu^u(x) \quad \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous.

3) For all $f : G \to \mathbf{C}$ continuous with compact support, and all $x \in G$,

$$\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$$

Let μ_0 be a quasi-invariant probability for the Haar system (Definition 3.2/p. 23 [4]), and let ν_0 be the measure on G induced by μ_0. let $\lambda \in [\nu_0]$ be a symmetric probability $\left(\lambda = \lambda^{-1}\right)$. We set $\tilde{\lambda} = d_*(\lambda) = r_*(\lambda)$. Let $\lambda = \int \lambda^u d\tilde{\lambda}(u)$ be the r-decomposition of λ in the sense of Theorem 2.1/ p. 5 [4]. If we set $\nu = \int \nu^u d\tilde{\lambda}(u)$, then $\nu \tilde{}\, \nu_0 \tilde{}\, \lambda$. If P is a positive Borel function such that $P = \frac{d\nu}{d\lambda}$, then $P = \frac{d\nu^u}{d\lambda^u}$ for -a.a. $u \in G^{(0)}$.

$(G, [\lambda])$ is a measure groupoid in the sense of P. Hahn [4] and $(\nu, \tilde{\lambda})$ is a Haar measure for $(G, [\lambda])$ (Definition 3.11/ p. 39 [4]). Let $\mathcal{E} = (r, d)(G)$ be the associated equivalence relation on $G^{(0)}$ and set $\lambda' = (r, d)_*(\lambda)$. Let $\lambda = \int \lambda_{u,v} d\lambda'(u, v)$ be a decomposition of λ relative to (r, d). Applying Theorem 3.9/ pg.17 [4] to the groupoid $(\mathcal{E}, [\lambda'])$, we obtain

Lemma 1. *There is a conull Borel set $U_0' \subset G^{(0)}$ and a Borel function $q : G|U_0' \to \mathbf{R}_+^*$ such that*

1) λ' has the r-decomposition $\lambda' = \int \lambda'^u d\tilde{\lambda}(u)$ on $\mathcal{E}_0 = \mathcal{E}|U_0'$.

2) For all $f : \mathcal{E}_0 \to \mathbf{R}_+$ Borel and all $(u, v) \in \mathcal{E}_0$,

$$\int f((u, v)(s, t))\, q(s, t)\, d\lambda'^v(s, t) = \int f(s, t)\, q(s, t)\, d\lambda'^u(s, t)$$

3) $(u, v) \to \frac{q(u,v)}{q(v,u)}$ is a strict homomorphism of $\mathcal{E}_0$ into $\mathbf{R}_+^*$.

The following lemma is derived from Theorem 4.4/pg.23[4] converted to left invariance.

LEMMA 2. *There exists a conull Borel set $U_0 \subset G^{(0)}$ such that*
(i) $(u,v) \to \frac{q(u,v)}{q(v,u)}$ *is a strict homomorphism of* $\mathcal{E}_0 = \mathcal{E}|U_0$ *into* $\mathbf{R}_+^*$.
(ii) $y \to \frac{P(y)}{P(y^{-1})} = \Delta = \frac{d\nu}{d\nu^{-1}}$ *is a strict homomorphism of* $G_0 = G|U_0$ *into* $\mathbf{R}_+^*$.
If we define $\delta : G_0 \to \mathbf{R}_+^*$, *by*

$$\delta(y) = \frac{P(y)}{P(y^{-1})} \frac{q(d(y), r(y))}{q(r(y), d(y))} = \Delta(y) \frac{q(d(y), r(y))}{q(r(y), d(y))},$$

then δ *is a strict homomorphism.*
On G_0 *the integral* $f \to \int f(y) P(y) d\lambda(y)$ *has a* (r,d)*- decomposition*

$$\int_{\mathcal{E}_0} \int_{G_0} f(y) d\nu_{u,v}(y) q(u,v) d\lambda'(u,v)$$

with respect to λ' *on* $\mathcal{E}_0$ *such that:*
1) $\nu_{u,v}$ *is* σ*-finite measure supported on* G_v^u, *for all* $(u,v) \in \mathcal{E}_0$.
2) *For all* $f \geq 0$ *0 Borel on* G,

$$(u,v) \mapsto \int f(y) d\nu_{u.v}(y) \ [: \mathcal{E}_0 \to \overline{\mathbf{R}}]$$

is an extended real-valued Borel function.
3) *For all* $f \geq 0$ *Borel on* G,

$$\int f(xy) d\nu_{d(x),v}(y) = \int f(y) d\nu_{r(x),v}(y)$$

for all $x \in G_0$, $v \in G^{(0)}$ *such that* $(d(x), v), (r(x), v) \in \mathcal{E}_0$.
4) *For all* $f \geq 0$ *Borel on* G,

$$\delta(x) \int f(yx) d\nu_{u,r(x)}(y) = \int f(y) d\nu_{u,d(x)}(y)$$

for all $x \in G_0$, $u \in G^{(0)}$ *such that* $(u, r(x)), (u, d(x)) \in \mathcal{E}_0$.
Thus $\nu_{u,u}$ *is a left Haar measure on* G_u^u *and* $\delta|_{G_u^u}$ *is its modular function for all* $u \in U_0$.

As we have noted in [2], using a similar argument as in Theorem 3.4/p. 329[7], the relations in the preceding lemma can be extended to a saturated conull Z set :

LEMMA 3. *On* G_0 *the integral* $f \to \int f(y) P(y) d\lambda(y)$ *has a* (r,d)*- decomposition*

$$\int_{\mathcal{E}_0} \int_{G_0} f(y) d\nu_{u,v}(y) q(u,v) d\lambda'(u,v)$$

with respect to λ' *on* $\mathcal{E}_0$ *such that for some saturated conull* σ*-compact* $Z \subset G^{(0)}$ *we have:*
1) $\nu_{u,v} \neq 0$, *for all* $u, v \in Z$, *with* $u \tilde{} v$.
2) *For all* $f \geq 0$ *Borel on* G,

$$\int f(xy) d\nu_{d(x),v}(y) = \int f(y) d\nu_{r(x),v}(y)$$

for all $x \in G|Z$, $v \in Z$ *such that* $v \tilde{} r(x)$.

3) *For all* $f \geq 0$ *Borel on* G,

$$\delta(x) \int f(yx)\, d\nu_{u,r(x)}(y) = \int f(y)\, d\nu_{u,d(x)}(y)$$

for all $x \in G|Z$, $u \in Z$ *such that* $u\tilde{\ } r(x)$.

4) $\delta, \Delta : G|Z \to \mathbf{R}_+^*$ *are strict homomorphisms*

5) *For all* $f \geq 0$ *Borel on* G,

$$\int f\left(x^{-1}\right) \delta\left(x^{-1}\right) d\nu_{u,v}(x) = \int f(x)\, d\nu_{v,u}(x)$$

.for all $u, v \in Z$, *with* $u\tilde{\ }v$.

In what follows we shall assume that $(r,d)(G)$ is a closed subset of $G^{(0)} \times G^{(0)}$.

Let $\pi : G^{(0)} \to G^{(0)}/G$ ($u\tilde{\ }v <=> \exists\ x \in G$ such that $r(x) = u$ and $d(x) = v$) be the canonical projection.

We shall prove that for each orbit $[u]$, there is a probability $\eta_{\pi(u)}$ supported on $[u]$ such that:

1) $d_*(\nu^v)\ \tilde{\ }\ \eta_{\pi(u)}$ for all $v \in [u]$

2) $\nu^v = \int \nu_{v,w} d\eta_{\pi(u)}(w)$ for all $v \in [u]$, where $\left\{\nu_{u,v},\ u, v \in G^{(0)},\ u\ \tilde{\ }\ v\right\}$ is a system of measures with the properties in the preceding section.

¿From this will easily follow that every quasi-invariant measure for $\left\{\nu^u,\ u \in G^{(0)}\right\}$ is equivalent with $\int \eta_{\dot{u}} d\tilde{\mu}(\dot{u})$, for some probability measure $\tilde{\mu}$ on $G^{(0)}/G$.

π is a continuous open map from the second countable, locally compact, Hausdorff space G onto the second countable, locally compact, Hausdorff space $G^{(0)}/G$. From Theorem 3.3[1] it follows that there is a full π-system of Radon measures on $G^{(0)}$, i.e. a family of positive Radon measures on $G^{(0)}$,$\left\{\mu_{\dot{u}}^1, \dot{u} \in G^{(0)}/G\right\}$, such that:

1) For all $\dot{u} \in G^{(0)}/G$, $\mathrm{supp}\left(\mu_{\dot{u}}^1\right) = [u]$.

2) For all f in $C_c\left(G^{(0)}\right)$,

$$\dot{u} \to \int f(v)\, d\mu_{\dot{u}}^1(v) \quad \left[: G^{(0)}/G \to \mathbf{R}\right]$$

is continuous with compact support.

Let

$$\nu_{\dot{u}}^1 = \int \nu^w d\mu_{\dot{u}}^1(w).$$

Then for all f in $C_c(G)$,

$$\dot{u} \to \int f(x)\, d\nu_{\dot{u}}^1(x) \quad \left[: G^{(0)}/G \to \mathbf{R}\right]$$

is continuous with compact support.

Let $(K_n)_n$ be an increasing sequence of compact sets with $\bigcup_n K_n = G$. For each n, let $f_n : G \to [0,1]$ be a continuous with compact support function such that $f_n(x) = 1$ for all $x \in K_n$. Let $a_n(\dot{u}) = \frac{1}{2^n \nu_{\dot{u}}^1(f_n)}$ if $\nu_{\dot{u}}^1(f_n) > 1$, and $a_n(\dot{u}) = \frac{1}{2^n}$ otherwise. It is not hard to see that $\dot{u} \to a_n(\dot{u})$ is continuous. Let

$$P_{\dot{u}}(x) = \sum_n a_n(\dot{u})\, f_n(x) \text{ for all } x \in G$$

Since $|a_n(\dot{u}) f_n(x)| \leq \frac{1}{2^n}$, it follows that $(\dot{u}, x) \to \sum_n a_n(\dot{u}) f_n(x)$ is uniformly convergent and therefore $(\dot{u}, x) \to P_{\dot{u}}(x)$ is continuous. Thus, for all f in $C_c(G)$,

$$\dot{u} \to \int f(x) P_{\dot{u}}(x) d\nu_{\dot{u}}^1(x) \quad \left[: G^{(0)}/G \to \mathbf{R}\right]$$

is continuous with compact support. If we set $M(\dot{u}) = \int P_{\dot{u}}(x) d\nu_{\dot{u}}^1(x)$, then $0 < M(\dot{u}) < \infty$ and $\dot{u} \to M(\dot{u})$ is continuous. Let $\lambda_{\dot{u}}^1$ define by

$$\lambda_{\dot{u}}^1(f) = \frac{1}{M(\dot{u})} \int f(x) P_{\dot{u}}(x) d\nu_{\dot{u}}^1(x)$$

for all f continuous with compact support. Then $\dot{u} \to \lambda_{\dot{u}}^1$ is continuous and consequently, $\dot{u} \to d_*\left(\lambda_{\dot{u}}^1\right)$is continuous. Let $\mu_{\dot{u}}^2 = d_*\left(\lambda_{\dot{u}}^1\right)$ and $\nu_{\dot{u}}^2 = \int \nu^w \, d\mu_{\dot{u}}^2(w)$. With the same argument, for each $\dot{u}$ we can choose a probability measure $\lambda_{\dot{u}}^2$ in the class of $\nu_{\dot{u}}^2$ such that $\dot{u} \to \lambda_{\dot{u}}^2$ is continuous. For each $\dot{u}$, let $\left(\lambda_{\dot{u}}^2\right)^{-1}$ the image of $\lambda_{\dot{u}}^2$ by the inverse map, and let $\lambda_{\dot{u}} = \frac{1}{2}\left(\lambda_{\dot{u}}^2 + \left(\lambda_{\dot{u}}^2\right)^{-1}\right)$. Then $\dot{u} \to \lambda_{\dot{u}}$ is continuous and consequently, $\dot{u} \to d_*(\lambda_{\dot{u}})$is continuous. Let $\eta_{\dot{u}} = d_*(\lambda_{\dot{u}})$. Then for each $\dot{u}$ $\eta_{\dot{u}}$ is a transitive quasi-invariant probability for the Haar system $\left\{\nu^u,\, u \in G^{(0)}\right\}$ and $\dot{u} \to \eta_{\dot{u}}$ is continuous.

Let μ a quasi-invariant measure for the Haar system $\left\{\nu^u,\, u \in G^{(0)}\right\}$ and let $\mu_1 = \int \eta_{\pi(u)} d\mu(u)$.

The quasi-invariance of μ and the quasi-invariance of $\eta_{\dot{u}}$ imply that $\mu_1 \tilde{\ } \mu$. Let $\lambda = \int \lambda_{\pi(u)} d\mu(u)$, and observe that for all f continuous with compact support

$$\begin{aligned}
\int f(x) d\lambda(x) &= \int f(x) d\lambda_{\pi(u)}(x) d\mu(u) = \int f\left(x^{-1}\right) d\lambda_{\pi(u)}(x) d\mu(u) \\
&= \int f\left(x^{-1}\right) d\lambda(x)
\end{aligned}$$

and

$$\begin{aligned}
\int f(u) d\mu_1(u) &= \int f(v) d\eta_{\pi(u)}(v) d\mu(u) = \int f(d(x)) d\lambda_{\pi(u)}(x) d\mu(u) \\
&= \int f(d(x)) d\lambda(x)
\end{aligned}$$

Therefore $\mu_1 = d_*(\lambda)$ with λ a symmetric probability measure on G. Let $\nu = \int \nu^u d\mu_1(u)$ and $\eta = \int \eta_{\pi(w)} \times \eta_{\pi(w)} d\mu(w) = \int \eta_{\pi(w)} d\mu_1(w)$. Let $\lambda = \int \lambda^u d\mu_1(u)$ be a r-decomposition of λ and $\lambda' = (r, d)_*(\lambda)$. The quasi-invariance of the transitive measure $\eta_{\dot{u}}$ implies that $d_*(\nu^u) \tilde{\ } \eta_{\pi(u)}$ and $\delta_u \times d_*(\nu^u) \tilde{\ } \delta_u \times \eta_{\pi(u)}$. Consequently,

$$\int \delta_u \times d_*(\lambda^u) d\mu_1(u) \tilde{\ } \int \delta_u \times d_*(\nu^u) d\mu_1(u) \tilde{\ } \int \delta_u \times \eta_{\pi(u)} d\mu_1(u).$$

On the other hand, for all f continuous with compact support, we have

$$\begin{aligned}
\int\int f(s,t) \delta_u \times d_*(\lambda^u)(s,t) d\mu_1(u) &= \int\int f(u, d(x)) \lambda^u(x) d\mu_1(u) \\
&= \int f(r(x), d(x)) \lambda(x) \\
&= \int f(s,t) \lambda'(s,t)
\end{aligned}$$

Therefore $\eta\,\tilde{}\,\lambda'$. If $q = \frac{d\eta}{d\lambda'}$, then q is positive Borel function and it easy to verify that for μ_1-a.a. u $q = \frac{d\left(\delta_u \times \eta_{\pi(u)}\right)}{d\lambda'^u}$, where $\lambda' = \int \lambda'^u d\mu_1(u)$ is a r-decomposition of λ'. So

$$\int f((u,v)(s,t))\, q(s,t)\, d\lambda'^v(s,t) = \int f(s,t)\, q(s,t)\, d\lambda'^u(s,t).$$

Applying Theorem 3.9/p. 17[4] and Corollary 3.14/p. 19[4], it results that

$$\frac{q(v,u)}{q(u,v)} = \frac{d\eta^{-1}}{d\eta} \text{ and } (u,v) \to \frac{q(u,v)}{q(v,u)} \text{ is an a.e. homomorphism.}$$

Since $\eta^{-1} = \eta$, it follows that $\frac{q(u,v)}{q(v,u)} = 1$ for η-a.a. (u,v). Let $\Delta = \frac{d\nu}{d\nu^{-1}}$ be a modular function associated to the Haar system $\{\nu^u, u \in G^{(0)}\}$ and the quasi-invariant measure μ_1, and let $\delta(y) = \Delta(y)\frac{q(d(y),r(y))}{q(r(y),d(y))}$. We have $\Delta(y) = \delta(y)$ ν-a.a. y because $\frac{q(u,v)}{q(v,u)} = 1$ for η-a.a. (u,v).

Applying Lemma 2 and Lemma 3, we obtain that there is a μ_1-conull saturated subset Z of $G^{(0)}$ such that on $G_0 = G|Z$ the integral $f \to \int f(y)\,\nu(y)$ has the (r,d)-decomposition

$$\int\int f(y)\, d\nu_{u,v}(y)\, q(u,v)\, d\lambda'(u,v)$$

with respect to λ'. Hence,

Lemma 4. *There exists a μ_1-conull saturated σ-compact subset Z of $G^{(0)}$ and a system of measures, $\{\nu_{u,v},\, u,v \in Z,\, u\,\tilde{}\,v\}$, with the following properties:*

1) *$\nu_{u,v}$ is supported on G_v^u, and $\nu_{u,v} \neq 0$, for all $u,v \in Z, u\,\tilde{}\,v$.*

2) *For all $f \geq 0$ Borel on $G_0 = G|Z$,*

$$\int f(y)\, d\nu(y) = \int\int f(y)\, d\nu_{u,v}(y)\, d\eta(u,v)$$
$$= \int\int\int\int f(y)\, d\nu_{u,v}(y)\, d\eta_{\pi(w)}(v)\, d\eta_{\pi(w)}(u)\, d\mu(w).$$

3) *For all $f \geq 0$ Borel on $G_0 = G|Z$,*

$$(u,v) \mapsto \int f(y)\, d\nu_{u.v}(y)\ \left[: (r,d)(G_0) \to \overline{\mathbf{R}}\right]$$

is an extended real-valued Borel function.

4) *For all $f \geq 0$ Borel on $G_0 = G|Z$,*

$$\int f(xy)\, d\nu_{d(x),v}(y) = \int f(y)\, d\nu_{r(x),v}(y) \quad \textit{for all } x \in G_0,\ v \in [d(x)]$$

5) *For all $f \geq 0$ Borel on $G_0 = G|Z$,*

$$\Delta(x)\int f(yx)\, d\nu_{u,r(x)}(y) = \int f(y)\, d\nu_{u,d(x)}(y) \quad \textit{for all } x \in G_0,\ u \in [d(x)]$$

6) *$\Delta : G_0 \to \mathbf{R}_+^*$ is a strict homomorphisms.*

7) *For all $f \geq 0$ Borel on $G_0 = G|Z$,*

$$\int f(y)\, d\nu_{u,v}(y) = \int f\left(y^{-1}\right)\Delta\left(y^{-1}\right) d\nu_{v,u}(y) \ \textit{ for all } (u,v) \in (r,d)(G_0)$$

8) *For* μ*-a.a.* $u \in G^{(0)} = G|Z$,

$$\nu^u = \int \nu_{u,v} d\eta_{\pi(u)}(v)$$

Proof. Let us apply Lemma 2 and Lemma 3 to the groupoid G, the Haar system $\left\{\nu^u,\, u \in G^{(0)}\right\}$ and the quasi-invariant probability $\mu_1 = \int \eta_{\pi(u)} d\mu(u) = d_*(\lambda)$. Now it is easy to see that the system $\{\nu_{u,v},\, u, v \in Z,\, u\tilde{\ }v\}$ fulfills 1)-7). Let us prove 8). Let $f, g \geq 0$ be two Borel functions on G. Then

$$\begin{aligned}
&\int g(u)\left(\int f(x)\, d\nu^u(x)\right) d\mu_1(u) \\
&= \int g(r(x))\, f(x)\, d\nu(x) \\
&= \int\int\int\int g(r(x))\, f(x)\, d\nu_{u,v}(x)\, d\eta_{\pi(w)}(v)\, d\eta_{\pi(w)}(u)\, d\mu(w) \\
&= \int\int g(u)\left(\int\int f(x)\, d\nu_{u,v}(x)\, d\eta_{\pi(u)}(v)\right) d\eta_{\pi(w)}(u)\, d\mu(w) \\
&= \int g(u)\left(\int\int f(x)\, d\nu_{u,v}(x)\, d\eta_{\pi(u)}(v)\right) d\mu_1(u)
\end{aligned}$$

Hence $\int f(x)\, d\nu^u(x) = \int\int f(x)\, d\nu_{u,v}(x)\, d\eta_{\pi(u)}(v)$ for μ_1-a.a. u. □

Remark 1. *With the notation in the preceding lemma,* $\Delta|_{G|[u]}$ *is the modular function associated to the Haar system* $\{\nu^u, u \in G^{(0)}\}$ *and the quasi-invariant measure* $\eta_{\pi(u)}$ *for* μ*-a.a.* u*. Indeed, let* $\nu_{\dot{u}} = \int \nu^u d\eta_{\dot{u}}$*, and* $g \geq 0$ *a Borel function on* $G^{(0)}/G$*,* $f \geq 0$ *a Borel function on* G*. We have*

$$\begin{aligned}
&\int g(\pi(u))\left(\int f(x)\, d\nu_{\pi(u)}(x)\right) d\mu(u) \\
&= \int g(\pi(u))\left(\int f(x)\, d\nu^u(x)\right) d\mu_1(u) \\
&= \int g(\pi(r(x)))\, f(x)\, d\nu(x) \\
&= \int g(\pi(r(x)))\, f(x)\, \Delta(x)\, d\nu^{-1}(x) \\
&= \int g(\pi(d(x)))\, f\left(x^{-1}\right) \Delta\left(x^{-1}\right) d\nu(x) \\
&= \int g(\pi(u))\left(\int f\left(x^{-1}\right) \Delta\left(x^{-1}\right) d\nu^u(x)\right) d\mu_1(u) \\
&= \int\int g(\pi(u))\left(\int f\left(x^{-1}\right) \Delta\left(x^{-1}\right) d\nu^u(x)\right) d\eta_{\pi(w)}(u)\, d\mu(w) \\
&= \int g(\pi(w))\left(\int f\left(x^{-1}\right) \Delta\left(x^{-1}\right) d\nu_{\pi(w)}(x)\right) d\mu(w)\,.
\end{aligned}$$

Hence $\int f(x)\, d\nu_{\pi(u)}(x) = \int f\left(x^{-1}\right) \Delta\left(x^{-1}\right) d\nu_{\pi(u)}(x)$ *for* μ*-a.a.* u*, and so* $\Delta = \frac{d\nu_{\pi(u)}}{d\left(\nu_{\pi(u)}\right)^{-1}}$.

3. The structure of the C^*-algebra of a locally compact groupoid

REMARK 2. *If $\left(\mu, G^{(0)} * \mathcal{H}, L\right)$ is a representation of the locally compact groupoid G (Definition 3.20 [5]), then for μ-a.a. u $\left(\eta_{\pi(u)}, \left(G^{(0)} * \mathcal{H}\right)|_{[u]}, L|_{G|[u]}\right)$ is a representation of the locally compact groupoid $G|[u]$. We shall denote the representation induced by $\left(\eta_{\pi(u)}, \left(G^{(0)} * \mathcal{H}\right)|_{[u]}, L|_{G|[u]}\right)$ by $L_{\pi(u)}$. Also $L_{\pi(u)}$ can be viewed as a representation of the entire groupoid with respect to the transitive measure $\eta_{\pi(u)}$.*

Let L be a representation of the space $C_c(G)$. L is unitarily equivalent to the integrated form of a representation on groupoid also denoted by L :

$$\left(\mu, G^{(0)} * \mathcal{H}, L\right) \tilde{} \left(\mu_1, G^{(0)} * \mathcal{H}, L\right)$$

(where $\mu_1 = \int \eta_{\pi(u)} d\mu(u) \tilde{} \mu$).

The relation between the two representation is:

$$\langle L(f)\xi_1, \xi_2\rangle = \int f(x) \langle L(x)\xi_1(d(x)), \xi_2(r(x))\rangle \Delta^{-\frac{1}{2}}(x)\, d\nu^u(x)\, d\mu_1(u)$$

where $f \in C_c(G)$, $\xi_1, \xi_2 \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\mu_1(u)$. ($\Delta$ is the modular function of the Haar system $\{\nu^u, u \in G^{(0)}\}$ and the quasi-invariant measure μ_1).

Also we shall denote by $L_{\pi(u)}$ the representation induced by

$$\left(\eta_{\pi(u)}, \left(G^{(0)} * \mathcal{H}\right)|_{[u]}, L|_{G|[u]}\right)$$

on $C_c(G|([u]))$. (on the locally compact groupoid $G|[u]$ we shall consider the Haar system $\{\nu^v, v \in [u]\}$). Then we have

$$\left\langle L_{\pi(u)}(f)\xi_1, \xi_2\right\rangle = \int f(x) \langle L(x)\xi_1(d(x)), \xi_2(r(x))\rangle \Delta^{-\frac{1}{2}}(x)\, d\nu^v(x)\, d\eta_{\pi(u)}(v)$$

where $f \in C_c(G|[u])$, $\xi_1, \xi_2 \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(v)\, d\eta_{\pi(u)}(v)$. (because $\Delta|_{G|_{[u]}}$ is the modular function of the Haar system $\{\nu^v, v \in [u]\}$ and the quasi-invariant measure $\eta_{\pi(u)}$ for μ_1=a.a. u).

For $f \in C_c(G)$, let us denote

$\|f\| = \sup\{\|L(f)\| : L$ representation of $C_c(G)\}$ (the C^*-norm of f)

$\|f\|_* = \sup\{\|L(f)\| : L$ representation of $C_c(G)$ with respect to a transitive measure$\}$

Obviously, .$\|f\|_* \leq \|f\|$ for any $f \in C_c(G)$.

LEMMA 5. *With the notation of Remark 2 $\|f\|_* = \|f\|$ for all $f \in C_c(G)$.*

PROOF. It is suffices to prove $\|f\|_* \geq \|f\|$. Let L be a representation of $C_c(G)$, which is the integrated form of $\left(\mu_1, G^{(0)} * \mathcal{H}, L\right)$. Let us denote $\|\xi\|_{\pi(u)} = \left(\int \|\xi(v)\|^2 d\eta_{\pi(u)}(v)\right)^{\frac{1}{2}}$ for $\xi \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\mu_1(u)$. Then $\|\xi\|^2 = \int \|\xi\|_{\pi(u)}^2 \mu(u)$.

Let $f \in C_c(G)$, $\xi_1, \xi_2 \in \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\mu_1(u)$.We have

$$
\begin{aligned}
&\left|\langle L(f)\,\xi_1, \xi_2\rangle\right| \\
&= \left| \int \int f(x) \langle L(x)\,\xi_1(d(x)), \xi_2(r(x))\rangle \Delta^{-\frac{1}{2}}(x)\, d\nu^u(x)\, d\mu_1(u) \right| \\
&= \left| \int \left(\int \int f(x) \langle L(x)\,\xi_1(d(x)), \xi_2(r(x))\rangle \Delta^{-\frac{1}{2}}(x)\, d\nu^u(x)\, d\eta_{\pi(w)}(u) \right) d\mu(w) \right| \\
&\leq \int \left| \left\langle L_{\pi(w)}(f)\,\xi_1, \xi_2 \right\rangle \right| d\mu(w) \\
&\leq \int \left\| L_{\pi(w)}(f) \right\| \|\xi_1\|_{\pi(w)} \|\xi_2\|_{\pi(w)}\, d\mu(w) \\
&\leq \|f\|_* \int \|\xi_1\|_{\pi(w)} \|\xi_2\|_{\pi(w)}\, d\mu(w) \\
&\leq \|f\|_* \left(\int \|\xi_1\|^2_{\pi(w)}\, d\mu(w) \right)^{\frac{1}{2}} \left(\int \|\xi_2\|^2_{\pi(w)}\, d\mu(w) \right)^{\frac{1}{2}} \\
&= \|f\|_* \|\xi_1\| \|\xi_2\|.
\end{aligned}
$$

Thus $\|f\| \leq \|f\|_*$. □

THEOREM 1. *Let G be a locally compact second countable groupoid which admits a continuous Haar system. As a consequence the range map, $r : G \to G^{(0)}$, is an open map. Assume that $(r, d)(G)$ is a closed subset of $G^{(0)} \times G^{(0)}$ (the graph of the equivalence relation induced on $G^{(0)}$ is closed).*

Then $C^*(G)$ is isomorphic to the completion of

$$\left\{ \left(f_{\pi(u)}\right)_{\pi(u) \in G^{(0)}/G} : (\exists)\ f \in C_c(G) \ \text{ such that } f|_{G|[u]} = f_{\pi(u)} \right\}$$

in the norm $\left\| \left(f_{\pi(u)}\right)_{\pi(u)} \right\| = \sup \left\| f_{\pi(u)} \right\|_{\pi(u)}$.

PROOF. Let $\{\nu^u, u \in G^{(0)}\}$be a continuous Haar system on G. Let $C^*(G)$ be the C^*-algebra of G with respect to the Haar system $\{\nu^u, u \in G^{(0)}\}$ and $\|\|$ be the C^*-norm. For each orbit $[u]$, let $C^*(G|\,[u])$ be the C^*-algebra of the locally compact groupoid $G|\,[u]$ with respect to the Haar system $\{\nu^v, v \in [u]\}$ and $\|\|_{\pi(u)}$ be the C^*-norm of this algebra.

Let $f \in C_c(G)$. Applying the preceding lemma, $\|f\| = \|f\|_* = \sup\limits_{[u]} \left\| f|_{G|[u]} \right\|_{\pi(u)}$.

Then it is easy to observe that $C^*(G)$ is isomorphic to the algebra obtained by completing

$$\left\{ \left(f_{\pi(u)}\right)_{\pi(u) \in G^{(0)}/G} : (\exists)\ f \in C_c(G) \ \text{such that } f|_{G|[u]} = f_{\pi(u)} \right\}$$

in the norm $\left\| \left(f_{\pi(u)}\right)_{\pi(u)} \right\| = \sup \left\| f_{\pi(u)} \right\|_{\pi(u)}$.

Since all the groupoids $G|\,[u]$ are transitive, the structures of the C^*- algebras $C^*(G|\,[u])$ are known (see [6]). □

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