M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 7, 2001

Paper 4
Received: Mar 15, 2001, Revised: Oct 17, 2001, Accepted: Dec 12, 2001
Editor: J. Avron

Number Operator Algebras

Fabien Besnard,
Dept. de Mathématiques
Univ. De Paris VII
75221 Paris
besnard@math.jussieu.fr

Abstract

Under some hypotheses (symmetry, confluence), we enumerate all quadratically presented algebras, generated by creation and destruction operators, in which number operators exist. We show that these are algebras of bosons, fermions, their immediate generalizations that we call pseudo-bosons and pseudo-fermions, and also matrix algebras, in the finitely generated case. We then recover q-bosons (and pseudo-q-bosons) by a completion operation.

1 Introduction

In [Bes1] we have proposed a new way of looking at quantization. In this point of view, one should quantize the equations of evolution rather than the canonical commutation relations, the latter being a consequence of the former. In the case of a system of harmonic oscillators, which is crucial for field theory, we want to find algebras (over **C**) of q-numbers generated by a set $\{a_i | i \in \mathcal{I}\}$ of so-called destruction operators, and a set $\{a_i^+ | i \in \mathcal{I}\}$ of creation operators, conjugate to the former by an anti-involution which we denote by J. Our algebras will then have the structure of $*$-algebras. It should be stressed that the word "operator" is just a convention here, since no Hilbert space is a priori fixed. We require the existence of elements N_i, for all $i \in \mathcal{I}$, so that the following equations hold :

$$[N_i, a_j] = -\delta_{ij} a_i \tag{1}$$

$$[N_i, a_j^+] = \delta_{ij} a_i^+ \tag{2}$$

Since the base field **C** does not play a particular role, we will replace it by any field of characteristic 0.

In this article we propose to give a detailed account of the results obtained in our thesis, as well as a few novelties concerning Fock algebras. We will begin in the second section by defining precisely what we call a number operator algebra, and work out the first consequences of the definition. In the third section we will restrict to the case of quadratically confluent number operator algebras of finite type, and state the classification theorem. The proof is quite long, so we will not give it here in full. However we give an account of the demonstration, as detailed as we can, in section four. Then, we will tackle to the problem of n.o.a. of infinite type in section five. In particular we will show that we can do without the confluence hypothesis. Furthermore, only four out of the six different kinds of algebras we have found in the finite case remain. These algebras are precisely those which can be obtained as deformations of ϵ-Poisson algebras (a generalization of Super-Poisson algebras about which one can consult [Sch] or [Bes2]). In section six, we will prove a generalization of the classification theorem for n.o.a. of infinite type, in which we let the number operators belong to a certain completion of the algebra. We will see that this completion operation is natural in order to have a Fock representation. In this case, we will find two more solutions, namely q-bosonic and pseudo-q-bosonic algebras.

2 Definitions and first consequences

2.1 Definitions

We fix once for all K a field of characteristic 0, τ an involution of K (possibly the identity), and R the sub-field of elements of K fixed by τ. For each cardinal number α we choose a representative set $\mathcal{I}_\alpha$. In particular if $\alpha = n$ is finite, we take $\mathcal{I}_n = \{1, \ldots, n\} \subset \mathbf{N}$. All algebras are unital K-algebras and all morphisms preserve units. If A is an algebra, we say that it is trivial iff $A = 0$ or $A = K$. We denote by $Z(A)$ the center of A.

Definition 1 *Let α be a cardinal number and B be a non-trivial K-algebra. Let $X_A = \{a_i | i \in \mathcal{I}_\alpha\}$, $X_{A^+} = \{a_i^+ | i \in \mathcal{I}_\alpha\}$, $N = \{N_i | i \in \mathcal{I}_\alpha\}$ be 3 sets indexed by $\mathcal{I}_\alpha$, with the a_i's and a_i^+'s in B, and N_i's in $B/Z(B)$. We call (B, X_A, X_{A^+}, N) a number operator algebra of type α if, and only if :*

(i) B is generated by $X_A \cup X_{A^+}$ as an algebra.

(ii) One uniquely defines an anti-involution J on B by setting $J(a_i) = a_i^+$.

(iii) Equations (1) and (2) are fulfilled.

<u>Remark</u> : N_i belongs to $B/Z(B)$ which is only a vector space, nevertheless the commutator of such an element with any element of B is well defined, so that equations (1) and (2) make sense.
We have to define morphisms between two n.o.a. : we will only need to do so for n.o.a. of the same type. For a more general definition, see [Bes1].

Definition 2 *Let (B, X_A, X_{A^+}, N) and (B', X'_A, X'_{A^+}, N') be two n.o.a. of type α, and let f be an algebra homomorphism from B to B'. We will say that f is a morphism of n.o.a. iff :*

(i) $f \circ J = J' \circ f$

(ii) $f(Z(B)) \subset Z(B')$

(iii) There exists a bijection $\phi : \mathcal{I}_\alpha \to \mathcal{I}_\alpha$ such that $f(N_i) \in N'_{\phi(i)} + Z(B')$

It is easy to see that n.o.a. of type α and their morphisms form a category. We are now going to define another category, in which all creation (resp. destruction) operators play a symmetric role. We will denote by $\mathcal{S}_\alpha$ the group of permutations of $\mathcal{I}_\alpha$ that leave all but a finite number of elements invariant.

Definition 3 *Let B be a n.o.a. of type α and $\mathcal{S}_\alpha$ be the group of permutations of $\mathcal{I}_\alpha$ with finite support. We will say that B is symmetric iff for all $\sigma \in \mathcal{S}_\alpha$ the prescription $a_i \mapsto a_{\sigma(i)}$ uniquely defines an automorphism σ^* of B commuting with J.*

Definition 4 *A n.o.a. morphism f between two symmetric n.o.a. of type α will be called symmetric iff it commutes with σ^*, $\forall \sigma \in \mathcal{S}_\alpha$.*

Let (B, X_A, X_{A^+}, N) be a n.o.a. of type α, set $X = X_A \cup X_{A^+}$, and let $\langle X \rangle$ be the free monoid, $L_\alpha = K\langle X \rangle$ the free algebra, generated by X. Thus, the elements of L_α are of the form $x = \sum_i \lambda_i x_i$, with $\lambda_i \in K$, $x_i \in \langle X \rangle$. The set of those monomials x_i such that $\lambda_i \neq 0$ is called the support of x, it is a finite set. The $\lambda_i x_i$'s are called the terms of x.
We call π the canonical projection from L_α onto B, and I the kernel of π.
There is an anti-involution on L_α sending a_i to a_i^+. We also denote it by J.
The symmetric group $\mathcal{S}_\alpha$ also acts on L_α in an obvious way. If B is symmetric this action commutes with π. Since everything is commuting with π we will often drop it from the notations : whether an element belongs to L_α or B should be clear from the context.

Definition 5 *If I is generated by quadratic elements we say that B is a quadratic n.o.a.*

Remark : What we call a quadratic element is an element of degree ≤ 2. If it has no term of degree ≤ 1 we call it homogeneous quadratic.

2.2 A few lemmas

We denote by $\mathbf{Z}^{(\alpha)}$ the direct sum $\bigoplus_{i \in \mathcal{I}_\alpha} \mathbf{Z}$. Its elements are mappings $p : \mathcal{I}_\alpha \to \mathbf{Z}$ with a finite support.
For all $i \in \mathcal{I}_\alpha$ we define the derivation $\mathcal{N}_i : L_\alpha \to L_\alpha$ on the generators by $\mathcal{N}_i(a_j) = -\delta_{ij} a_i$ and $\mathcal{N}_i(a_j^+) = \delta_{ij} a_i^+$. The following trivial lemma has important consequences :

Lemma 1 *If B is a n.o.a. then the diagram of vector spaces :*

$$\begin{array}{ccc} L_\alpha & \xrightarrow{\mathcal{N}_i} & L_\alpha \\ {\scriptstyle\pi}\Big\downarrow & & {\scriptstyle\pi}\Big\downarrow \\ B & \xrightarrow{ad(N_i)} & B \end{array}$$

commutes for all $i \in \mathcal{I}_\alpha$.

Corollary 1 *B is a $\mathbf{Z}^{(\alpha)}$-graded algebra. More precisely : $L_\alpha = \bigoplus_{p\in\mathbf{Z}^{(\alpha)}} L_\alpha^p$, $B = \bigoplus_{p\in\mathbf{Z}^{(\alpha)}} B^p$, $I = \bigoplus_{p\in\mathbf{Z}^{(\alpha)}} (I \cap L_\alpha^p)$, with $L_\alpha^p = \{x \in L_\alpha | \forall i\, \mathcal{N}_i(x) = p(i)x\}$, and $B^p = \{y \in B | ad(N_i)(y) = p(i)y\}$.*

Corollary 2

$$\forall i, j \quad [N_i, N_j] = 0$$

Proof :
To prove the lemma, one just has to verify that $\pi(\mathcal{N}_i(x)) = [N_i, \pi(x)]$ for $x \in X$ since $\mathrm{ad}(N_i) = [N_i, .]$ is a (inner) derivation.
For the corollary 1 just notice that $\mathcal{N}_i\mathcal{N}_j = \mathcal{N}_j\mathcal{N}_i$ for all i, j. Then according to lemma 1 we also have $[\mathrm{ad}(N_i),\mathrm{ad}(N_j)] = 0$, where $[,]$ is the commutator in $\mathrm{End}(B)$. Thus we can decompose each space into common eigenspaces for the appropriate family of commuting endomorphisms.
Let us prove corollary 2 : $\forall x \in B$, $0 = [\mathrm{ad}(N_i),\mathrm{ad}(N_j)](x) = \mathrm{ad}([N_i, N_j])(x)$. Thus $[N_i, N_j] \in Z(B)$. But the central elements commute with all N_i, thus $Z(B) \subset B^0$. Now, if $N_j = \sum_{p\in\mathbf{Z}^\alpha} N_j^p$, then $\forall i$:

$$[N_i, N_j] = \sum_{p\in\mathbf{Z}^\alpha} p(i) N_j^p \in B^0$$

$$\Rightarrow \forall i, \forall p \neq 0, \quad p(i)N_j^p = 0$$

$$\Rightarrow N_j \in B^0 \Rightarrow [N_i, N_j] = 0$$

QED.

Let us notice that the action of the derivations $\mathcal{N}_i$ (resp. $[N_i, .]$) on a monomial x is to multiply it by the integer $n_i(x)$, which is the number of a_i^+ minus the number of a_i appearing in x. We will call $n_i(x)$ the *i-number* of x. More generally if x belongs to an eigenspace of $\mathcal{N}_i$ or $\mathrm{ad}(N_i)$ we call the corresponding eigenvalue the i-number of x.

Lemma 2 *If B is symmetric, B^0 is a sub-representation space for $\mathcal{S}_\alpha$.*

Proof :
Indeed, if $x \in B^0$ it is a linear combination of monomials of zero i-number for all i, and for $\sigma \in \mathcal{S}_\alpha$, $\sigma(x)$ shares the same property. QED.

3 Number Operator Algebras of Finite Type

3.1 The confluence hypothesis

In order to state the fundamental confluence hypothesis, we have to introduce some combinatorial terminology. In this subsection we do not have to suppose yet that α is finite, although we will only need the confluence hypothesis in this case. We refer the reader to [Berg] or [Ufn] for a more formal presentation. In the sequel by a *reduction system* we mean a subset of $\langle X\rangle \times L_\alpha$. Its elements are called reductions : they are couples (m, f) for which we will use the notation $m \to f$.
Given a presentation of an ideal I (i.e. a set of generators) and a monoid ordering $<$, that is an ordering on $\langle X\rangle$ which is compatible with multiplication, it is sometimes possible (always if $<$ is total) to construct a reduction system by isolating the leading monomial of every element of the presentation. We say that this reduction system is associated with the presentation and $<$. More precisely, if P is the presentation, then the reduction system associated with P and $<$ is $S_{I,<} = \{\mathrm{lm}(g) \to -\frac{1}{\mathrm{lc}(g)}(g - \mathrm{lt}(g)) | g \in P\}$ where we used the following notations : lm stands for "leading monomial", lc stands for "leading coefficient" and lt stands for "leading term".
A reduction system is useful if it gives a way of rewriting elements of B so as to give them a unique normal form. Indeed, let $x = \sum_i \lambda_i x_i$ be an element of L_α. If $m \to f$ is a reduction of $S_{I,<}$, then every occurrence of m as a subword of any monomial x_i may be replaced by f without changing the class of x modulo I. The aim is then to apply every possible reduction to x until we

get an *irreducible* element, that is to say an element we cannot reduce any further.

Of course this is not always a well defined procedure. First if we have two reductions $m \to f$ and $m' \to f'$ it can happen that the same monomial x_i can be written $x_i = abc$ with $ab = m$ and $bc = m'$. In this case we say that there is an *overlap ambiguity*. It is called solvable if there are two sequences of reductions, s_1 and s_2, such that applying s_1 on fc and s_2 on af' gives the same result. This can be visualized on the following diagram :

$$\begin{array}{ccccc} & & abc & & \\ & \swarrow & & \searrow & \\ fc & & & & af' \\ & s_1 \searrow & & \swarrow s_2 & \\ & & \text{same result} & & \end{array}$$

There can also be *inclusion ambiguities* : $x_i = abc$ with $abc = m$ and $b = m'$. It is said to be solvable if there are two sequences of reductions s_1 and s_2 such that s_1 applied on f is equal to s_2 applied on $af'b$.

When all ambiguities are solvable, the reduction system is said to be *confluent*.

There is one last problem to solve : we must be sure that the procedure will stop, and will not give an infinite cycle of reductions. This is achieved by using orderings satisfying the descending chain condition (DCC) : all decreasing sequences are stationary. Among such orderings, the most natural ones are the so-called "deglex" (degree-lexicographic) orderings, obtained from a total ordering $<_0$ on the generators, that is : $x < y$ iff $d^\circ(x) < d^\circ(y)$ or ($d^\circ(x) = d^\circ(y)$ and x is before y in the lexicographic order induced by $<_0$).

So, if $S_{I,<}$ is confluent and if $<$ is a monoid ordering satisfying DCC, Bergman's diamond lemma [Berg] states that the set of irreducible monomials is a K-basis for B.

For instance take $\alpha = 1$, $X_A = \{a\}$, $X_{A^+} = \{a^+\}$, so that $L_1 = K\langle a, a^+\rangle$, and denote by $<$ the deglex-ordering coming from $a^+ <_0 a$. Let us consider the ideal I generated by $P = \{a^2, {a^+}^2, aa^+ + a^+a - 1\}$. The reduction system associated to P and $<$ is $S = \{a^2 \to 0, {a^+}^2 \to 0, aa^+ \to 1 - a^+a\}$. This system is easily seen to be confluent. For instance the overlap ambiguity coming from a^2a^+ is solvable because $(a^2)a^+ \to 0$ and $a(aa^+) \to a(1 - a^+a) = a - (aa^+)a \to a - (1 - a^+a)a = a^+(a^2) \to 0$. By Bergman's lemma we find that the irreducible monomials 1, a, a^+ and a^+a form a K-basis of $B = L_1/I$.

Remark : It is always possible to avoid inclusion ambiguities in a reduction system (see [Berg] or [Bes1]). In this case we say that the reduction system is simplified. It is also always possible to assume that every element of a confluent reduction system is of the form $m \to r$ with r irreducible. We shall say that such a reduction system is reduced.

Definition 6 *We say that a presentation P of an ideal I is quadratically confluent (resp. deglex-quadratically confluent) iff the elements of P are at most of degree two, and there exists a monoid ordering $<$ satisfying DCC (resp. a deglex ordering), such that the reduction system associated with P and $<$ is confluent.*

3.2 The Main Theorem

We can now state our main result for n.o.a. of finite type :

Theorem 1 *Let n be a finite number and let $B = L_n/I$ be a symmetric deglex-quadratically confluent n.o.a., i.e. I satisfies the following properties*

(P_0) $I \neq L_n$, $I \neq \langle X \rangle$.

(P_1) $J(I) \subset I$.

(P_2) $\forall \sigma \in \mathcal{S}_n$, $\sigma^*(I) \subset I$.

(P_3) $\exists N_1, \ldots, N_n \in B$ *s.t. (1) and (2) hold.*

(P_4) : $\exists <_0$, *a total ordering on X s.t. I admits a quadratic and confluent reduction system, adapted to the deglex ordering coming from $<_0$.*

then, if $n = 1$, there exists $h \in R \setminus \{0\}$ such that I is generated by one of the following sets :

(a) $\{a^2, {a^+}^2, aa^+ + a^+a - h\}$

(b) $\{aa^+ - a^+a - h\}$

if $n \geq 2$ there exists $h \in R \setminus \{0\}$ such that I is generated by one of the following sets :

(a) $\{{a_i}^2, {a_i^+}^2, a_ia_j + a_ja_i, a_i^+a_j^+ + a_j^+a_i^+, a_ia_j^+ + a_j^+a_i, a_ia_i^+ + a_i^+a_i - h | 1 \leq i \neq j \leq n\}$

(a') $\{{a_i}^2, {a_i^+}^2, a_ia_j - a_ja_i, a_i^+a_j^+ - a_j^+a_i^+, a_ia_j^+ - a_j^+a_i, a_ia_i^+ + a_i^+a_i - h | 1 \leq i \neq j \leq n\}$

(b) $\{a_i^2, {a_i^+}^2, a_i a_j, a_i^+ a_j^+, a_i a_j^+, a_i a_i^+ + \sum_k a_k^+ a_k - h | 1 \leq i \neq j \leq n\}$

(b') $\{a_i^2, {a_i^+}^2, a_i a_j, a_i^+ a_j^+, a_i^+ a_j, a_i^+ a_i + \sum_k a_k a_k^+ - h | 1 \leq i \neq j \leq n\}$

(c) $\{a_i a_j - a_j a_i, a_i^+ a_j^+ - a_j^+ a_i^+, a_i a_j^+ - a_j^+ a_i, a_i a_i^+ - a_i^+ a_i - h | 1 \leq i \neq j \leq n\}$

(c') $\{a_i a_j + a_j a_i, a_i^+ a_j^+ + a_j^+ a_i^+, a_i a_j^+ + a_j^+ a_i, a_i a_i^+ - a_i^+ a_i - h | 1 \leq i \neq j \leq n\}$

Remark 1 : In the case $n = 1$, the hypothesis (P_4) can be loosened to :

$$I \text{ is generated by elements of degree } \leq 2 \qquad (P_4')$$

It is also true for n infinite that we can replace (P_4) with (P_4'), as we shall see later. However, in the case $2 \leq n < \infty$ there exist ideals I satisfying $(P_0), \ldots, (P_3)$ and (P_4') but not (P_4) (see [Bes1]).

Remark 2 : In the physical case, h is a positive real number and we can set h to 1 by rescaling the units, which amounts to the symmetric n.o.a. isomorphism $\phi_\lambda : a_i \mapsto \lambda a_i$, with $\lambda \in \mathbf{R}$. Then, we get :

- (a) The tensor product of n Weyl algebras (boson case), $A_n = A_1 \otimes \ldots \otimes A_1$, with $A_1 = L_1 / \langle a a^+ - a^+ a - 1 \rangle$.
- (a') The graded tensor product of n Weyl algebras (we call it the pseudo-boson case), $\hat{A}_n = A_1 \hat{\otimes} \ldots \hat{\otimes} A_1$.
- (b) The matrix algebra $\mathcal{M}_{n+1}(K)$.
- (b') The same as above but with the creation and destruction operators exchanged.
- (c) The graded tensor product of n Clifford algebras (the fermion case), $\hat{C}_n = C_1 \hat{\otimes} \ldots \hat{\otimes} C_1$, with $C_1 = L_1 / \langle a^2, {a^+}^2, a a^+ + a^+ a - 1 \rangle$.
- (c') The tensor product of n Clifford algebras (the pseudo-fermion case), $C_n = C_1 \otimes \ldots \otimes C_1$.

Particles whose creation and destruction operators form the algebra (c) or (c') satisfy Pauli's exclusion principle : only one particle of that kind can be found in a given state $(a_i^2 = {a_i^+}^2 = 0)$. Particles of type (b) or (b') follow a more extreme exclusion principle : only one such particle can be found, regardless of its state $(a_i a_j = a_i^+ a_j^+ = 0)$.

Remark 3 : We see that all these algebras depend only on a single constant h. Thanks to this fact we can see them as deformations of the "classical" algebras obtained by taking $h = 0$. This point of view is developed in [Bes2].

4 Sketch of proof of theorem 1

We do not have the space here to give a full proof. Nevertheless, we will give enough indications (we hope) for the reader to fill in the blanks. For a detailed proof see [Bes1].

4.1 A few more lemmas

The way we prove the theorem is by reducing the number of cases enough to be able to treat them all. This is done with the help of the following lemmas.

Lemma 3 *If (P_3) and (P_4') are satisfied, then I can be generated by a set of elements of the form : ${a_i}^2$ (1), ${a_i^+}^2$ (1'), $\alpha a_i a_j + \beta a_j a_i$ (2), $\alpha a_i^+ a_j^+ + \beta a_j^+ a_i^+$ (2'), $\alpha a_i a_j^+ + \beta a_j^+ a_i$ (3), $\sum_{1\leq i\leq n} \alpha_i a_i a_i^+ + \sum_{1\leq i\leq n} \beta_i a_i^+ a_i - \lambda$ (4), a_i (5), or a_i^+ (5').*

Proof :
Let P be a quadratic presentation of I and let $r \in P$. We can write $r = \sum_{p\in\mathbf{Z}^n} r^p$, where p, seen as a function of i, must be $\pm 2\delta_{ij}$, or $\pm(\delta_{ij} + \delta_{ik})$, or 0, or $\pm\delta_{ij}$. Now, from lemma 1, $r \in I \Leftrightarrow r^p \in I$, $\forall p$, and the forms (1) to (5') correspond to the different possibilities for the p's. QED.

Lemma 4 *If I fulfills (P_0), (P_3), and (P_4'), then it must contain at least one set of generators of type (4) with $\lambda \neq 0$.*

Proof :
Let us suppose that it is not so. Then, by lemma 3 and (P_4'), I must be generated by elements of the form : a_i, a_i^+, or r, with r homogeneous quadratic. Now by (P_0), $\exists i$ such that a_i or a_i^+ is not in I. Suppose $a_i \notin I$, and let $\tilde{N}_i$ belong to $\pi^{-1}(N_i)$. By (P_3), I must contain $\tilde{N}_i a_i - a_i\tilde{N}_i + a_i$, whose only term of degree one is a_i. But every element of I can be written as $\sum xry + \sum s_j a_j t_j + \sum u_k a_k^+ v_k$, r being homogeneous quadratic, and the second sum running over $j \neq i$. This is a contradiction. QED.

Lemma 5 *If $B = L_n/I$ satisfies (P_0), (P_3) and (P_4'), and if C is a commutative algebra, then $Hom_{K-alg}(B,C) = \{0\}$.*

Proof :
We have $\forall i$, $\phi([N_i, \pi(a_i^+)]) = \phi(\pi(a_i^+)) = 0$, since C is commutative. For the

same reason, $\phi(\pi(a_i)) = 0$. Now, by lemma 4, I contains an element of the form $x+\lambda.1$, $\lambda \neq 0$, x homogeneous quadratic. We have : $0 = \phi(\pi(x+\lambda.1)) = \phi(\pi(x)) + \lambda\phi(1) = 0 + \lambda.\phi(1)$. Therefore $\phi(1) = 0$, and $\phi = 0$. QED.

Lemma 6 *Let $B = L_n/I$, $B' = L_n/I'$, π and π' the respective projections. If $\exists\phi$: $B \to B'$, an algebra homomorphism such that $\phi(\pi(a_i)) = \pi'(a_i)$ and $\phi(\pi(a_i^+)) = \pi'(a_i^+)$, then : I fulfills $(P_3) \Rightarrow I'$ fulfills (P_3).*
(In particular, this is the case if $I \subset I'$ and if ϕ is induced by the identity map of L_n)

Proof :
It is easily verified that the images by ϕ of the number operators of B are number operators for B'. QED.

Lemma 7 *Let $B = L_n/I$ such that (P_3) holds, let B' be any algebra, and $\phi \in \mathrm{Hom}_{K-\mathrm{alg}}(B, B')$ such that $\forall i$, $\phi(a_i) = \phi(a_i^+)$. Then, $\forall i$ $\phi(a_i) = 0$.*

Proof :
Set $x_i := \phi(a_i) = \phi(a_i^+)$. On one hand $[N_i, a_i] = -a_i \Rightarrow [\phi(N_i), x_i] = -x_i$, and on the other hand $[N_i, a_i^+] = a_i^+ \Rightarrow [\phi(N_i), x_i] = x_i$. So $x_i = 0$. QED.

Lemma 8 *Let $n \geq 2$, $B = L_n/I$ such that (P_3) holds, B' any algebra, and $\phi \in \mathrm{Hom}_{K-\mathrm{alg}}(B, B')$ such that $\forall i, j$, $\phi(a_i) = \phi(a_j)$ (resp. $\phi(a_i^+) = \phi(a_j^+)$). Then, $\forall i$ $\phi(a_i) = 0$ (resp. $\phi(a_i^+) = 0$).*

Proof :
Let us examine the first case, the other one being similar. Let $i \neq j$ and let $x := \phi(a_i) = \phi(a_j)$. Then $[N_i, a_i] = -a_i \Rightarrow [\phi(N_i), x] = -x$, and $[N_i, a_j] = 0 \Rightarrow [\phi(N_i), x] = 0$. Thus $x = 0$. QED.

We now have to work out the consequences of (P_2). It is a bit long, but very easy. We only state the results, leaving the details to the reader (one could also see [Bes1]).
In what follows, we suppose $n \geq 2$, V is a n dimensional vector space with basis $e_1, \ldots, e_n$ and coordinates $\epsilon_1, \ldots, \epsilon_n$. Let $\rho : \mathcal{S}_n \to \mathrm{End}(V)$ be the representation given by $\rho(\sigma)(e_i) := \sigma.e_i := e_{\sigma(i)}$, let H be the hyperplane of equation $\epsilon_1 + \ldots + \epsilon_n = 0$. W is the vector space $W = V \oplus V \oplus \mathbf{1}$, where $\mathbf{1}$ is the trivial representation of $\mathcal{S}_n$ of dimension 1. W bears the representation $\rho \oplus \rho \oplus 1$. Finally, we set $x_i = e_i \oplus 0 \oplus 0$, $y_i = 0 \oplus e_i \oplus 0$, and let 1 be a non-zero vector of $\mathbf{1}$.

Lemma 9 *Let $w \in W$, $w \neq 0$, and let $O(w)$ be the linear span of the orbit of w under the action of $\mathcal{S}_n$. Then $O(w)$ is isomorphic as a representation space to : $\mathbf{1}$, H, $H \oplus \mathbf{1}$, $H \oplus H$ or $H \oplus H \oplus \mathbf{1}$. Furthermore $O(w)$ has a basis of the form :*

- *in the 1st case : $\{\mu_x.1_x + \mu_y 1_y + \mu_1.1\}$, with $1_x = x_1 + \ldots + x_n$, $1_y = y_1 + \ldots + y_n$, and $\mu_x, \mu_y, \mu_1 \in K$, not all zero.*
- *in the 2nd case : $\{\lambda_x(x_i - x_1) + \lambda_y(y_i - y_1) | i > 1\}$, $(\lambda_x, \lambda_y) \neq (0,0)$.*
- *in the 3rd case : $\{\mu_x 1_x + \mu_y 1_y + \mu_1 1, \lambda_x(x_i - x_1) + \lambda_y(y_i - y_1) | i > 1\}$ $(\mu_x, \mu_y, \mu_1) \neq (0,0,0)$, $(\lambda_x, \lambda_y) \neq (0,0)$.*
- *in the 4th case : $\{x_i - x_1, y_i - y_1 | i > 1\}$.*
- *in the 5th case : $\{\mu_x 1_x + \mu_y 1_y + \mu_1 1, x_i - x_1, y_i - y_1 | i > 1\}$, $(\mu_x, \mu_y, \mu_1) \neq (0,0,0)$.*

If we take an element x of I, we can make $\mathcal{S}_n$ act upon it to get others, so that the whole orbit of x belongs to I, and of course, so does its linear span. Using this and noticing that the last lemma apply to our situation if we set $x_i = a_i a_i^+$, $y_i = a_i^+ a_i$, and $1 = 1$, we arrive at the following result :

Lemma 10 *If I fulfills (P_0), (P_1), (P_2), (P_3) and (P_4'), then I can be generated by a union of sets, each having one of the following forms :*

- *form $(2,0)$: $\{a_i{}^2, a_i^{+2} | 1 \leq i \leq n\}$*
- *form $(1,1)_a$: $\{a_i a_j + a_j a_i, a_i^+ a_j^+ + a_j^+ a_i^+ | i < j\}$*
- *form $(1,1)_b$: $\{a_i a_j - a_j a_i, a_i^+ a_j^+ - a_j^+ a_i^+ | i < j\}$*
- *form $(1,1)_c$: $\{a_i a_j, a_i^+ a_j^+ | i \neq j\}$*
- *form $(1,-1)_a$: $\{r a_i a_j^+ + s a_j^+ a_i | i \neq j\}$, with $(r,s) \neq (0,0)$, $r, s \in R$.*
- *form $(1,-1)_b$: $\{a_i a_j^+, a_j^+ a_i | i \neq j\}$*
- *form $(0,0)$: $\{\sum_i \alpha_i a_i a_i^+ + \sum_i \beta_i a_i^+ a_i - \lambda\}$*

Furthermore, each set of the form (0,0) can be replaced by a union of sets of the form :

- *form* A_1 : $\{a_i a_i^+ - a_1 a_1^+ | i > 1\}$
- *form* B_1 : $\{a_i^+ a_i - a_1^+ a_1 | i > 1\}$
- *form* A_2 : $\{a_i a_i^+ - \lambda | 1 \le i \le n\}$
- *form* B_2 : $\{a_i^+ a_i - \mu | 1 \le i \le n\}$
- *form* C : $\{\sum_{1 \le i \le n} a_i a_i^+ - \lambda\}$
- *form* D : $\{\sum_{1 \le i \le n} a_i^+ a_i - \mu\}$
- *form* E_1 : $\{\alpha(a_i a_i^+ - a_1 a_1^+) + \beta(a_i^+ a_i - a_1^+ a_1) | i > 1\}$, $\alpha\beta \neq 0$
- *form* F : $\{\alpha \sum_{1 \le i \le n} a_i a_i^+ + \beta \sum_{1 \le i \le n} a_i^+ a_i - \lambda\}$, $\alpha\beta \neq 0$

In every case, we can assume that α, β, λ, μ *and* ν *belong to* R.

The next step is to combine the different sets of generators enumerated by lemma 10. For instance, if we are given the set $A_1 \cup C$, we can replace it with a set of the form A_2. Obviously, some combinations, such as the union of two sets of the form A_2 with different values of lambda, give a trivial result and we can get rid of them. The next proposition sum up the results which are non-trivial.

Proposition 1 *Let* $I_2 = \{x \in I | d^\circ(x) \le 2\}$, $I_2^{(2,0)} = I \cap \mathrm{Span}\{{a_i}^2, {a_i^+}^2 | 1 \le i \le n\}$, $I_2^{(1,1)} = I \cap \mathrm{Span}\{a_i a_j, a_i^+ a_j^+ | i \neq j\}$, $I_2^{(1,-1)} = I \cap \mathrm{Span}\{a_i a_j^+, a_i^+ a_j | i \neq j\}$, $I_2^{(0,0)} = I_2 \cap L_n^{(0,\dots,0)}$.
Under the hypotheses of lemma 10, there exists a presentation R *for* I*, of the form* $R = R^{(2,0)} \coprod R^{(1,1)} \coprod R^{(1,-1)} \coprod R^{(0,0)}$*, such that* $R^{(i,j)}$ *is a basis of* $I_2^{(i,j)}$*.*
Furthermore :

- $R^{(2,0)} = (2,0)$ *or the empty set.*
- $R^{(1,1)} = (1,1)_a$ *or* $(1,1)_b$ *or* $(1,1)_c$ *or* $\emptyset$*.*
- $R^{(1,-1)} = (1,-1)_a$ *or* $(1,-1)_b$ *or* $\emptyset$*.*

and $R^{(0,0)}$ *is one of the following sets :*

$A_2 = \{a_1a_1^+ - \lambda, \dots, a_na_n^+ - \lambda\}$
$A_2 \cup B_1 = \{a_1a_1^+ - \lambda, \dots, a_na_n^+ - \lambda, a_2^+a_2 - a_1^+a_1, \dots, a_n^+a_n - a_1^+a_1\}$
$A_2 \cup B_2 = \{a_1a_1^+ - \lambda, \dots, a_na_n^+ - \lambda, a_1^+a_1 - \lambda, \dots, a_n^+a_n - \lambda\}$
$A_2 \cup D = \{a_1a_1^+ - \lambda, \dots, a_na_n^+ - \lambda, \sum_i a_i^+a_i - \mu\}$
$A_3 = \{a_ia_i^+ + \beta\sum_j a_j^+a_j - \lambda | 1 \leq i \leq n\}$
$A_1 \cup D = \{a_2a_2^+ - a_1a_1^+, \dots, a_na_n^+ - a_1a_1^+, \sum_i a_i^+a_i - \lambda\}$
$C = \{\sum_i a_ia_i^+ - \lambda\}$
$C \cup D = \{\sum_i a_ia_i^+ - \lambda, \sum_i a_i^+a_i - \mu\}$
$E_2 = \{a_ia_i^+ + \beta_1a_i^+a_i + \beta_2\sum_{j\neq i}a_j^+a_j - \lambda | 1 \leq i \leq n\}, \beta_1 \neq \beta_2, \beta_1 + (n-1)\beta_2 \neq 0$ *or* $\{a_i^+a_i + \alpha_1a_ia_i^+ + \alpha_2\sum_{j\neq i}a_ja_j^+ - \mu | 1 \leq i \leq n\}, \alpha_1 \neq \alpha_2, \alpha_1 + (n-1)\alpha_2 \neq 0$
$E_2' = \{a_ia_i^+ + \beta_1a_i^+a_i + \frac{\beta_1}{1-n}\sum_{j\neq i}a_j^+a_j - \lambda | 1 \leq i \leq n\}, \beta_1 \neq 0$ *or* $\{\sum_i a_ia_i^+ - \lambda, \alpha(a_ia_i^+ - a_1a_1^+) + \beta(a_i^+a_i - a_1^+a_1) | i > 1\}$
$E_1 \cup C \cup D = \{\sum_j a_ja_j^+ - \lambda, \sum_j a_j^+a_j - \mu, \alpha a_ia_i^+ + \beta a_i^+a_i - \nu | i > 1\}$, *with* $\nu = (\alpha\lambda + \beta\mu)/n$
$F = \{\alpha\sum_i a_ia_i^+ + \beta\sum_i a_i^+a_i - \lambda\}$
$A_1 \cup B_1 \cup F = \{a_ia_i^+ - a_1a_1^+, a_i^+a_i - a_1^+a_1, \alpha a_1a_1^+ + \beta a_1^+a_1 - \lambda | i > 1\}$ *and also :* B_2, $A_1 \cup B_2$, $B_2 \cup C$, D, B_3, $B_1 \cup C$, *or* E_2'', *which are respectively symmetrical to* A_2, $A_2 \cup B_1$, $A_2 \cup D$, C, A_3, $A_1 \cup D$, E_2', *by the exchange of* a_i *and* a_i^+. *Moreover, in each case, at least one of the constants* λ, μ *or* ν *is non-zero,* α *and* β *are non-zero, and all the constants belong to* R.
Such a presentation is unique, except in the cases E_2, E_2' *and* E_2'' *for which we give two forms, and is called "standard".*

Even if a standard presentation is given and if the hypothesis (P_4) is satisfied, the standard presentation could happen to be non-confluent for any ordering. The next proposition shows that it is not so. Indeed, if a confluent reduction system exists for some deglex-ordering $<$, then this system is associated with a standard presentation. Furthermore, all orderings are not allowed.

Proposition 2 *Let* I *be an ideal such that* (P_0),...,(P_4) *hold, and let* S *be a quadratic confluent reduction system, which is adapted to some deglex ordering* $<$, *and associated with* I. *We also assume that* S *is simplified and reduced. Let* $R = R^{(2,0)} \coprod R^{(1,1)} \coprod R^{(1,-1)} \coprod R^{(0,0)}$ *be a standard presentation of* I, *and* R_S *the presentation associated with* S *and* $<$.

Then, $R_S = R_S^{(2,0)} \coprod R_S^{(1,1)} \coprod R_S^{(1,-1)} \coprod R_S^{(0,0)}$*,* $S = S^{(2,0)} \coprod S^{(1,1)} \coprod S^{(1,-1)} \coprod S^{(0,0)}$*, where* $S^{(i,j)}$ *is the reduction system associated with* $R^{(i,j)}$*, and* $R_S^{(2,0)} = R^{(2,0)}$*,* $R_S^{(1,1)} = \{x/\mathrm{lc}(x) | x \in R^{(1,1)}\}$*,* $R_S^{(1,-1)} = \{x/\mathrm{lc}(x) | x \in R^{(1,-1)}\}$*. Moreover, depending on* $R^{(0,0)}$*, the* $S^{(0,0)}$ *part is :*

A_2 *:* $S^{(0,0)} = \{a_i a_i^+ \rightarrow \lambda | 1 \leq i \leq n\}$*.*

$A_1 \cup B_2$ *:* $\exists i_0$ *s.t.* $S^{(0,0)} = \{a_i a_i^+ \rightarrow a_{i_0} a_{i_0}^+, a_j^+ a_j \rightarrow \mu | i \neq i_0, 1 \leq j \leq n\}$

$A_2 \cup B_2$ *:* $S^{(0,0)} = \{a_i a_i^+ \rightarrow \lambda, a_i^+ a_i \rightarrow \mu\}$*.*

A_3 *: there are two possibilities (a) :* $S^{(0,0)} = \{a_i a_i^+ \rightarrow \lambda - \beta \sum_j a_j^+ a_j)\}$*, or*
(b) : $\exists i_0, j_0$ *s.t.* $S^{(0,0)} = \{a_i a_i^+ \rightarrow a_{i_0} a_{i_0}^+, a_{j_0}^+ a_{j_0} \rightarrow \frac{\lambda}{\beta} - \frac{1}{\beta} a_{i_0} a_{i_0}^+ - \sum_{j \neq j_0} a_j^+ a_j | i \neq i_0\}$

C *:* $\exists i_0$ *s.t.* $S^{(0,0)} = \{a_{i_0} a_{i_0}^+ \rightarrow \lambda - \sum_{i \neq i_0} a_i a_i^+\}$

$C \cup D$ *:* $\exists i_0, j_0$ *s.t.* $S^{(0,0)} = \{a_{i_0} a_{i_0}^+ \rightarrow \lambda - \sum_{i \neq i_0} a_i a_i^+, a_{j_0}^+ a_{j_0} \rightarrow \mu - \sum_{j \neq j_0} a_j^+ a_j\}$

$A_1 \cup D$ *:* $\exists i_0, j_0$ *s.t.* $S^{(0,0)} = \{a_i a_i^+ \rightarrow a_{i_0} a_{i_0}^+, a_{j_0}^+ a_{j_0} \rightarrow \mu - \sum_{j \neq j_0} a_j^+ a_j | i \neq i_0\}$

$A_2 \cup D$ *:* $\exists i_0$ *s.t.* $S^{(0,0)} = \{a_i a_i^+ \rightarrow \lambda, a_{i_0}^+ a_{i_0} \rightarrow \mu - \sum_{j \neq i_0} a_j^+ a_j | 1 \leq i \leq n\}$

E_2*, with* $n \geq 3$ *: (a)* $\{a_i a_i^+ \rightarrow \lambda - \beta_1 a_i^+ a_i - \beta_2 \sum_{j \neq i} a_j^+ a_j | 1 \leq i \leq n\}$*, or*
(b) $\{a_i^+ a_i \rightarrow \mu - \alpha_1 a_i a_i^+ - \alpha_2 \sum_{j \neq i} a_j a_j^+ | 1 \leq i \leq n\}$

E_2 *with* $n = 2$ *: (a), or (b) as in the previous case, or*
(c) $\{a_j^+ a_j \rightarrow \frac{1}{\beta_2}(\lambda - \beta_1 a_i^+ a_i - a_i a_i^+), a_j a_j^+ \rightarrow \lambda(1 - \frac{\beta_1}{\beta_2}) + (\frac{\beta_1^2}{\beta_2} - \beta_2) a_i^+ a_i + \frac{\beta_1}{\beta_2} a_i a_i^+\}$*, with* $\{i, j\} = \{1, 2\}$*, or*
(d) $\{a_i^+ a_i \rightarrow \frac{1}{\beta_1}(\lambda - \beta_2 a_j^+ a_j - a_i a_i^+), a_j a_j^+ \rightarrow \lambda(1 - \frac{\beta_2}{\beta_1}) + (\frac{\beta_2^2}{\beta_1} - \beta_1) a_j^+ a_j + \frac{\beta_2}{\beta_1} a_i a_i^+\}$*, with* $\{i, j\} = \{1, 2\}$

E_2' *: (a)* $\{a_i a_i^+ \rightarrow \lambda - \beta_1 a_i^+ a_i + \frac{\beta_1}{n-1} \sum_{j \neq i} a_j^+ a_j | 1 \leq i \leq n\}$*, or*
(b) $\{a_{i_0} a_{i_0}^+ \rightarrow \lambda - \sum_{j \neq i_0} a_j a_j^+, a_i^+ a_i \rightarrow -\alpha\lambda + a_{i_0}^+ a_{i_0} + 2\alpha a_i a_i^+ + \alpha \sum_{j \neq i, i_0} a_j a_j^+ | i \neq i_0\}$*, for* $i_0 \in [1..n]$*, or*
(c) $\{a_{i_0} a_{i_0}^+ \rightarrow \lambda - \sum_{j \neq i_0} a_j a_j^+, a_{i_0}^+ a_{i_0} \rightarrow -\alpha\lambda + a_{j_0}^+ a_{j_0} + 2\alpha a_{j_0} a_{j_0}^+ + \alpha \sum_{j \neq i_0, j_0} a_j a_j^+, a_i^+ a_i \rightarrow a_{j_0}^+ a_{j_0} + \alpha a_{j_0} a_{j_0}^+ - \alpha a_i a_i^+ | i \neq i_0, j_0\}$ *for some* i_0*, and with* $\alpha = (1-n)/\beta_1$

F *: (a)* $\{a_{i_0} a_{i_0}^+ \rightarrow \frac{\lambda}{\alpha} - \sum_{i \neq i_0} a_i a_i^+ - \frac{\beta}{\alpha} \sum_{1 \leq j \leq n} a_j^+ a_j\}$*, for some* i_0*, or*
(b) $\{a_{j_0}^+ a_{j_0} \rightarrow \frac{\lambda}{\beta} - \frac{\alpha}{\beta} \sum_{1 \leq i \leq n} a_i a_i^+ - \sum_{j \neq j_0} a_j^+ a_j\}$ *for some* j_0*.*

$A_1 \cup B_1 \cup F$ *: (a)* $\{a_i a_i^+ \rightarrow a_{i_0} a_{i_0}^+, a_j^+ a_j \rightarrow \frac{\lambda}{\beta} - \frac{\alpha}{\beta} a_{i_0} a_{i_0}^+ | i \neq i_0, 1 \leq j \leq n\}$*, or*
(b) $\{a_i^+ a_i \rightarrow a_{i_0}^+ a_{i_0}, a_j a_j^+ \rightarrow \frac{\lambda}{\alpha} - \frac{\beta}{\alpha} a_{i_0}^+ a_{i_0} | i \neq i_0, 1 \leq j \leq n\}$

$E_1 \cup C \cup D$ *:* $\exists \mathcal{K}, \mathcal{L} \subset [1, .., n]$*,* $\mathcal{K} \cap \mathcal{L} = \emptyset$*,* $i_0 \notin \mathcal{K} \cup \mathcal{L}$*,* $\mathcal{K} \cup \mathcal{L} \cup \{i_0\} = [1..n]$ *such that* $S^{(0,0)} = \{a_i a_i^+ \rightarrow \frac{\nu}{\alpha} - \frac{\beta}{\alpha} a_i^+ a_i, a_j^+ a_j \rightarrow \frac{\nu}{\beta} - \frac{\alpha}{\beta} a_j a_j^+, a_{i_0} a_{i_0}^+ \rightarrow \lambda' -$

$\sum_{\mathcal{K}} a_k a_k^+ + \frac{\beta}{\alpha} \sum_{\mathcal{L}} a_k^+ a_k, a_{i_0}^+ a_{i_0} \rightarrow \mu' - \sum_{\mathcal{L}} a_k^+ a_k + \frac{\alpha}{\beta} \sum_{\mathcal{K}} a_k a_k^+ | i \in \mathcal{L}, j \in \mathcal{K}\}$, $\lambda' = \lambda - Card(\mathcal{L})\frac{\nu}{\alpha}$, $\mu' = \mu - Card(\mathcal{K})\frac{\nu}{\beta}$.
By the exchange of a_i and a_i^+ we also find the forms corresponding to B_2, $A_2 \cup B_1$, B_3, and E_2''.

For the proof, see [Bes1].
The next proposition will allow us to use symmetry between a_i and a_i^+ and fix the value of some constant when needed.

Proposition 3 *Let ϕ be one of the following automorphisms of L_n :*

$$\begin{array}{rcl} \epsilon : L_n & \rightarrow & L_n \\ a_i & \longmapsto & a_i^+ \\ a_i^+ & \longmapsto & a_i \end{array} \qquad \begin{array}{rcl} \phi_{\lambda,\mu} : L_n & \rightarrow & L_n \\ a_i & \longmapsto & \lambda a_i \\ a_i^+ & \longmapsto & \mu a_i^+ \end{array}$$

with $\lambda\mu \in R \setminus \{0\}$. Then if I fulfills (P_0),...,(P_4), so does $\phi(I)$.

Proof :
We do it for $\phi_{\lambda,\mu}$ the case of ϵ is even easier and is left to the reader.
For (P_0), it is obvious. Let us show that $\phi_{\lambda,\mu}(I)$ fulfills (P_1) : it is only needed to verify that the image under $\phi_{\lambda,\mu}$ of a standard presentation of I, which is a presentation of $\phi_{\lambda,\mu}(I)$, is sent into $\phi_{\lambda,\mu}(I)$ by J.
If x is a homogeneous quadratic element of a standard presentation of I, $\phi_{\lambda,\mu}(x)$ is proportional to x, so $J(\phi_{\lambda,\mu}(x))$ is proportional to $\phi_{\lambda,\mu}(J(x))$ and therefore belongs to $\phi_{\lambda,\mu}(I)$. Now if $x = \sum \alpha_i a_i a_i^+ + \sum \beta_i a_i^+ a_i - \nu.1$, with α_i, β_i, $\nu \in R$, is a generator of I, then $\phi_{\lambda,\mu}(x) = \lambda\mu(\sum \alpha_i a_i a_i^+ + \sum \beta_i a_i^+ a_i) - \nu.1$ is stable under J, because $\lambda\mu \in R$.
It is clear by its definition that $\phi_{\lambda,\mu}$ commutes with the action of $\mathcal{S}_n$. Thus, $\sigma^*(\phi_{\lambda,\mu}(I)) = \phi_{\lambda,\mu}(\sigma^*(I)) = \phi_{\lambda,\mu}(I)$
Lastly, let S whose elements we denote by $m_s \rightarrow f_s$, be a quadratic confluent reduction system, adapted to $<$ and associated with I, and let W be the vector space spanned by the irreducible monomials relatively to S. If we set $\phi_{\lambda,\mu}(S) = \{m_s \rightarrow \frac{1}{\lambda^k \mu^l} \phi_{\lambda,\mu}(f_s) | s \in S$, m_s of degree k in a_i and l in $a_i^+\}$, then $\phi_{\lambda,\mu}(W) = W$ is also the linear span of monomials that are irreducible under $\phi_{\lambda,\mu}(S)$. Moreover $\phi_{\lambda,\mu}(S)$ is clearly adapted to $<$ and $I \oplus W = L_n \Rightarrow \phi_{\lambda,\mu}(I) \oplus \phi_{\lambda,\mu}(W) = L_n$. Thus, by Bergman's lemma, $\phi_{\lambda,\mu}(S)$ is confluent. QED.

The last of our lemmas will help us to reduce even further the number of cases.

Lemma 11 *If I fulfills (P_0),...,(P_4), and if $n \geq 2$, then I must contain a set of generators of type (1,1) or (1,-1).*

Proof :
Let S be a quadratic confluent reduction system for I, associated with some deglex-ordering $<$, R be the associated presentation, T the basis of irreducible monomials, and $W = \mathrm{Span}(T)$. Let us denote by $\tilde{N}_1, \dots, \tilde{N}_n$ the representatives of $N_1, \dots, N_n$ in W, and write $\tilde{N}_1 = \lambda_1 \xi_1 + \dots + \lambda_k \xi_k + u$, with $d^\circ(\xi_1) = \dots = d^\circ(\xi_k) > d^\circ u$ and $\xi_1 > \dots > \xi_k$, $\lambda_1, \dots, \lambda_k \in K \setminus \{0\}$.
We first need to show the following formula :

$$\forall i, j, \quad \mathcal{N}_i \tilde{N}_j = 0 \tag{3}$$

This is true because $[N_i, N_j] = 0 \Leftrightarrow \mathcal{N}_i \tilde{N}_j \in I$. But T is made of eigenvectors for $\mathcal{N}_i$, so W is stable under $\mathcal{N}_i$, consequently $\mathcal{N}_i \tilde{N}_j \in W \cap I = \{0\}$.
Obviously, we also have :

$$\forall i, j, \quad \mathcal{N}_i \xi_j = 0 \tag{4}$$

Write $\xi_1 = x\eta y$, with $x, y \in X$. There are two cases :

- There exists $b \in X$ with an index different from the indices of x and y. Suppose first that $b > x$. Then $b\xi_1 > \xi_1 b$, and since $b\xi_1 > b\xi_2 > \dots$ and $\xi_1 b > \xi_2 b > \dots$ we have $\mathrm{lm}([\tilde{N}_1, b]) = b\xi_1$. Thus $b\xi_1 = bx\eta y$ must be reducible, but since $\xi_1 = x\eta y$ is not and the reduction system is quadratic, it is only possible if bx is reducible. We then have relations of type $(1,1)$ or $(1,-1)$ in R. Now if $b < x$, we have $\mathrm{lm}([\tilde{N}_1, b]) = \xi_1 b = x\eta y b$, and we deduce that yb is reducible. Therefore the relations $(1,1)$ or $(1,-1)$ are in R.

- There is no such b. Then $n = 2$ and $X = \{x, x^+, y, y^+\}$. If xy or yx are reducible we have relations $(1,1)$ or $(1,-1)$, so we suppose they are irreducible. For k large enough, we have $(xy)^k \xi_1 \neq \xi_1 (xy)^k$. Indeed, if not we would conclude that ξ_1 divides $(xy)^k$ but it is impossible since ξ_1 must contain x^+ or y^+ by the formula (4). So we deduce that $\mathrm{lm}([\tilde{N}_1, (xy)^k]) = \xi_1 (xy)^k$ or $(xy)^k \xi_1$. In both cases it is irreducible, so this is a contradiction. QED.

Even with the help of these lemmas, there still remain about a hundred cases to study. This is done quite in details in appendix A.

5 Number Operator Algebras of Infinite Type

5.1 The Classification Theorem

As we have said, in this case we do not need the confluence hypothesis.

Theorem 2 *Let α be an infinite cardinal number and let $B = L_\alpha/I$ be a symmetric n.o.a. of quadratic presentation, i.e. I satisfies the following properties*

(P_0) $I \neq L_\alpha$, $I \neq \langle X \rangle$.

(P_1) $J(I) \subset I$.

(P_2) $\forall \sigma \in \mathcal{S}_\alpha$, $\sigma^*(I) \subset I$.

(P_3) $\forall i \in \mathcal{I}_\alpha$, $\exists N_i \in B$ *s.t. (1) and (2) hold.*

(P_4') *: I is generated by elements of degree two or less.*
Then there exists $h \in R \setminus \{0\}$ such that I is generated by one of the following sets :

(a) $\{a_i{}^2, a_i^{+2}, a_ia_j + a_ja_i, a_i^+a_j^+ + a_j^+a_i^+, a_ia_j^+ + a_j^+a_i, a_ia_i^+ + a_i^+a_i - h | i \neq j\}$

(a') $\{a_i{}^2, a_i^{+2}, a_ia_j - a_ja_i, a_i^+a_j^+ - a_j^+a_i^+, a_ia_j^+ - a_j^+a_i, a_ia_i^+ + a_i^+a_i - h | i \neq j\}$

(c) $\{a_ia_j - a_ja_i, a_i^+a_j^+ - a_j^+a_i^+, a_ia_j^+ - a_j^+a_i, a_ia_i^+ - a_i^+a_i - h | i \neq j\}$

(c') $\{a_ia_j + a_ja_i, a_i^+a_j^+ + a_j^+a_i^+, a_ia_j^+ + a_j^+a_i, a_ia_i^+ - a_i^+a_i - h | i \neq j\}$

The algebras of case (a) (resp. (a'), (c), (c')) are called fermionic (resp. pseudo-fermionic, bosonic, pseudo-bosonic) algebras, and are denoted by $\hat{C}_\alpha$ (resp. C_α, A_α, $\hat{A}_\alpha$).

5.2 The Lemmas

The lemmas 1, 2, 4, 5, 6, 7, and 8 are valid both in the finite and infinite cases.
Due to the action of the symmetric group, the terms with a sum cannot survive in the infinite case, or else the sum should be infinite, which is meaningless in our purely algebraic setting. For this reason, the lemma 10 gets replaced by :

Lemma 12 *If I fulfills (P_0), (P_1), (P_2), (P_3) and (P_4') then it is generated by a union of sets, each being of one of the forms (with α, β, λ, r, $s \in R$) :*

- $(2,0)$: $\{{a_i}^2, {a_i^+}^2 | i \in \mathcal{I}_\alpha\}$
- $(1,1)_a$: $\{a_i a_j + a_j a_i, a_i^+ a_j^+ + a_j^+ a_i^+ | i,j \in \mathcal{I}_\alpha, i \neq j\}$
- $(1,1)_b$: $\{a_i a_j - a_j a_i, a_i^+ a_j^+ - a_j^+ a_i^+ | i,j \in \mathcal{I}_\alpha, i \neq j\}$
- $(1,1)_c$: $\{a_i a_j, a_i^+ a_j^+ | i,j \in \mathcal{I}_\alpha, i \neq j\}$
- $(1,-1)_a$: $\{r a_i a_j^+ + s a_j^+ a_i | i,j \in \mathcal{I}_\alpha, i \neq j\}$, $(r,s) \neq (0,0)$, $r,s \in R$.
- $(1,-1)_b$: $\{a_i a_j^+, a_j^+ a_i | i,j \in \mathcal{I}_\alpha, i \neq j\}$
- A_1 : $\{a_i a_i^+ - a_j a_j^+ | i,j \in \mathcal{I}_\alpha, i > j\}$
- A_2 : $\{a_i a_i^+ - \lambda | i \in \mathcal{I}_\alpha\}$
- E : $\{\alpha a_i a_i^+ + \beta a_i^+ a_i - \lambda | i \in \mathcal{I}_\alpha\}$
- E_1 : $\{\alpha(a_i a_i^+ - a_j a_j^+) + \beta(a_i^+ a_i - a_j^+ a_j) | i,j \in \mathcal{I}_\alpha, i \neq j\}$

and the forms B_1, B_2 symmetric to A_1, A_2, by the exchange of a_i and a_i^+.

We must combine these different sets of generators to enumerate all possible presentations. The equivalent of the proposition 1 is the following :

Proposition 4 *If I fulfills (P_0), (P_1), (P_2), (P_3) and (P_4') then there exists a presentation R of I, of the form $R = R^{(2,0)} \coprod R^{(1,1)} \coprod R^{(1,-1)} \coprod R^{(0,0)}$, such that :*

- $R^{(2,0)} = (2,0)$ *or the empty set.*
- $R^{(1,1)} = (1,1)_a$ *or* $(1,1)_b$ *or* $(1,1)_c$ *or* $\emptyset$.
- $R^{(1,-1)} = (1,-1)_a$ *or* $(1,-1)_b$ *or* $\emptyset$.

and $R^{(0,0)}$ is one of the following sets :
$A_2 = \{a_i a_i^+ - \lambda | i \in \mathcal{I}_\alpha\}$
$A_2 \cup B_1 = \{a_i a_i^+ - \lambda, a_j^+ a_j - a_0^+ a_0 | i,j \in \mathcal{I}_\alpha, j > 0\}$
$A_2 \cup B_2 = \{a_i a_i^+ - \lambda, a_i^+ a_i - \lambda | i \in \mathcal{I}_\alpha\}$
$E = \{a_i a_i^+ + \beta a_i^+ a_i - \lambda | i \in \mathcal{I}_\alpha\}$
$A_1 \cup B_1 \cup E = \{a_i a_i^+ - a_0 a_0^+, a_i^+ a_i - a_0^+ a_0, a_0 a_0^+ + \beta a_0^+ a_0 - \lambda | i > 0\}$
as well as B_2, $B_2 \cup B_1$. In each case λ and β are non-zero, and belong to R.

We refer to appendix B for the exposition of the proof of theorem 2.

6 Topological Number Operator Algebras

6.1 Definitions

In order to encompass the case of q-bosons, we must relax the conditions we impose on number operator algebras so as to let the number operators belong to some completion of the algebra. We are led to the following definition :

Definition 7 *Let B be a non-trivial K-algebra. Let $X_A = \{a_i | i \in \mathcal{I}_\alpha\}$ and $X_{A^+} = \{a_i^+ | i \in \mathcal{I}_\alpha\}$ be 2 sets of distinct elements of B. Let $V_n := BX_A^n$ be the left ideal of B generated by X_A^n and let $\tilde{B} := \varprojlim_{n\in\mathbf{N}^*} B/V_n$. If the following conditions hold :*

(H_1) : $\bigcap_{n\in\mathbf{N}^*} V_n = \{0\}$

(H_2) : $\forall b \in X_A^+, \forall n \in \mathbf{N}, \exists N \in \mathbf{N}$ such that $V_N b \subset V_n$
then the canonical morphism $B \to \tilde{B}$ is an embedding of algebras. If, in addition to this, we have :

(i) B is generated by $X_A \cup X_{A^+}$ as an algebra.

(ii) One uniquely defines an anti-involution J on B by setting $J(a_i) = a_i^+$.

(iii) For all $i \in \mathcal{I}_\alpha$, there exists $\tilde{N}_i \in \tilde{B}$ such that for all $j \in \mathcal{I}_\alpha$:

$$[\tilde{N}_i, a_j] = -\delta_{ij} a_j \tag{5}$$

$$[\tilde{N}_i, a_j^+] = \delta_{ij} a_j^+ \tag{6}$$

then $(B, X_A, X_{A^+}, (\tilde{N}_i)_{i\in\mathcal{I}_\alpha})$ is called a topological number operator algebra, and $\tilde{B}$ is the completion of B for the topology generated by the neighborhoods V_n of the origin.

In order for this definition to make sense, we must prove a few things. First, since the V_n's are left ideals of B satisfying $V_m \subset V_n$ whenever $m \geq n$ they form a projective system of left ideals and $\tilde{B}$ is a left B-module. We also know that they form a basis of neighborhoods of zero of a topology for which the sum and the left multiplication are continuous. The property (H_1) shows that the canonical map is into and that the topology is separated. We still have to show that the multiplication is a continuous mapping from $B \times B$ to B. For this, it is easy to see that we only need to show the continuity of right multiplication. This is assured by the property (H_2) for multiplication

by elements of X_{A^+}. Since it is obviously true for elements of X_A, we can show that it is true for any monomial, and then for any element of B.
With this definition we can expect the elements of $\tilde{B}$ to be expressed as normal ordered series, that is to say with the creations to the left.
In the next lemmas, B is a topological n.o.a.

Lemma 13 *Let $S_0 = \{0\}$ and $\forall k \in \mathbf{N}^*$, $S_k \supset S_{k-1}$ a subspace of B supplementary to V_k. Then $\tilde{B}$ is isomorphic to the set of power series of the type*

$$S = \sum_{k\in\mathbf{N}} u_k \text{ with } u_k \in S_{k+1} \cap V_k$$

endowed with the obvious laws.

Proof :
Let us define the natural projections $\pi_k : B \to B/V_k$, and $\pi_{k,j} : B/V_k \to B/V_j$, for $j \leq k$. Every $x \in \tilde{B}$ is given by a sequence $(x_k)_{k\in\mathbf{N}^*}$ such that $x_k \in B/V_k$ and $\forall j \leq k$, $\pi_{k,j}(x_k) = x_j$. Let s_k be the linear section of π_k associated with S_k. We set $u_k = s_{k+1}(x_{k+1}) - s_k(x_k)$, and $S(x) = \sum_{k\in\mathbf{N}} u_k$. Conversely if $S = \sum_k u_k$ we set $x_k = \pi_k(\sum_{j=0}^{k-1} u_j)$.
It is trivial to verify that we have defined two linear maps, inverse to each other. Indeed, the lemma is just a restatement of the definition of the projective limit, with in addition the condition (H_2) assuring that the product of two series is well defined. QED.

Remark : If it happens that $V_n = V_{n+1}$ for some n, then $V_m = V_n$ for every $m \geq n$. We have $S_m = S_n$ and the series are just finite sums. Thus, in this case $\tilde{B}$ is embedded in B. ¿From (H_1) we finally get $B = \tilde{B}$, and $V_n = \{0\}$.

Let us see now a particular case.

Lemma 14 *Set $T = \{a^+_{i_1} \dots a^+_{i_k} a_{j_1} \dots a_{j_l} | k, l \geq 0\}$ and $T_l = \{a^+_{i_1} \dots a^+_{i_k} a_{j_1} \dots a_{j_l} | k \geq 0\}$. Suppose that T generates B as a K-space. Then there exist $T'_0 \subset T_0$, $T'_1 \subset T_1$, ... such that for all k, $T_0 \coprod \dots \coprod T'_k$ is a basis of B/V_{k+1}.*
Moreover, every $x \in \tilde{B}$ can be written in a unique way :

$$x = \sum_{l=0}^{\infty} \sum_{t\in T'_l} \lambda_{t,l} t$$

with the condition that $\forall l$, the set $\{t \in T'_l | \lambda_{t,l} \neq 0\}$ is finite.

Proof :
Since $V_k = BX_A^k$ and T is a generating family for B, we have $V_k = \text{Span}\{T_j | j \geq k\}$. We choose for $T'_0 \subset T_0$ a basis of B/V_1. Then we choose T'_1 so that $T'_0 \coprod T'_1$ is a basis of B/V_2, and so on. Then we set $S_k = \text{Span}\{T'_j | j < k\}$ and apply the previous lemma. QED.

Remark : It is not assumed that $T' = \bigcup_{n \in \mathbf{N}} T'_n$ is a basis of B. In fact this assumption implies (H_1) and seems to be strictly stronger.

Of course a topological n.o.a. B is $\mathbf{Z}^{\mathcal{I}_\alpha}$-graded, and so will be its completion $\tilde{B}$. For every $n \in \mathbf{Z}^{\mathcal{I}_\alpha}$, we write $\tilde{B}^n = \{x \in \tilde{B} | \forall i \in \mathcal{I}_\alpha \ [N_i, x] = n(i)x\}$.

Lemma 15 *Let $n \in \mathbf{Z}^{\mathcal{I}_\alpha}$. Then $\forall x \in \tilde{B}^n$, $\exists (x_k) \in B^{\mathbf{N}^*}$ such that for all k $x_k \in B^n$ and $x = \lim_k x_k$.*

Proof :
Let $B' := \bigoplus_{p \in \mathbf{Z}^{\mathcal{I}_\alpha}, p \neq n} B^p$. $\forall k \in \mathbf{N}^*$, $\exists z_k \in B$ such that $x - z_k \in V_k$. Now $z_k = x_k + y_k$ with $x_k \in B^n$ and $y_k \in B'$. Since V_k is stable under $\text{ad}(N_i)$ for all i, we have $x - x_k \in V_k$ and $y_k \in V_k$. QED.

To give another motivation for the definition 7, let us introduce a new kind of algebra, that would seem more natural in a physicist's point of view :

Definition 8 *Let B be a non-trivial K-algebra satisfying (i) and (ii) of definition 7. Let us call F the left module $F = B/BV_1$ and let us define $\rho : B \rightarrow End(F)$ such that $\rho(x)(u) = x.u$. If the following properties are satisfied :*

(a) $Ker(\rho) = \{0\}$.

(b) F is generated by the monoid $X_{A^+}^*$ as a K-space.

(c) $\forall i \in \mathcal{I}_\alpha$, $\exists N_i \in \text{End}(F)$ s.t.

$$[N_i, \rho(a_j)] = -\delta_{ij} \rho(a_j) \tag{7}$$

$$[N_i, \rho(a_j^+)] = \delta_{ij} \rho(a_j^+) \tag{8}$$

(d) $N_i(|1\rangle) = 0$, where $|1\rangle = 1[V_1]$.

then $(B, X_A, X_{A^+}, (N_i)_{i \in \mathcal{I}_\alpha})$ is called a Fock algebra and we call ρ the Fock representation of B.

Thanks to (7) and (8), a Fock algebra is graded in exactly the same way as a n.o.a., and we can define B^p and the i-numbers for i-homogeneous elements. In the next lemmas, B is a Fock algebra.

Lemma 16 *Let $i \in \mathcal{I}_\alpha$ and $x \in B$ such that x is i-homogeneous and $n_i(x) \leq 0$. Then $\exists \lambda \in K$ such that $x = \lambda\ [V_1]$. Furthermore, if $n_i(x) < 0$, $\lambda = 0$.*

Proof :
By (b) of definition (8), there exists a linear combination of elements of X^*_{A+}, let us say $\lambda + \xi$, where λ is the scalar part, such that $x - \lambda - \xi \in V_1$. The result is then clear using the fact that V_1 is graded. QED.

Lemma 17 $\forall i, j \in \mathcal{I}_\alpha$, $\forall k \in \mathbf{N}^*$, $\exists\ y^1_{ij}, \ldots, y^{k-1}_{ij}$ *such that $y^n_{ij} \in X^n_{A+}X^n_A$, y^n_{ij} has the same numbers as $a_i a^+_j$ and $a_i a^+_j = \sum_{n=0}^{k-1} y^n_{ij}\ [V_k]$.*

Proof :
By the previous lemma the result holds for $k = 1$. If it holds for k, then $a_i a^+_j = \sum_{n=0}^{k-1} y^n_{ij} + v_k$, with $v_k \in V_k$. Write $v_k = \sum w_t z_t\ [V_{k+1}]$ with $w_t \in B$ and $z_t \in X^k_A$. Then each w_t can be decomposed as a linear combination of some elements in X^*_{A+} modulo V_1. Thus we can write $v_k = \sum w'_t z_t\ [V_{k+1}]$ with $w'_t \in X^*_{A+}$. Now the numbers of $w'_t z_t$ are the same as those of $a_i a^+_j$. Thus z_t must contain a_i and w'_t must contain a^+_j, at least once. If we remove one copy of these two generators in z_t and w'_t respectively, we find that the numbers of the remaining two monomials must cancel. Since one consists of generators only, and the other of destructions only, they must be of the same length. So $w'_t \in X^k_{A+}$. QED.

Proposition 5 *If B is a Fock algebra, then B fulfills (H_1) and (H_2).*

Proof :
Let $x \in \bigcap_{n \in \mathbf{N}^*} V_n$ and m be a monomial. Take n larger than the length of m, and any n-uple $\mathcal{I}$. Then $n_i(a_\mathcal{I} m) < 0$, so $a_\mathcal{I} m = 0\ [V_1]$ by lemma 16. Thus $V_n m = 0[V_1]$ and $\rho(x)(m) = 0$, consequently $x \in \mathrm{Ker}(\rho) = \{0\}$, and (H_1) is fulfilled.
Let us show that $V_k a^+_i \subset V_{k-1}$ for all i, which clearly entails (H_2). This is true for $k = 1$. If it is true for k, then $\forall p \leq k-1$, $V_k X^p_{A+} \subset V_{k-p}$. Let x be a monomial in V_{k+1}. Write $x = x' a_j$. We have $a_j a^+_i = \sum_{p=0}^{k-1} \lambda_{k,p} v_{k,p} w_{k,p} + n_k$, with $n_k \in V_k$, $v_{k,p} \in X^p_{A+}$ and $w_{k,p} \in X^p_A$, by lemma 17. Thus $x' v_{k,p} \in V_{k-p}$ and $x \in V_k$. So by induction, we find the claimed result. QED.

Remark : Let B be a topological n.o.a. and a Fock algebra, and put the discrete topology on F. Then the topology of $\tilde{B}$ is stronger than the topology of pointwise convergence on $\mathrm{End}(F)$. In all the cases we will investigate, these topologies are in fact the same.

6.2 The classification theorem

With the help of the preceding lemmas and the calculations already done in proving theorems 1 and 2, there only remains a little work to prove the following theorem.

Theorem 3 *Let α be an infinite cardinal number. If $B = L_\alpha / I$ is a topological n.o.a. of type α that is symmetric and quadratically presented, then either I is one of the ideals enumerated in theorem 2 or there exist $h \in R \setminus \{0\}$, $q \in R \setminus \{-1, 1\}$ such that I is generated by*

- *(d)* $\{a_i a_j - a_j a_i, a_i^+ a_j^+ - a_j^+ a_i^+, a_i a_j^+ - a_j^+ a_i, a_i a_i^+ - q a_i^+ a_i - 1 | i \neq j\}$
- *(d')* $\{a_i a_j + a_j a_i, a_i^+ a_j^+ + a_j^+ a_i^+, a_i a_j^+ + a_j^+ a_i, a_i a_i^+ - q a_i^+ a_i - 1 | i \neq j\}$

In the case (d), the algebra is called a q-boson algebra, and denoted by A_α^q. In the case (d') it is called a pseudo-q-boson algebra, and denoted by $\hat{A}_\alpha^q$. In both cases we have $N_i = \sum_{k=0}^{\infty} \frac{(1-q)^k}{1-q^k} {a_i^+}^k {a_i}^k + \lambda_i$, $\lambda_i \in K$, the algebras $\tilde{B}$ are central and the number operators are unique up to an additive constant.

For the proof we refer to appendix C.

7 Concluding Remarks

In this article, we have tried to explore the algebraic constraints that a free field theory must abide by. Of course this approach have raised as many questions as it has answered. Imposing quadratic relations (and confluence in the finite case) seems to be just as restrictive as needed in order to state a classification theorem. In this way we have recovered all the known cases, plus two new ones if the number of degrees of freedom is finite. The virtue of this method is also to put on an equal footing bosons, fermions, pseudo-bosons and pseudo-fermions, which shows that ϵ-symmetry (see [Bes2]) appears in a natural way.
There are at least two directions towards which we can try to go further : incorporating infinite sums in the defining relations and allowing cubic relations in order to recover para-statistics. These subjects are under investigations but what we have done so far indicates that other algebraic hypotheses must be imposed to keep the problem feasible.

A End of Proof of Theorem 1

If I is such that $(P_0), \ldots, (P_4)$ hold, it has a standard presentation by proposition 1. We have to study every such presentation that is not yet ruled out by lemma 11. In most cases, it is possible to show that (P_0), (P_3), or (P_4) cannot hold by making use of our different lemmas. Nonetheless, it is sometimes necessary to call upon a reduction system and calculate in a basis of irreducible monomials.
For convenience, we will deal with relations in B rather than with generators of the ideal I.
When the relations $(0,0)$ depend on a single constant term λ (which must be non-zero) we assume that $\lambda = 1$.
In cases containing $(1,-1)_a$, and if $r \neq 0$, we set $q = -s/r$. If $s \neq 0$, we set $q' = -r/s$.
¿From now on we assume $n \geq 2$. We will look at the case $n = 1$ afterwards.
$\underline{(1,1)_c \cup (1,-1)_b \cup (2,0) :}$
So I is generated by the relations $(1,1)_c \cup (1,-1)_b \cup (2,0)$ together with some relations of type $(0,0)$. If relations of type A_2, C, B_2 or D with λ or $\mu \neq 0$ are present, it is easily seen by multiplying them on the left or on the right by a_i that $a_i = 0$. Thus (P_0) is not satisfied. Let us see the other cases :

- A_3 : In B we have :

$$\begin{cases} a_1 a_1^+ + \beta \sum a_i^+ a_i = 1 & (l_1) \\ \quad \ldots & \\ a_n a_n^+ + \beta \sum a_i^+ a_i = 1 & (l_n) \end{cases}$$

  Let's multiply (l_1) on the right by a_n, we get $a_n = 0$, thus $B = 0$. This also rules out the case $A_1 \cup B_1 \cup F = A_3 \cup B_1$.

- E_2 :

$$\begin{cases} a_1 a_1^+ + \beta_1 a_1^+ a_1 + \beta_2 \sum_{i \neq 1} a_i^+ a_i = 1 & (l_1) \\ \quad \ldots & \\ a_n a_n^+ + \beta_1 a_n^+ a_n + \beta_2 \sum_{i \neq n} a_i^+ a_i = 1 & (l_n) \end{cases}$$

  We do as above.

- F :

$$\alpha \sum a_i a_i^+ + \beta \sum a_i^+ a_i = 1$$

We first multiply the relation F by a_i on the left, then on the right, and we get :

$$\beta a_i a_i^+ a_i = \alpha a_i a_i^+ a_i = a_i$$

If $\alpha \neq \beta$, we have $B = 0$. Thus we can assume that $\alpha = \beta = 1$.

We have $a_i a_i^+ a_i = a_i$, $\forall i$. But if there exists a quadratic confluent reduction system for I, it is of type (a) or (b) (see proposition 2). In both cases, there exists i such that $a_i a_i^+$ and $a_i^+ a_i$ are irreducible, so $a_i a_i^+ a_i$ must be irreducible, and this is a contradiction.

$\underline{(1,1)_{a \text{ or } b} \cup (1,-1)_b \cup (2,0) :}$

Making use of either $(1,-1)_{a,b}$ or $(2,0)$, we see that A_2 or B_2 with $\lambda \neq 0$ leads to $B = 0$.

- C : Multiplying the relation (C) by a_1 on the left we get :

$$\begin{aligned} {a_1}^2 a_1^+ + \sum_{i>1} a_1 a_i a_i^+ &= a_1 \\ \Rightarrow \sum_{i>1} \pm a_i a_1 a_i^+ &= a_1 \\ \Rightarrow 0 = a_1 &\Rightarrow B = 0 \end{aligned}$$

This also rules out all other presentations containing relations of type C (in particular $E_2' = E_1 \cup C$).

- F : We have

$$\begin{aligned} \alpha \sum_{1 \leq i \leq n} a_1 a_i a_i^+ + \beta \sum_{1 \leq i \leq n} a_1 a_i^+ a_i &= a_1 \\ \Rightarrow \alpha \sum_{i>1} \pm a_i a_1 a_i^+ + \beta a_1 a_1^+ a_1 &= a_1 \\ \Rightarrow \beta a_1 a_1^+ a_1 &= a_1 \end{aligned}$$

Multiplying by a_1 on the right we would obtain in the same way :

$$\alpha a_1 a_1^+ a_1 = a_1 \tag{9}$$

thus $\alpha = \beta$. Furthermore, multiplying (9) by a_i, $i \neq 1$ we get :

$$a_1 a_i = \alpha a_1 a_1^+ a_1 a_i = \pm \alpha a_1 a_1^+ a_i a_1 = 0$$

But, the presentation being standard, $\{a_i a_j - a_j a_i, a_i^+ a_j^+ - a_j^+ a_i^+ | i < j\}$ is a basis of $I_2^{(1,1)}$. Now $a_1 a_i \notin \mathrm{Span}\{a_i a_j - a_j a_i, a_i^+ a_j^+ - a_j^+ a_i^+\}$, a contradiction.

We can do the same for F, and thus for A_3, B_3 and E_2.

$\underline{(1,1)_c \cup (1,-1)_a \cup (2,0)}$:

- A_3 : Let's multiply (l_1) on the right by a_1 :

$$a_1 a_1^+ a_1 = a_1 \tag{10}$$

  Now $a_1 a_1^+ = a_i a_i^+ \Rightarrow a_1 = a_i a_i^+ a_1$, $\forall i$. Then if $s \neq 0$, we have $a_1 = -q' a_i a_1 a_i^+ = 0$, for $i \neq 1$, and $B = 0$. We can thus assume that $s = 0$.
  Now $a_i a_j^+ = 0$, and multiplying (l_1) to the left by a_1, we find $a_1 = \beta a_1 a_1^+ a_1$, consequently we have $\beta = 1$, by (10).
  We are then in the case (b) of the theorem. It is easily seen that the reduction system $\{a_i a_j \to 0, a_i^+ a_j^+ \to 0, {a_i}^2 \to 0, {a_i^+}^2 \to 0, a_i a_j^+ \to 0, a_i a_i^+ \to 1 - \sum a_k^+ a_k | 1 \leq i \neq j \leq n\}$ is confluent and adapted to the deglex-ordering coming from $a_1^+ < \ldots < a_n^+ < a_1 < \ldots < a_n$, which we will denote by $<_n$ in the rest of the section. We let the reader verify that (P_3) holds, with $N_i = a_i^+ a_i + \lambda_i.1$, $\lambda_i \in K$.

- F : Let's multiply F by a_1 on the right, then on the left :

$$\begin{cases} \beta a_1 (\sum_{1 \leq i \leq n} a_i^+ a_i) = a_1 \\ \alpha (\sum_{1 \leq i \leq n} a_i a_i^+) a_1 = a_1 \end{cases}$$

$$\Rightarrow \begin{cases} r a_1 = r \beta a_1 a_1^+ a_1 + \beta \sum_{i>1} (-s a_i^+ a_1) a_i \\ s a_1 = s \alpha a_1 a_1^+ a_1 + \alpha \sum_{i>1} a_i (-r a_1 a_i^+) \end{cases}$$

$$\Rightarrow \begin{cases} r a_1 = r \beta a_1 a_1^+ a_1 \\ s a_1 = s \alpha a_1 a_1^+ a_1 \end{cases}$$

$$\Rightarrow \begin{cases} r a_1 a_2^+ = -s \beta a_1 a_1^+ a_2^+ a_1 = 0 \\ s a_2^+ a_1 = -r \alpha a_1 a_2^+ a_1^+ a_1 = 0 \end{cases}$$

  Thus $r = 0$ or $s = 0$ (the presentation is standard). If $r = 0$, $a_j^+ (\alpha \sum_i a_i a_i^+) + a_j^+ \beta (\sum a_i^+ a_i) = \alpha a_j^+ a_j a_j^+ = a_j^+$. Now $\exists j$ such that $a_j^+ a_j a_j^+$ is irreducible, and we come to a contradiction. The case $s = 0$ is symmetrical.

- $A_1 \cup B_1 \cup F$: From case F we know that we can assume that $r = 0$, then we have $a_n = \alpha a_1 a_1^+ a_n + \beta a_1^+ a_1 a_n = 0$. Consequently $B = 0$.

- E_2 : By F, we have $rs = 0$. In each case it is easy to show that $B = 0$.

$\underline{(1,1)_b \cup (1,-1)_a \cup (2,0)}$:

- C : Let's show that $B=0$:

$$\sum a_i a_i^+ = 1$$

$$\Rightarrow \sum a_1 \dots a_n a_i a_i^+ = a_1 \dots a_n$$

$$\Rightarrow \sum a_1 \dots a_{i-1} a_{i+1} \dots a_n a_i a_i a_i^+ = a_1 \dots a_n$$

$$\Rightarrow 0 = a_1 \dots a_n$$

  Suppose that every product $a_{i_1} \dots a_{i_k}$ of length k is zero. This is true for $k=n$. Then :

$$a_{i_1} \dots a_{i_{k-1}} = \sum a_{i_1} \dots a_{i_{k-1}} a_i a_i^+$$

  and the sum is zero, since all terms $a_{i_1} \dots a_{i_{k-1}} a_i$ vanish. Thus, by induction, we see that $B=0$.

  Since we made no use of relations $(1,-1)_a$, we can get rid of the case $(1,1)_b \cup (2,0) \cup C$ by the same method.

- F : Thanks to all the relations we have, one can show that

$$(\alpha-\beta)(r+s)a_1 \dots a_n = 0$$

  Then, if $(\alpha-\beta)(r+s) \neq 0$, it can be proved by induction that $B=0$. If only one of the factors $\alpha-\beta$ or $r+s$ vanishes then it is possible to show that $a_1 a_n = 0$. Let's see the case $\alpha=\beta$ and $r+s=0$ in more details. We can assume $\alpha=1$.

  If $n \geq 2$, it is easily verified that the natural projection $L_n \to L_2$ gives a surjective homomorphism from B_n onto B_2. Therefore, if (P_3) does not hold for $n=2$, it will not hold for any $n \geq 2$.

  Let $<$ be the only deglex-ordering such that $a_1^+ < a_1 < a_2^+ < a_2$. The reduction system $S = \{a_2 a_1 \to a_1 a_2, a_2^+ a_1^+ \to a_1^+ a_2^+, {a_1}^2 \to 0, {a_1^+}^2 \to 0, {a_2}^2 \to 0, {a_2^+}^2 \to 0, a_2 a_1^+ \to a_1^+ a_2, a_2^+ a_1 \to a_1 a_2^+, a_2 a_2^+ \to 1 - a_1 a_1^+ - a_2^+ a_2 - a_1^+ a_1\}$ is confluent and adapted to $<$. If we call T the basis of irreducible monomials and $T_0 := T \cap B_2^0$, then it is clear that $T_0 = \{1, a_2^+ a_2, (a_1^+ a_1)^k, (a_1^+ a_1)^k a_2^+ a_2, (a_1 a_1^+)^k, (a_1 a_1^+)^k a_2^+ a_2) | k \geq 1\}$. Indeed, no a_2 can be on the left of another term, an a_2^+ can only be on the left of an a_2, etc... Now, for $k \geq 1$: $[(a_1^+ a_1)^k, a_1^+] = (a_1^+ a_1)^k a_1^+$, $[(a_1^+ a_1)^k a_2^+ a_2, a_1^+] =$

$(a_1^+a_1)^k a_1^+ a_2^+ a_2$, $[(a_1a_1^+)^k a_2^+ a_2, a_1^+] = -a_1^+(a_1a_1^+)^k a_2^+ a_2$, $[(a_1a_1^+)^k, a_1^+] = -a_1^+(a_1a_1^+)^k$ and $[a_2^+a_2, a_1^+] = 0$.
We see that a_1^+ never appears in these commutators. Consequently, (P_3) cannot hold.

- A_3 : From F we have $r+s=0$, then :

$$\begin{array}{rcl} a_1\dots a_n & = & a_1a_1^+a_1\dots a_n + \beta\sum_{i\neq 1} a_i^+a_ia_1\dots a_n \\ & = & a_1a_1^+a_1\dots a_n \\ & = & a_2a_2^+a_1\dots a_n \\ & = & a_1a_2a_2^+a_2\dots a_n \\ & = & a_1(a_1a_1^+)a_2\dots a_n \\ & = & 0 \end{array}$$

  So by induction $B=0$.

- E_2 : From F, we can assume that $r+s=0$ and $\beta_1+(n-1)\beta_2=n$ (which corresponds to $\beta=1$ in the case F).
  $a_1(l_1)-(l_1)a_1$ gives :

$$(\beta_1-1)a_1a_1^+a_1 + \beta_2\sum_{i\neq 1}(a_1a_i^+ - a_i^+a_1)a_i = 0$$

$$\Rightarrow (\beta_1-1)a_1a_1^+a_1 = 0 \tag{11}$$

  But we also have :

$$\begin{aligned} a_1(l_n)-(l_n)a_1 = a_1a_na_n^+ + \beta_1a_1a_n^+a_n + \beta_2\sum_{1<i<n} a_1a_i^+a_i - a_na_n^+a_1 \\ +\beta_2a_1a_1^+a_1 - \beta_1a_n^+a_na_1 - \beta_2\sum_{1<i<n} a_i^+a_1a_i \end{aligned}$$

$$\Rightarrow 0 = \beta_2a_1a_1^+a_1 \tag{12}$$

  - if $\beta_2\neq 0$, (12) $\Rightarrow$ $a_1a_1^+a_1=0$. Thanks to $a_1\dots a_n(l_i)$ we get $a_1\dots a_n=0$ and by induction $B=0$.
  - if $\beta_2=0$, we can assume $\beta_1=1$, or else $B=0$ by (11). We get the reduction system $\{{a_i}^2\to 0, {a_i^+}^2\to 0, a_ja_i\to a_ia_j, a_j^+a_i^+\to a_i^+a_j^+, a_ia_j^+\to a_j^+a_i, a_ia_i^+\to 1-a_i^+a_i | 1\leq i<j\leq n\}$. It is confluent and adapted to $<_n$. B is a solution to our problem, with $N_i = a_i^+a_i+\lambda_i.1$, $\lambda_i\in K$, and we are in case (a) of theorem 1.

$\underline{(1,1)_a \cup (1,-1)_a \cup (2,0)}$:
This case is similar to the preceding one. In the case E_2, we find the solution (a').
$\underline{(1,1)_c \cup (1,-1)_b}$:

- C : Let $\phi : L_n \to C$, with $C = \bigoplus_{i=1}^{n} K[x_i, y_i]/\langle x_i y_i - 1\rangle$, defined by : $\phi(a_i) = x_i$, $\phi(a_i^+) = y_i$. ϕ goes to the quotient, indeed : $\forall i \neq j$, $\phi(a_i a_j) = x_i x_j = 0$, $\phi(a_i^+ a_j^+) = y_i y_j = 0$, $\phi(a_i a_j^+) = \phi(a_j^+ a_i) = x_i y_j = 0$, $\phi(\sum a_i a_i^+ - 1) = (\sum x_i y_i) - 1 = 0$. But $\phi \neq 0$ then by lemma 5, (P_3) does not hold.

- $C \cup D$: We have $(C)a_1 \Rightarrow a_1 a_1^+ a_1 = \lambda a_1$ and $a_1(D) \Rightarrow a_1 a_1^+ a_1 = \mu a_1$, then if $\lambda \neq \mu$, $B = 0$. If $\lambda = \mu$, we do as above, using the same ϕ.

- A_2, $\lambda = 0$, $\cup D$: Computing $(D)a_1$ we are led to ${a_1}^2 = 0$.

- A_3 or $A_1 \cup D$: With $\alpha = 1$ or 0 :

$$\alpha a_1 a_1^+ + \beta \sum a_i^+ a_i = 1$$
$$\Rightarrow \alpha {a_1}^2 a_1^+ + \beta a_1 a_1^+ a_1 = a_1$$
$$\Rightarrow \alpha a_1 a_2 a_2^+ + \beta a_1 a_1^+ a_1 = a_1$$
$$\Rightarrow \beta a_1 a_1^+ a_1 a_1^+ = a_1 a_1^+$$
$$\Rightarrow \beta a_1 a_1^+ a_2 a_2^+ = 0 = a_1 a_1^+$$

  This is impossible in both cases. We can do the same for $A_1 \cup B_1 \cup F$.

- E_2 or E_2' : Let's calculate $a_1(l_1)$, $(l_1)a_1$, $a_1(l_n)$, and $(l_n)a_1$. We get :

$$\begin{cases} {a_1}^2 a_1^+ + \beta_1 a_1 a_1^+ a_1 = a_1 & (i) \\ a_1 a_1^+ a_1 + \beta_1 a_1^+ {a_1}^2 = a_1 & (ii) \\ \beta_2 a_1 a_1^+ a_1 = a_1 & (iii) \\ \beta_2 a_1^+ {a_1}^2 = a_1 & (iv) \end{cases}$$

  Thanks to the three last formulas we get $(\beta_2 - \beta_1 - 1)a_1 = 0$.
  So, if $\beta_2 - \beta_1 - 1 \neq 0$, i.e. $\beta_1 \neq \frac{n-1}{n}$ in the case E_2', $B = 0$.
  Moreover, (iii) and (iv) $\Rightarrow \beta_2 \neq 0$, or else $B = 0$.
  The only remaining case is $\beta_2 = \beta_1 + 1 \neq 0$. We define $\phi : B \to$

$\bigoplus_{i=1}^{n} K[x_i, y_i]/\langle x_i y_i - 1/\beta_2 \rangle$ by $\phi(a_i) = x_i$, $\phi(a_i^+) = y_i$. ϕ is well defined since $\phi(a_i a_i^+ + \beta_1 a_i^+ a_i + \beta_2 \sum_{j \neq i} a_j^+ a_j) = (1+\beta_1) x_i y_i + \beta_2 \sum_{j \neq i} x_j y_j = 1$. Thus (P_3) cannot hold.

- $E_1 \cup C \cup D$:
$$\begin{cases} a_1 a_1^+ + \ldots + a_n a_n^+ = \lambda & (l_1) \\ a_1^+ a_1 + \ldots + a_n^+ a_n = \mu & (l_2) \\ \alpha a_i a_i^+ + \beta a_i^+ a_i = \nu & (l_{3,i}) \end{cases}$$
We must have $\lambda = \mu$ by $C \cup D$. Now, if we multiply $(l_{3,i})$ by a_j with $j \neq i$, we find $\nu a_j = 0$, therefore $\nu = 0$. But $\lambda(\alpha + \beta) = n\nu = 0$ and λ must be non-zero by lemma 4, then $\alpha + \beta = 0$. We can assume that $\lambda = 1$ and $\alpha = -\beta = 1$. We then use the same ϕ as in case E_2.

- F :
$$\alpha \sum a_i a_i^+ - \beta \sum a_i^+ a_i = 1$$
$$\Rightarrow \alpha {a_i}^2 a_i^+ - \beta a_i a_i^+ a_i = a_i$$
Now (P_4) implies that there must always exist i s.t. $a_i a_i^+$ and $a_i^+ a_i$ are irreducible. A contradiction.

$\underline{(1,1)_a \text{ or } {}_b \cup (1,-1)_b}$: Such a presentation is never standard. Indeed, we always have relations of the form :

$$\sum_{1 \leq i \leq n} \alpha_i a_i a_i^+ + \sum_{1 \leq i \leq n} \beta_i a_i^+ a_i + \lambda = 0$$

with $\lambda \neq 0$. Therefore :

$$a_1 \sum_{1 \leq i \leq n} \alpha_i a_i a_i^+ + \beta_1 a_1 a_1^+ a_1 + \lambda a_1 = 0$$

$$\Rightarrow \alpha_2 a_1 a_2 a_2^+ a_2 + \beta_1 a_1 a_1^+ a_1 a_2 + \lambda a_1 a_2 = 0$$
$$\Rightarrow \pm \alpha_2 a_2 (a_1 a_2^+) a_2 \pm \beta_1 a_1 (a_1^+ a_2) a_2 + \lambda a_1 a_2 = 0$$
$$\Rightarrow \lambda a_1 a_2 = 0$$
$$\Rightarrow a_1 a_2 = 0$$

This is impossible.

$\underline{(1,1)_c \cup (1,-1)_a}$: This case is very easy, and we only state the results.

- C, $C \cup D$, $A_1 \cup D$: Multiplying the relation C or D by a_i, one can prove that the presentation is not standard.

- A_3 or $A_1 \cup B_1 \cup F$: It can be proved that ${a_i}^2 = 0$, thus the presentation is not standard.

- E_2 or E_2' : It can be shown that :

$$rs(\beta_2 - \beta_1 - 1)a_1 = 0$$

  Therefore we have $rs(\beta_2 - \beta_1 - 1) = 0$. If $\beta_2 - \beta_1 - 1 = 0$ the homomorphism ϕ of $(1,1)_c \cup (1,-1)_b \cup E_2$ can be used to exclude this case. If $rs = 0$ and $\beta_2 - \beta_1 - 1 \neq 0$, the presentation is shown to be non-standard.

- F : We have, $\forall i$: $\alpha {a_i}^2 a_i^+ a_i + \beta a_i a_i^+ {a_i}^2 = {a_i}^2$, so we conclude that the presentation is not standard.

$\underline{(1,1)_b \cup (1,-1)_a}$:

- A_2 :

$$\begin{aligned} ra_2 = ra_2 a_1 a_1^+ = ra_1 a_2 a_1^+ \\ \Rightarrow -sa_1 a_1^+ a_2 = ra_2 \\ \Rightarrow (r+s)a_2 = 0 \end{aligned}$$

  Thus $r + s \neq 0 \Rightarrow B = 0$. If $r + s = 0$ an homomorphism $\delta : B \to C$, with $C = K[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle x_1 y_1 - 1, \ldots, x_n y_n - 1\rangle$, is defined by setting $\delta(a_i) = x_i$, $\delta(a_i^+) = y_i$. Therefore (P_3) cannot hold, by lemma 5. We do the same for $A_2 \cup B_1$ and $A_2 \cup B_2$.

- A_3, $A_1 \cup D$ and $A_2 \cup D$: In these three cases we have :

$$\begin{cases} a_i a_i^+ = a_j a_j^+ \\ \alpha a_i a_i^+ + \beta \sum_j a_j^+ a_j = 1 \end{cases}$$

  with $\alpha = 0$ for $A_2 \cup D$ and $A_1 \cup D$, and $\alpha = 1$ for A_3.
  If we set $x = a_1 a_1^+ = \ldots = a_n a_n^+$, we easily show that $\forall i$, $ra_i x + sxa_i = 0$, and $\forall i \neq 1$, $ra_1 a_i^+ a_i + sa_i^+ a_i a_1 = 0$. Then :

$$\begin{aligned}
&\beta(ra_1a_1^+a_1 + sa_1^+{a_1}^2) = (r+s)a_1 \\
\Rightarrow\ &\beta(r{a_1}^2a_1^+a_1 + sa_1a_1^+{a_1}^2) = (r+s){a_1}^2 \\
\Rightarrow\ &\beta(ra_1x + sxa_1)a_1 = (r+s){a_1}^2 \\
\Rightarrow\ &0 = (r+s){a_1}^2
\end{aligned}$$

Therefore $r+s=0$. Let's show that $x := a_ia_i^+$ and $y_i := a_i^+a_i$ are central elements of B. For x it is trivial : we calculate the commutator of x with a_k or a_k^+ by writing $x = a_ia_i^+$ with $i \neq k$. Now y_i clearly commutes with a_j and a_j^+ for $j \neq i$. As for $j = i$, we just need to write

$$y_i = \frac{1}{\beta} - \frac{1}{\beta}x - \sum_{k\neq i} a_k^+a_k$$

Now we show that $B^0 \subset Z(B)$. Let $m \in B^0$ be a monomial, and write m as $m = b_1 \dots b_{2l}$, with $b_i = a_i$ or a_i^+. There must exist $i < j$ such that $b_j = b_i^+$ $(:= J(b_i))$ and $\forall k$, $i < k < j$, $b_k \neq b_i$ and $b_k \neq b_i^+$. Then b_i commutes with every b_k s.t. $i < k < j$, and we can write :

$$\begin{aligned}
m &= b_1 \dots b_{i-1}b_ib_{i+1} \dots b_{j-1}b_jb_{j+1} \dots b_{2l} \\
&= b_1 \dots b_{i-1}b_{i+1} \dots b_{j-1}(b_ib_j)b_{j+1} \dots b_{2l}
\end{aligned}$$

Now $b_ib_j = b_ib_i^+ = x$ or y_i, thus $b_ib_i^+ \in Z(B)$. Consequently $m = b_1 \dots b_{i-1}b_{i+1} \dots b_{j-1}b_{j+1} \dots b_{2l}(b_ib_i^+)$. Let $m = m'(b_ib_i^+)$, with $b_ib_i^+ \in Z(B)$ and $m' \in B^0$. By an easy induction, we find $m \in Z(B)$. Then $B^0 \subset Z(B)$ (the other inclusion is always true). As a consequence, (P_3) cannot hold (unless $B = 0$).

- $A_1 \cup B_1 \cup F$: As above, we must have $r+s=0$. If $\alpha+\beta \neq 0$, lemma 5 can be used, with the help of $\gamma : B \to K[x,y]/\langle xy - 1/(\alpha+\beta)\rangle$, defined by $\gamma(a_i) = x$, $\gamma(a_i^+) = y$. If $\alpha+\beta = 0$ we have (with $\alpha = 1$):

$$\begin{aligned}
&a_1a_1^+a_1 - a_1^+{a_1}^2 = a_1 \\
\Rightarrow\ &a_1a_i^+a_i - a_1^+{a_1}^2 = a_1,\ i \neq 1 \\
\Rightarrow\ &a_i^+a_ia_1 - a_1^+{a_1}^2 = a_1 \\
\Rightarrow\ &a_1^+{a_1}^2 - a_1^+{a_1}^2 = 0 = a_1 \\
\Rightarrow\ &B = 0
\end{aligned}$$

- C : We do as in case $(1,1)_c \cup (1,-1)_b \cup C$.
- $C \cup D$:

$$ra_1(C) + s(C)a_1 \Rightarrow r{a_1}^2 a_1^+ + sa_1 a_1^+ a_1 = \lambda(r+s)a_1 \tag{13}$$

$$ra_1(D) + s(D)a_1 \Rightarrow ra_1 a_1^+ a_1 + sa_1^+ {a_1}^2 = \mu(r+s)a_1 \tag{14}$$

From these two relations we get :

$$\begin{cases} r{a_1}^2 a_1^+ a_1 + sa_1 a_1^+ {a_1}^2 = \lambda(r+s){a_1}^2 \\ r{a_1}^2 a_1^+ a_1 + sa_1 a_1^+ {a_1}^2 = \mu(r+s){a_1}^2 \end{cases}$$

$$\Rightarrow (r+s)(\lambda-\mu){a_1}^2 = 0$$

This shows that $(r+s)(\lambda-\mu) = 0$.

  - $\lambda = \mu$: We can use ϕ as in case C.
  - $\lambda \neq \mu, r+s = 0$ then (13) and (14) $\Rightarrow {a_1}^2 a_1^+ = a_1 a_1^+ a_1 = a_1^+ {a_1}^2$. Thus :

$$\begin{array}{rcl} \lambda a_1 \ldots a_n & = & a_1 \ldots a_n \sum a_i a_i^+ \\ & = & \sum {a_i}^2 a_i^+ a_1 \ldots a_{i-1} a_{i+1} \ldots a_n \\ & = & \sum a_i^+ a_i (a_1 \ldots a_n) \\ & = & \mu a_1 \ldots a_n \end{array}$$

    Therefore $a_1 \ldots a_n = 0$, and, by induction : $B = 0$.

- E_2 or E_2' : On one hand $ra_1(l_1) + s(l_1)a_1$ gives

$$(r+s)a_1 = s\beta_1 a_1^+ {a_1}^2 + r{a_1}^2 a_1^+ + (s + r\beta_1)a_1 a_1^+ a_1 \tag{15}$$

on the other hand $ra_1(l_2) + s(l_2)a_1$ gives :

$$s\beta_2 a_1^+ {a_1}^2 + r\beta_2 a_1 a_1^+ a_1 = (r+s)a_1 \tag{16}$$

¿From (16) we get :

$$\beta_2 = 0 \Rightarrow r+s = 0$$

Furthermore :

$$\begin{cases} \beta_2(15) - \beta_1(16) \\ \quad (16) \end{cases} \Leftrightarrow (S) : \begin{cases} \beta_2 a_1 (ra_1 a_1^+ + sa_1^+ a_1) = (\beta_2 - \beta_1)(r+s)a_1 \\ \beta_2 (ra_1 a_1^+ + sa_1^+ a_1)a_1 = (r+s)a_1 \end{cases}$$

Now :

$$(S) \Rightarrow \begin{cases} \beta_2 a_1(r a_1 a_1^+ + s a_1^+ a_1)a_1 = (\beta_2 - \beta_1)(r+s){a_1}^2 \\ \beta_2 a_1(r a_1 a_1^+ + s a_1^+ a_1)a_1 = (r+s){a_1}^2 \end{cases}$$

$$\Rightarrow (\beta_2 - \beta_1 - 1)(r+s){a_1}^2 = 0$$

Then we must have $(r+s)(\beta_2 - \beta_1 - 1) = 0$.

- $\beta_2 - \beta_1 - 1 = 0$:
  - $\beta_2 \neq 0$, we define $\rho : L_n \to C$, with $C = K[x,y]/\langle xy - \frac{1}{\beta_2}\rangle$, such that $\rho(a_1) = x$, $\rho(a_1^+) = y$, and $\rho(a_i) = \rho(a_i^+) = 0$, $\forall i > 1$. We then see that ρ is well defined non-zero homomorphism to C.
  - $\beta_2 = 0$ (which entails $r + s = 0$), $\beta_1 = -1$. We then have the presentation $\{a_i a_j - a_j a_i, a_i^+ a_j^+ - a_j^+ a_i^+, a_i a_j^+ - a_j^+ a_i, a_i a_i^+ - a_i^+ a_i - 1\}$. We recognize the Weyl algebra A_n, which is a well known solution to our problem, with $N_i = a_i^+ a_i + \lambda_i.1$, $\lambda_i \in K$.
- $1 + \beta_1 - \beta_2 \neq 0$, $r + s = 0$

  $$(S) \Leftrightarrow \begin{cases} \beta_2 a_1[a_1, a_1^+] = 0 \\ \beta_2[a_1, a_1^+]a_1 = 0 \end{cases}$$

  where $[.,.]$ denotes the commutator.
  - $1+\beta_1+(n-1)\beta_2 \neq 0$ (always true in the case E_2') : We can then define a homomorphism ξ from B to $C = K[x_1,\ldots,x_n]/\langle x_i^2 - \dfrac{1}{1+\beta_1+(n-1)\beta_2}\rangle$ by setting : $\xi(a_i) = \xi(a_i^+) = x_i$.
  - $1 + \beta_1 + (n-1)\beta_2 = 0$

    $$(l_1)+\ldots+(l_n) \Leftrightarrow \sum a_i a_i^+ + \beta_1 \sum a_i^+ a_i + (n-1)\beta_2 \sum a_i^+ a_i = n$$

    $$\Leftrightarrow \sum a_i^+ a_i - \sum a_i a_i^+ = n$$

    $$\Leftrightarrow \sum [a_i, a_i^+] = -n$$

    But $\beta_2 \neq 0$, (or else $1 + \beta_1 - \beta_2 = 0$), then from (S) we get :

    $$-n a_1 \ldots a_n = \sum [a_i, a_i^+] a_1 \ldots a_n$$

$$-na_1 \dots a_n = \sum_i ([a_i, a_i^+] a_i a_1 \dots a_{i-1} a_{i+1} \dots a_n) = 0$$
$$\Rightarrow a_1 \dots a_n = 0$$

And we can iterate to get $B = 0$.

- $E_1 \cup C \cup D$: From $C \cup D$, the only remaining case to study is $\lambda = \mu$, that is to say $n\nu = (\alpha + \beta)\lambda$. Moreover (by the case E_2), $\beta_2 = 0$, we must have $r + s = 0$. Then we see that we can use ξ as in the case E_2, setting $\xi(a_i) = \xi(a_i^+) = x_i$, with $x_i^2 = \dfrac{\lambda}{n}$.
- F : We have :

$$\begin{aligned} (r+s)a_j &= ra_j(\alpha \textstyle\sum a_i a_i^+ + \beta \sum a_i^+ a_i) + s(\alpha \sum a_i a_i^+ + \beta \sum a_i^+ a_i)a_j \\ &= \alpha \textstyle\sum_{i \neq j} a_i (r a_j a_i^+ + s a_i^+ a_j) + \beta \sum_{i \neq j} (r a_j a_i^+ + s a_i^+ a_j) a_i \\ &\qquad + (r\beta + s\alpha) a_j a_j^+ a_j + r\alpha {a_j}^2 a_j^+ + s\beta a_j^+ {a_j}^2 \\ &= (r\beta + s\alpha) a_j a_j^+ a_j + r\alpha {a_j}^2 a_j^+ + s\beta a_j^+ {a_j}^2 \end{aligned}$$

  Since there always exists j such that the monomials in the last expression are irreducible, we must have $r\alpha = s\beta = 0$ and $r\beta + s\alpha = 0$, that entails that either r and s or α and β must be zero, which is impossible.

$(1,1)_a \cup (1,-1)_a$: This case is quite similar to the previous one, and we leave it to the reader. In the case E_2 we find the pseudo-boson solution.

$(1,1)_c \cup (2,0)$:

- A_3 : Let us distinguish between the different forms of $S^{(0,0)}$.
  - (a) On one hand we have ${a_j}^2 a_j^+ \to 0$, and on the other hand we find ${a_j}^2 a_j^+ \to (1-\beta) a_j - \beta \sum_{i \neq j} a_j a_i^+ a_i$, which is irreducible. This system cannot be confluent.
  - (b) : We can use the same argument by reducing ${a_{j_0}}^2 a_{j_0}^+$ in two different ways.
- $A_1 \cup B_1 \cup F$: From $(l_3)a_i$ and $a_i(l_3)$ we get $\alpha = \beta$. We set α to 1, and we define $\delta : L_n \to L_1 = K\langle a, a^+ \rangle$ by $\delta(a_i) = a$, $\delta(a_i^+) = a^+$. This δ induces a non-zero morphism from B to the Clifford algebra $\mathrm{Cl}(1,1) := K\langle a, a^+ \rangle / \langle a^2, {a^+}^2, aa^+ + a^+ a - 1 \rangle$. Indeed, $\delta(a_i a_i^+ - a_j a_j^+) = 0 = \delta(a_i^+ a_i - a_j^+ a_j)$, $\delta(a_i a_j) = a^2 = 0$, $\delta(a_i^+ a_j^+) = {a^+}^2 = 0$, $\delta(a_i a_i^+ + a_i^+ a_i) = aa^+ + a^+ a = 1$. We conclude by lemma 8.

- E_2 : $(l_1)a_2a_2^+ \Leftrightarrow a_1a_1^+a_2a_2^+ = \lambda a_2a_2^+$. Moreover, $a_1a_1^+(l_2) \Leftrightarrow a_1a_1^+a_2a_2^+ = \lambda a_1a_1^+$. Thus $a_ia_i^+ = a_ja_j^+$, which is not possible.

- $E_1 \cup C \cup D$: From C and D we have $\lambda = \mu = 0$. Then $n\nu = \alpha\lambda + \beta\mu = 0$, which is impossible.

- F : We have $\alpha\sum_i a_ia_i^+a_j - a_j = 0$ and $\beta\sum_i a_ja_i^+a_i - a_j = 0$, but at least one one of these two expressions is irreducible.

$\underline{(1,1)_b \cup (2,0)}$:

- C : $B = 0$ (see case $(1,1)_b \cup (1,-1)_a \cup (2,0)$).

- E_2 :

  - (a) : We have $({a_i}^2)a_i^+ \to 0$, and on the other hand $a_i(a_ia_i^+) \to \lambda(1-\beta_1)a_i + \beta_1\beta_2\sum_{\mathcal{I}} a_j^+a_ja_i + \beta_1\beta_2\sum_{\mathcal{J}} a_j^+a_ia_j - \beta_2\sum_{j\neq i} a_ia_j^+a_j$ where $\mathcal{I}$ is the set of indices j s.t. $a_j < a_i$ and $\mathcal{J}$ the set of all j's s.t. $a_j > a_i$. The last expression being irreducible, we come to a contradiction.
  - (b) : We do the same with $a_i{a_i^+}^2$.
  - (c) : $(n = 2)$ We have $a_j^+{a_j}^2 \to 0$, and also $(a_j^+a_j)a_j \to \frac{1}{\beta_2}(\lambda a_j - \beta_1a_i^+a_ia_j - a_ia_i^+a_j)$. The term $a_i^+a_ia_j$ may be reduced to $a_i^+a_ja_i$, but the whole expression cannot reduce to 0.
  - (d) : We have $a_i^+{a_i}^2 \to 0$ and $a_i^+{a_i}^2 \to \frac{1}{\beta_1}(\lambda(1-\frac{1}{\beta_1})a_i - \beta_2a_j^+a_ja_i + \frac{\beta_2}{\beta_1}a_ia_j^+a_j)$. Confluence implies that $\beta_1 = 1$, $\beta_2 = 0$. In this case, I is generated by the relations $a_1a_1^+ + a_1^+a_1 = \lambda$, $a_2a_2^+ + a_2^+a_2 = \lambda$, ${a_1}^2 = {a_2}^2 = {a_1^+}^2 = {a_2^+}^2 = 0$, $a_ia_j - a_ja_i = a_i^+a_j^+ - a_j^+a_i^+ = 0$, $\forall i \neq j$. We can thus send B onto $\mathrm{Cl}(1,1)$ by $a_i \mapsto a$, $a_i^+ \mapsto a^+$. We then use lemma 8.

- E_2' :

  - (a) : Same method as in (a) of case E_2.
  - (b) or (c) : We have $0 \leftarrow {a_{i_0}}^2a_{i_0}^+ \to \lambda a_{i_0} - \sum_{\mathcal{I}} a_{i_0}a_ia_i^+ - \sum_{\mathcal{J}} a_ia_{i_0}a_i^+$ with $\mathcal{I} = \{i | a_{i_0} < a_i\}$ and $\mathcal{J} = \{i | a_{i_0} > a_i\}$. This contradicts confluence.

- F :

- (a) : We can assume without loss of generality that $i_0 = n$, so that $a_n a_i \to a_i a_n$. One can then show that $0 \leftarrow {a_n}^2 a_n^+ \to \frac{\lambda}{\alpha}(1 - \frac{\beta}{\alpha}) a_n + \frac{\beta}{\alpha} \sum_{i<n} a_i a_i^+ a_n + \frac{\beta^2}{\alpha^2} \sum_{i<n} a_i^+ a_i a_n - \sum_{i<n} a_i a_n a_i^+ - \frac{\beta}{\alpha} \sum_{i<n} a_n a_i^+ a_i$, which is irreducible.
- (b) : Symmetrical computation with ${a_n^+}^2 a_n$.

- A_3 or $A_1 \cup B_1 \cup F$: It is easy to show that $a_1 \ldots a_n = 0$. In any case the normal form of $a_1 \ldots a_n$ looks like $a_{i_1} \ldots a_{i_n} \neq 0$. Therefore, (P_4) does not hold.

$\underline{(1,1)_a \cup (2,0)}$: This case is similar to the previous one.

$\underline{(1,-1)_b \cup (2,0)}$:

- A_3 : By multiplying $a_1 a_1^+ + \beta \sum_i a_i^+ a_i = 1$ to the left by a_1, we find $\beta a_2 a_2^+ a_1 = a_1$, thus $B = 0$. We use a similar method in cases $A_1 \cup D$, $A_2 \cup D$, and $A_1 \cup B_1 \cup F$.

- C or $C \cup D$: We can assume that $i_0 = n$. We have $0 \leftarrow {a_n}^2 a_n^+ \to \lambda a_n - a_n a_1 a_1^+ - \ldots - a_n a_{n-1} a_{n-1}^+$, consequently the reduction system cannot be confluent.

- E_2 :
  - $n \geq 3$: From $a_1(l_1)a_2$ we find $a_1 a_2 = \beta_1 a_1 a_1^+ a_1 a_2$, and from $a_1(l_3)a_2$, $a_1 a_2 = \beta_2 a_1 a_1^+ a_1 a_2$, thus $a_1 a_2 = 0$.
  - $n = 2$: In every case we can easily prove that the reduction system is not confluent.

- E_2' : Same methods as above.

- F : Let's assume we have a reduction system of type (a) (the case (b) is symmetrical), and suppose $i_0 = n$. We have $0 \leftarrow a_n a_n^+ a_1 \to - a_1 a_1^+ a_1 - \frac{\beta}{\alpha} \sum_{i>1} a_i^+ a_i a_1 + \frac{1}{\alpha} a_1$, which is irreducible.

- $E_1 \cup C \cup D$: $\forall i \in \mathcal{L}$, $0 \leftarrow a_i a_i^+ a_{i_0} \to \frac{1}{\alpha}(\nu a_{i_0} - \beta a_i^+ a_i a_{i_0})$, which is irreducible, and $\forall i \in \mathcal{K}$, $0 \leftarrow a_{i_0} a_j^+ a_j \to \frac{1}{\beta}(\nu a_{i_0} - \alpha a_{i_0} a_j a_j^+)$, irreducible too. Since at least one of two sets of indices is not empty, we conclude that the reduction system is not confluent.

$\underline{(1,-1)_a \cup (2,0)}$:

- $A_2 \cup D$, $\lambda = 0$: If $a_k^+ = \sup\{a_i^+\}$, we have $0 \leftarrow a_k^+ {a_k}^2 \rightarrow \mu a_k - \sum_{i \neq k} a_i^+ a_i a_k$, which is irreducible.

- C or $C \cup D$: see $(1,-1)_b \cup (2,0) \cup C$.

- A_3 : We know from proposition 2 that at most one of the $a_i^+ a_i$'s is reducible. But we have $0 \leftarrow {a_i}^2 a_i^+ \rightarrow (1-\beta) a_i + \beta^2 \sum_{j \neq i} a_j^+ a_j a_i - \beta \sum_{j \neq i} a_i a_j^+ a_j$. Now we can choose i such that $\forall j \neq i$, $a_j^+ a_j$ is irreducible. Then if $r = 0$, the last expression is irreducible. If $r \neq 0$, some of the $a_i a_j^+ a_j$'s can be reduced to $q a_j^+ a_i a_j$, but in any case we get a non-zero irreducible quantity.

- $A_1 \cup B_1 \cup F$: $a_i(l_{3,1}) a_j$ gives $a_i(\alpha a_1 a_1^+ + \beta a_1^+ a_1) a_j = \alpha a_i (a_i a_i^+) a_j + \beta a_i (a_j^+ a_j) a_j$, but this is excluded.

- $A_1 \cup D$: We have, $\forall i$, $a_i a_1 a_1^+ = a_i a_i a_i^+ = 0 \Rightarrow \mu a_1 a_1^+ = \sum a_i^+ a_i a_1 a_1^+ = 0$. A contradiction.

- E_2 :

  - (a) : By reducing ${a_1}^2 a_1^+$ in two different ways, one can prove that we must have $r \neq 0$, $\beta_2 = 0$ and $\beta_1 = 1$ for the system to be confluent. Thus we get $\{{a_i}^2 \rightarrow 0, {a_i^+}^2 \rightarrow 0, a_i a_j^+ \rightarrow q a_j^+ a_i, a_i a_i^+ \rightarrow 1 - a_i^+ a_i | 1 \leq i \neq j \leq n\}$ which is a confluent reduction system adapted to $<_n$. Let $\mathcal{I} = (i_1, \ldots, i_k) \in [1..n]^k$ be a k-uple of indices. $a_{\mathcal{I}}$ will stand for $a_{i_1} \ldots a_{i_k}$, and $a_{\mathcal{I}}^+$ for $a_{i_1}^+ \ldots a_{i_k}^+$, $|\mathcal{I}| = k$. If $\mathcal{I} = \emptyset$, we set $a_\emptyset = a_\emptyset^+ = 1$. With these notations, the basis of irreducible monomials for our reduction system is the set of all $a_{\mathcal{I}}^+ a_{\mathcal{J}}$, $\mathcal{I}$ and $\mathcal{J}$ running over all possible t-uple of indices such that $i_m \neq i_{m+1}$, $j_m \neq j_{m+1}$, $\forall m$. It is then possible to use this basis to explicitly calculate the commutator of an element of B^0 with an a_i and an a_i^+ (see [Bes1]). Doing this, one sees that (P_3) does not hold.
  - (b) : This case is symmetrical to the latter.
  - (c) : $0 \leftarrow a_j^+ {a_j}^2 \rightarrow \frac{1}{\beta_2}(\lambda a_j - \beta_1 a_i^+ a_i a_j - q' a_i a_j a_i^+)$, thus the reduction system is not confluent.
  - (d) : $a_i^+ {a_i}^2 \rightarrow \frac{1}{\beta_1}(\lambda(1 - \frac{1}{\beta_1}) a_i - \beta_2 a_j^+ a_j a_i + \frac{\beta_2}{\beta_1} a_i a_j^+ a_j)$. Now this last expression is irreducible, except for the term $a_i a_j^+ a_j$, which can possibly be reduced to $q a_j^+ a_i a_j$. So, we must have $\beta_1 = 1$ and $\beta_2 = 0$ and then $rs \neq 0$. Indeed, if $r = 0$ we have $0 \leftarrow a_j a_j^+ a_i \rightarrow \lambda a_i -$

$a_j^+ a_j a_i$, and if $s = 0$ we have $0 \leftarrow a_j a_i^+ a_i \rightarrow \lambda a_j - a_j a_i a_i^+$ both being impossible. There are two cases :

- $*$ If $a_i a_j^+$ is reducible, then $q a_i^+ a_j^+ a_i \leftarrow a_i^+ a_i a_j^+ \rightarrow \lambda a_j^+ - a_i a_i^+ a_j^+$. Since both expressions are irreducible, we conclude that the system is not confluent.
- $*$ If $a_j^+ a_i$ is reducible, then we have $q' a_j a_i a_j^+ \leftarrow a_j a_j^+ a_i \rightarrow \lambda a_i - a_j^+ a_j a_i$, and arrive at the same conclusion.

- E_2' : Case (b) or (c) $0 \leftarrow {a_{i_0}}^2 a_{i_0}^+ \rightarrow \lambda a_{i_0} - \sum_{j \neq i_0} a_{i_0} a_j a_j^+$, which is irreducible.

- $E_1 \cup C \cup D$: $\lambda' a_{i_0} + \frac{\beta}{\alpha} \sum_{\mathcal{L}} a_k^+ a_k a_{i_0} - \sum_{\mathcal{K}} a_k a_k^+ a_{i_0} \leftarrow a_{i_0} a_{i_0}^+ a_{i_0} \rightarrow \mu' a_{i_0} + \frac{\alpha}{\beta} \sum_{\mathcal{K}} a_{i_0} a_k a_k^+ - \sum_{\mathcal{L}} a_{i_0} a_k^+ a_k$ and the only further reductions we can possibly do are : $a_k a_k^+ a_{i_0} \rightarrow q' a_k a_{i_0} a_k^+$ and $a_{i_0} a_k^+ a_k \rightarrow q a_k^+ a_{i_0} a_k$. This shows that the two expressions cannot be reduced to a common normal form.

- F : Let's do the case (a), the other one being symmetrical. Suppose a_n is the largest of the a_i's, then $0 \leftarrow {a_n}^2 a_n^+ \rightarrow a_n (1 - \alpha \sum_{i<n} a_i a_i^+ - \beta \sum_{i \leq n} a_i^+ a_i)$. If $r = 0$ we see that confluence is impossible. And if $r \neq 0$, we have $0 = a_n - \alpha \sum_{i<n} a_n a_i a_i^+ - q\beta \sum_{i<n} a_i^+ a_i a_n$ and we come to the same conclusion.

$\underline{(1,-1)_b}$:

- A_3 : From $a_1 a_n = a_1 (\alpha a_1 a_1^+ + \beta \sum_i a_i^+ a_i) a_n$ we are led to $a_1 a_n = 0$. Thus, the presentation is not standard. The same method can be used for the cases $A_1 \cup D$, $A_2 \cup D$, $\lambda = 0$ and $A_1 \cup B_1 \cup F$.

- C, $C \cup D$, E_2, E_2', $E_1 \cup C \cup D$, F : These rather easy cases are left to the reader (it can be shown each time that the systems cannot be confluent).

$\underline{(1,-1)_a}$: Let us first notice that in all cases containing A_2 we have $r a_i a_j^+ + s a_j^+ a_i = 0 \Rightarrow r a_j a_i a_j^+ + s \lambda a_i = 0$. Then, if $r = 0$, $a_i = 0$, and (P_0) is not fulfilled. So $r \neq 0$, and $a_i a_j^+$ must be reducible.

- A_2 : According to the remark, we must have $r \neq 0$. Then we see that $S = \{a_i a_i^+ \rightarrow 1, a_i a_j^+ \rightarrow q a_j^+ a_i | i \neq j\}$ is confluent (there are no ambiguities) and adapted to $<_n$. The basis of irreducible monomials is

$T := \{a^+_{\mathcal{I}} a_{\mathcal{J}} | \mathcal{I}, \mathcal{J}$ run over all tuples of indices$\}$. If we write p for the projection on W =Span(T) in the direction of I, then $p(x)$ =nf(x). If the lengths of $\mathcal{I}$ and $\mathcal{J}$ are at least two, and if n belongs to $\mathcal{J}$, we have $p([a^+_{\mathcal{I}} a_{\mathcal{J}}, a^+_n]) = q^{n_{\mathcal{J}}} a^+_{\mathcal{I}} a_{\mathcal{J}'} - a^+_n a^+_{\mathcal{I}} a_{\mathcal{J}}$, where $n_{\mathcal{J}}$ is the number of a_i's in $a_{\mathcal{J}}$ that are to the right of the a_n which is the most to the right, and $\mathcal{J}' = \mathcal{J}$ where the n the most to the right has been left out. If n does not appear in $\mathcal{J}$, $p([a^+_{\mathcal{I}} a_{\mathcal{J}}, a^+_n]) = q^{|\mathcal{J}|} a^+_{\mathcal{I}} a^+_n a_{\mathcal{J}} - a^+_n a^+_{\mathcal{I}} a_{\mathcal{J}}$. In both cases, if $\mathcal{I} \neq (n, \ldots, n)$, we have $\mathrm{lm}(p([a^+_{\mathcal{I}} a_{\mathcal{J}}, a^+_n])) = a^+_n a^+_{\mathcal{I}} a_{\mathcal{J}}$. Now if $a^+_{\mathcal{I}} a_{\mathcal{J}} \in L^0_n$, $\mathcal{I} = (n, \ldots, n) \Rightarrow \mathcal{J} = (n, \ldots, n)$, then if $a^+_{\mathcal{I}} a_{\mathcal{J}} \in L^0_n$ we always have $\mathrm{lm}(p([a^+_{\mathcal{I}} a_{\mathcal{J}}, a^+_n])) = a^+_n a^+_{\mathcal{I}} a_{\mathcal{J}}$. Let $\tilde{N}_1$ be a representative of N_1 in W, and let $a^+_{\mathcal{I}} a_{\mathcal{J}}$ be the leading term of $\tilde{N}_1$. Then we see that $a^+_n a^+_{\mathcal{I}} a_{\mathcal{J}}$ is bigger than any other term of $p([\tilde{N}_1, a^+_n])$, thus $\mathrm{lm}(p([\tilde{N}_1, a^+_n])) = a^+_n a^+_{\mathcal{I}} a_{\mathcal{J}}$, which proves that (P_3) does not hold.

- $A_2 \cup B_1$, $A_2 \cup B_2$: As before, we have $r \neq 0$. For $i \neq i_0$, we get $a_i = (a_i a^+_i) a_i = a_i (a^+_{i_0} a_{i_0}) = q a^+_{i_0} a_i a_{i_0}$. Then $a_i - q a^+_{i_0} a_i a_{i_0} = 0$, which is irreducible.

- $A_2 \cup D$: It can readily be seen that we need $\lambda = \mu$ and $q = 0$ in order to have a confluent system. In this case, an argument like the one we used in case A_2 allows us to prove that (P_3) is not satisfied.

- A_3 :

  - If $r \neq 0$: $S = \{a_i a^+_i {\rightarrow} 1 - \beta \sum a^+_j a_j, a_i a^+_j {\rightarrow} q a^+_j a_i | i \neq j\}$ is confluent and adapted to $<_n$ (no ambiguities). Using the basis $T = \{a^+_{\mathcal{I}} a_{\mathcal{J}}\}$ it can be shown that (P_3) does not hold (see [Bes1]).
  - si $r = 0$: Let's look at the two possible reduction systems :
    * (a) : we have $0 {\leftarrow} a_n a^+_n a_1 {\rightarrow} a_1 - \beta \sum_i a^+_i a_i a_1$, and there is no confluence.
    * (b) : For $i \neq i_0$, $0 {\leftarrow} a_i a^+_i a_{i_0} {\rightarrow} a_{i_0} a^+_{i_0} a_{i_0}$, which is irreducible.

- $A_1 \cup B_1 \cup F$:

  - (a) We have $a_{i_0} a^+_{i_0} a_i \leftarrow a_i a^+_i a_i \rightarrow \dfrac{1}{\beta}(a_i - \alpha a_i a_{i_0} a^+_{i_0})$. The term on the left may be reducible to $q' a_{i_0} a_i a^+_{i_0}$. In any case, the ambiguity is not solvable.

  - (b) : With $i \neq i_0$ we have $\frac{1}{\alpha}(a_i - \beta a_{i_0}^+ a_{i_0} a_i) \leftarrow a_i a_i^+ a_i \rightarrow a_i a_{i_0}^+ a_{i_0}$. We conclude as before.

- $A_1 \cup D$: If $a_i a_{j_0}^+$ is reducible then $q a_{j_0}^+ a_i a_{j_0} \leftarrow a_i a_{j_0}^+ a_{j_0} \rightarrow a_i - \sum_{j \neq j_0} a_i a_j^+ a_j$. Now these two expressions cannot be reduced to a common form, so $\forall i \neq j_0$, $a_i a_{j_0}^+$ is irreducible. Consequently $a_{j_0}^+ a_i$ is reducible.

  - If $n \geq 3$, or $n = 2$ and $i_0 = j_0$, we can take $i \neq i_0, j_0$. We have $q' a_i a_{j_0}^+ a_i^+ \leftarrow a_{j_0}^+ a_i a_i^+ \rightarrow a_{j_0}^+ a_{i_0} a_{i_0}^+$. Then if $i_0 \neq j_0$, $a_{j_0}^+ a_{i_0} a_{i_0}^+ \rightarrow q' a_{i_0} a_{j_0}^+ a_{i_0}^+$ and the reduction system is not confluent. If $i_0 = j_0$, $a_{i_0}^+ a_{i_0} a_{i_0}^+ \rightarrow a_{i_0}^+ - \sum_{j \neq i_0} a_j^+ a_j a_{i_0}^+$, which is irreducible.
    We leave the case $n = 2$ and $i_0 \neq j_0$ to the reader (the system is not confluent).

- C : As in case $(1,1)_c \cup (1,-1)_b$ the homomorphism ϕ can be defined and used to rule out this case.

- $C \cup D$:

  - If $\exists k \neq j_0$ such that $a_k a_{j_0}^+$ is reducible, then $q a_{j_0}^+ a_k a_{j_0} \leftarrow a_k a_{j_0}^+ a_{j_0} \rightarrow \mu a_k - q \sum_{j \in \mathcal{K}} a_j^+ a_k a_j - \sum_{j \in \mathcal{L}} a_k a_j^+ a_j - a_k a_k^+ a_k$, with $\mathcal{K} = \{j \neq j_0 | a_k a_j^+$ reducible$\}$, and $\mathcal{L} = \{j \neq j_0 | a_k a_j^+$irreducible$\}$. We see that the expression is irreducible if $k \neq i_0$. If $k = i_0$, it can be reduced to $(\mu - \lambda) a_k - q \sum_{j \in \mathcal{K}} a_j^+ a_k a_j - \sum_{j \in \mathcal{L}} a_k a_j^+ a_j + \sum_{j \in \mathcal{K}'} a_j a_j^+ a_k + q' \sum_{j \in \mathcal{L}'} a_j a_k a_j^+$. Therefore we must have $\mu = \lambda$. But in this case we can use ϕ, as in case (C), and (P_3) is not satisfied. Thus we must have $\forall k \neq j_0$, $a_k a_{j_0}^+$ is irreducible, which entails $a_{j_0}^+ a_k$ is reducible.
  - If $\exists k \neq i_0$ such that $a_{i_0}^+ a_k$ is reducible, we can reduce $a_{i_0} a_{i_0}^+ a_k$ in two different ways, and come to a contradiction.
  - If $n > 2$, or $n = 2$ and $i_0 = j_0$.
    Let $k \neq i_0, j_0$, then $q' a_k a_{j_0}^+ a_{i_0}^+ \leftarrow a_{j_0}^+ a_k a_{i_0}^+ \rightarrow q a_{j_0}^+ a_{i_0}^+ a_k$. Since these two expressions are irreducible, this case is ruled out.
  - $n = 2$, $i_0 = 1$, $j_0 = 2$. We leave this case to the reader (reduce $a_2^+ a_2 a_1^+$ in two different ways).

- E_2 or E_2':

- If $r \neq 0$, $S = \{a_i a_i^+ \to 1 - \beta_1 a_i^+ a_i - \beta_2 \sum a_j^+ a_j, a_i a_j^+ \to q a_j^+ a_i | i \neq j\}$ is confluent and adapted to $<_n$. $T = \{a_{\mathcal{I}}^+ a_{\mathcal{J}}\}$ is the corresponding basis of irreducible monomials. By using the same method as in case A_3, we find that (P_3) does not hold.
- If $r = 0$, $S = \{a_i^+ a_j \to 0, a_i^+ a_i \to 1 - \alpha_1 a_i a_i^+ - \alpha_2 \sum_{k \neq j} a_k a_k^+\}$ is confluent and adapted to $<_n$. The corresponding basis is $T = \{a_{\mathcal{I}} a_{\mathcal{J}}^+\}$. If $\mathcal{I}$ does not begin with n, we have $[a_{\mathcal{I}} a_{\mathcal{J}}^+, a_n^+] = a_{\mathcal{I}} a_{\mathcal{J}}^+ a_n^+$. Moreover, $[a_n a_{\mathcal{I}} a_{\mathcal{J}}^+, a_n^+] = a_n a_{\mathcal{I}} a_{\mathcal{J}}^+ a_n^+ - a_{\mathcal{I}} a_{\mathcal{J}}^+ + \alpha_1 a_n a_n^+ a_{\mathcal{I}} a_{\mathcal{J}}^+ + \alpha_2 \sum_{i<n} a_i a_i^+ a_{\mathcal{I}} a_{\mathcal{J}}^+$. Thus we see that if $\mathcal{I}$ does not begin with n, $\mathrm{lm}([a_n a_{\mathcal{I}} a_{\mathcal{J}}^+, a_n^+]) = a_n a_{\mathcal{I}} a_{\mathcal{J}}^+ a_n^+$, and we get by induction : if $\mathcal{I} \neq (n, \ldots, n)$, $\mathrm{lm}([a_n a_{\mathcal{I}} a_{\mathcal{J}}^+, a_n^+]) = a_n a_{\mathcal{I}} a_{\mathcal{J}}^+ a_n^+$, and $[{a_n}^k {a_n^+}^k, a_n^+] = (1 - (-1)^k \alpha_1^k) {a_n}^k {a_n^+}^{k+1} + u$, with $u < {a_n}^k {a_n^+}^{k+1}$. Then if $\mathrm{lm}(N_1) \neq {a_n}^k {a_n^+}^k$, $\mathrm{lm}([N_1, a_n^+]) = \mathrm{lm}(N_1) a_n^+$, which is irreducible : a contradiction. And if $\mathrm{lm}(N_1) = {a_n}^k {a_n^+}^k$, we show that $\mathrm{lm}([N_1, {a_1^+}^k]) = {a_n}^k {a_n^+}^k {a_1^+}^k$, which is irreducible as well. Thus (P_3) cannot hold.

- $E_1 \cup C \cup D$:

  - $r = 0$: If $i \in \mathcal{L}$, $0 = a_i a_i^+ a_j = \frac{1}{\alpha}(\nu a_j - \beta a_i^+ a_i a_j)$, contradiction. Thus $\mathcal{L} = \emptyset$. But now, with $j \neq i_0$, $a_{i_0} a_{i_0}^+ a_j = \lambda a_j - \sum_{j \neq i_0} a_i a_i^+ a_j = \lambda a_j - a_j a_j^+ a_j$, which also contradicts the confluence.
  - $s = 0$: Symmetrical case.
  - $rs \neq 0$: We have $a_{i_0} > a_j^+$, $\forall j \in \mathcal{L}$, and $a_{i_0}^+ > a_j$, $\forall j \in \mathcal{K}$. The reader can then easily verify that $a_{i_0}^+ a_{i_0} a_j^+$ can be reduced in two different ways which lead to distinct irreducible expressions.

- F :

  - $\alpha \neq -\beta$: We define an homomorphism $\phi : B \to \bigoplus K[x_i, y_i]/\langle\{x_i y_i - \frac{1}{\alpha+\beta}\}\rangle$, and we use lemma 5.
  - $\alpha = -\beta$: Suppose $\alpha = 1$. We have $\sum a_i a_i^+ - \sum a_i^+ a_i = 1$ If $r = 0$ then $a_i a_i^+ a_i - \sum_j a_j^+ a_j a_i = a_i$, and $a_i \sum_j a_j a_j^+ - a_i a_i^+ a_i = a_i$. But both expressions are irreducible. This is impossible. Now if $r \neq 0$, $S = \{a_n a_n^+ \to 1 + \sum a_i^+ a_i - \sum_{i<n} a_i a_i^+, a_i a_j^+ \to q a_j^+ a_i | i \neq j\}$ is confluent and adapted to $<_n$. The basis T is made of monomials of the form $a_{\mathcal{I}_1}^+ a_{\mathcal{J}_1} a_{i_1} a_{i_1}^+ a_{\mathcal{I}_2}^+ a_{\mathcal{J}_2} a_{i_2} a_{i_2}^+ \ldots$, with $i_1, i_2, \ldots \neq n$. If t is such a monomial we have $[t, a_n^+] = a_{\mathcal{I}_1}^+ a_{\mathcal{J}_1} \ldots (a_{i_j} a_{i_j}^+) \ldots a_{\mathcal{I}_k}^+ a_{\mathcal{J}_k} a_n^+ - a_n^+ a_{\mathcal{I}_1}^+ a_{\mathcal{J}_1} \ldots$ Set $t = \mathrm{lm}(\tilde{N}_1)$.

- * If $t = a_j^+ u$, $j \neq n$, then $\mathrm{lm}(p([t, a_j])) = a_j t$, thus (P_3) does not hold.
- * If $t = a_n^+ u$, then $\mathrm{lm}(p([t, a_n])) = a_{n-1} a_{n-1}^+ u$, (P_3) does not hold.
- * If $t = a_j u$, $\mathrm{lm}(p([t, a_n])) = a_n t$, (P_3) does not hold.

$\underline{(1,1)_c}$: This case is rather easy, and we only state the results.

- A_3, $A_3 \cup B_1$, $A_1 \cup D$, and $A_2 \cup D$: It is easily shown that ${a_i}^3 = 0$, but since ${a_i}^3$ is irreducible, this contradicts (P_4).

- C : Use the same ϕ as in case $(1,1)_c \cup (1,-1)_b$.

- $C \cup D$: If $\lambda = \mu$, we can do the same as above. If $\lambda \neq \mu$, it can be shown that ${a_i}^3 = 0$.

- E_2, E_2', F, $E_1 \cup C \cup D$: (P_4) does not hold.

$\underline{(1,1)_b}$:

- A_2, $A_2 \cup B_1$, $A_2 \cup B_2$: Let $\phi : B \to C$, $C := K[x,y]/\langle xy - 1\rangle$, be the only homomorphism such that $\phi(a_i) = x$, $\phi(a_i^+) = y$. ϕ is clearly well defined and we conclude by lemma 5.

- $A_2 \cup D$: With $i_0 = n$, we have $\lambda a_n \leftarrow a_n a_n^+ a_n \to \mu a_n + \sum_{i<n} a_n a_i^+ a_i$. Thus (P_4) cannot hold.

- A_3 :

  - $\beta \neq -1/n$: As in A_2 we can define $\phi : B \to C := K[x,y]/\langle xy - 1/(1+n\beta)\rangle$.
  - $\beta = -1/n$: We define $\phi : B \to A_1 = K\langle a, a^+\rangle/\langle aa^+ - a^+a - 1\rangle$ by setting $\phi(a_i) = a$, $\phi(a_i^+) = a^+$. We then use lemma 8.

- $A_1 \cup B_1 \cup F$: Same method as above.

- $A_1 \cup D$: We send B to $K[x,y]/\langle xy - 1/n\rangle$ by $a_i \mapsto x$, $a_i^+ \mapsto y$, and we use lemma 5.

- C : Here we send B to $\bigoplus_i K[x_i, y_i]/\langle x_i y_i - 1\rangle$, as in case $(1,1)_c \cup (1,-1)_b$.

- $C \cup D$: We are going to show that (P_3) does not hold if $n = 2$. We will then be able to deduce that (P_3) never holds with the help of lemma 6, since the projection $L_n \to L_2$ goes over to the quotient.
  $S = \{a_1 a_1^+ \to \lambda - a_2 a_2^+, a_2^+ a_2 \to \mu - a_1^+ a_1, a_1 a_2 \to a_2 a_1, a_2^+ a_1^+ \to a_1^+ a_2^+\}$ is confluent (no ambiguity) and adapted to the deglex ordering coming from $a_2 < a_1 < a_1^+ < a_2^+$. The corresponding basis is $T = \{{a_2}^{k_1} {a_1^+}^{l_1} \ldots {a_2}^{k_r} {a_1^+}^{l_r} {a_1}^{m_1} {a_2^+}^{n_1} \ldots {a_1}^{m_s} {a_2^+}^{n_s}\}$, with $k_i, l_i, m_i, n_i \geq 0$.
  Let $x \in T \cap L_2^{(0,0)}$, $x \neq 1$. Then x begins with an a_2 or an a_1^+, and ends with an a_1 or an a_2^+, therefore $x a_1^+$ is reducible, meanwhile $a_1^+ x$ is irreducible. Moreover $a_1^+ x > x a_1^+$, so $\mathrm{lm}(p([N_1, a_1^+])) = a_1^+ \mathrm{lm}(N_1)$. This is absurd : (P_3) cannot hold.

- E_2 or E_2' : This case looks like A_3. We have to use different kinds of morphisms depending on whether $\beta_1 + (n-1)\beta_2 + 1$ vanishes or not. We leave the details to the reader.

- $E_1 \cup C \cup D$: Let's deal with the case $\mathcal{K} \neq \emptyset$, the case $\mathcal{L} \neq \emptyset$ being symmetrical. Let i be such that $a_i = \min\{a_j | 1 \leq j \leq n\}$. We must have $i \neq i_0$. We get $a_{i_0}^+ a_i a_{i_0} \leftarrow a_{i_0}^+ a_{i_0} a_i \to \mu' a_i - \sum_{\mathcal{L}} a_j^+ a_i a_j + \frac{\alpha}{\beta} \sum_{\mathcal{K}} a_j a_j^+ a_i$ and this contradicts (P_4).

- F : Quotienting out by the ideal generated by $\sum a_i a_i^+ - \lambda_1$ and $\sum a_i^+ a_i - \lambda_2$, with $\lambda_1 + \lambda_2 = \lambda \neq 0$, we are brought back to case $C \cup D$, and we conclude by lemma 6.

<u>$(1,1)_a$</u> :

- A_2, $A_2 \cup B_1$, $A_2 \cup B_2$: Let i, j be s.t. $a_i > a_j$. Then $-a_j a_i a_j^+ \leftarrow a_i a_j a_j^+ \to a_i$ and (P_4) cannot hold.

- $A_2 \cup D$: We have $\lambda a_{i_0} \leftarrow a_{i_0} a_{i_0}^+ a_{i_0} \to \mu a_{i_0} - \sum_{i \neq i_0} a_{i_0} a_i^+ a_i$ and (P_4) cannot hold.

- $A_1 \cup D$: We define $\delta : B \to \mathrm{Cl}(n, 0) := K\langle x_1, \ldots, x_n \rangle / \langle x_i x_j + x_j x_i - \delta_{ij} \rangle$ by $\delta(a_i) = \delta(a_i^+) = \frac{1}{\sqrt{n}} x_i$, we then use lemma 7.

- A_3 :
  - (a) : Suppose that a_i is the sup of the a_k's, and choose $j \neq i$. We find : $-a_j a_i a_j^+ \leftarrow a_i a_j a_j^+ \to (1-\beta) a_i + \beta^2 \sum a_k^+ a_k a_i - \beta \sum_{k \neq i} a_i a_k^+ a_k$. This contradicts (P_4).

- (b) : If $n \geq 3$, let $i, j \neq i_0$ be s.t. $a_i < a_j$. Then $-a_i a_j a_i^+ \leftarrow -a_j a_i a_i^+ \rightarrow -a_{i_0} a_j a_{i_0}^+$, which contradicts (P_4). If $n = 2$, one can easily show that $i_0 \neq j_0$ leads to a contradiction. We can then assume that $i_0 = j_0 = 2$. If we define $<$ by $a_2 < a_1 < a_1^+ < a_2^+$, we see that $S = \{a_1 a_2 \rightarrow - a_2 a_1, a_2^+ a_1^+ \rightarrow - a_1^+ a_2^+, a_1 a_1^+ \rightarrow a_2 a_2^+, a_2^+ a_2 \rightarrow \lambda - \alpha a_2 a_2^+ - a_1^+ a_1\}$ is confluent and adapted to $<$.
  $T = \{{a_2}^{k_1} {a_1^+}^{l_1} \ldots {a_2}^{k_r} {a_1^+}^{l_r} {a_1}^{m_1} {a_2^+}^{n_1} \ldots {a_1}^{m_s} {a_2^+}^{n_s}\}$. We proceed exactly as in case $(1,1)_b \cup C \cup D$.

- $A_1 \cup B_1 \cup F$: It can easily be shown that neither the system (a) nor the system (b) is confluent.
- C or $C \cup D$: We do as in case $(1,1)_b$.
- E_2 or E_2' : Once again, if $n \geq 3$, we can easily show that (P_4) does not hold. If $n = 2$, we find a confluent reduction system, and we show that (P_3) does not hold.
- $E_1 \cup C \cup D$: See $(1,1)_b$ ((P_4) does not hold).
- F : See $(1,1)_b$ ((P_3) does not hold).

This is the end of the demonstration for the case $n \geq 2$. Let us know look at the case $n = 1$.
The only relations we can have are : $(2,0)$, (A_2), (B_2), and (F) (by lemma 1).
<u>$(2,0)$</u> :

- A_2 : $a^2 a^+ = 0 = a \Rightarrow B = 0$
- B_2 : Symmetrical to the above case.
- F : Multiplying $\alpha a a^+ + \beta a^+ a = 1$ to the right, then to the left by a we get $\alpha = \beta$, and we recognize the Clifford algebra C_1 which is a solution to our problem.

<u>A_2 or B_2</u> : We can send B to $C = K[x,y]/\langle xy - 1\rangle$, by a non-zero isomorphism, and we conclude by lemma 5.
<u>F</u> : If $\alpha + \beta \neq 0$, we can send B to $K[x,y]/\langle xy - (1/\alpha + \beta)\rangle$ and use lemma 5.
If $\beta = -\alpha$, we find the Weyl algebra A_1, which is the bosonic solution.

QED.

B End of Proof of Theorem 2

We define a well-ordering $<$ on X by $a_i < a_j \Leftrightarrow a_i^+ < a_j^+ \Leftrightarrow i < j$ and $a_i^+ < a_j$ for all $i, j \in \mathcal{I}_\alpha$. We call $<_n$ the deglex-ordering defined by $<$.

As we did in the finite case, we can easily get rid of the ideals that contain relations $(1,1)_c$, $(1,-1)_b$ or $(2,0)$, on the one hand, and A_2, on the other hand. Furthermore, the relations $(2,0)$ together with E or $A_1 \cup B_1 \cup E$ imply $\beta = 1$, as is easily seen by multiplying $a_0 a_0^+ + \beta a_0^+ a_0 = 1$ on the left, then on the right, by a_0.

$\underline{(1,1)_c \cup (1,-1)_b \cup (2,0)}$: The only case to study is E. If we multiply the relation E on the left by a_j, with $i \neq j$, we get $\lambda a_j = 0$. Thus $B = 0$, in contradiction with (P_0).

$\underline{(1,1)_{a \text{ or } b} \cup (1,-1)_b \cup (2,0)}$:

- E or $A_1 \cup B_1 \cup E$: Let us multiply $a_k a_k^+ + \beta a_k^+ a_k = \lambda$ on the left by a_i and on the right by a_j, with i, j, k distinct. We get $a_i a_j = 0$, and we are back to the previous case.

$\underline{(1,1)_c \cup (1,-1)_a \cup (2,0)}$:

- E, $E \cup A_1 \cup B_1$: We have $rs = 0$ (see the finite case F). Now if $s = 0$, we see that $\lambda a_i = a_i a_j a_j^+ + \beta a_i a_j^+ a_j$ vanishes if $i \neq j$. The case $r = 0$ is symmetrical.

$\underline{(1,1)_a \cup (1,-1)_a \cup (2,0)}$:

- E: We have (with i, j, k distinct) :

$$\lambda r a_i a_j^+ = r(a_i a_k a_k^+ a_j^+ + \beta a_i a_k^+ a_k a_j^+) = s \lambda a_i a_j^+$$

$$\Rightarrow (r - s) a_i a_j^+ = 0$$

If $a_i a_j^+ = 0$, we easily find that $B = 0$, thus we can assume $r - s = 0$. We also have :

$$\begin{cases} \lambda a_i = {a_i}^2 a_i^+ + \beta a_i a_i^+ a_i = \beta a_i a_i^+ a_i \\ \lambda a_i = a_i a_i^+ a_i + \beta a_i^+ {a_i}^2 = a_i a_i^+ a_i \end{cases}$$

So $\beta = 1$ by (P_0). Thus B is a fermionic algebra $\hat{C}_\alpha$.

- $E \cup A_1 \cup B_1$: Thanks to the previous case we have $r - s = 0$ and $\beta = 1$. We let the reader use the relations A_1 and B_1 to show that $B = 0$.

$\underline{(1,1)_b \cup (1,-1)_a \cup (2,0)}$: This case is similar to the preceding one, except that we find pseudo-fermionic instead of fermionic algebras.
$\underline{(1,1)_c \cup (1,-1)_b}$: We can do as in case $(1,1)_c \cup (1,-1)_b \cup (2,0)$.
$\underline{(1,1)_{a \text{ or } b} \cup (1,-1)_b}$: We must have $a_i a_j = 0$ (see the finite case).
$\underline{(1,1)_c \cup (1,-1)_a}$:

- E : We have : $ra_i a_j^+ = r(a_i a_k a_k^+ a_j^+ + \beta a_i a_k^+ a_k a_j^+) = -s\beta a_i a_k^+ a_j^+ a_k = 0$. It entails that $sa_j^+ a_i = 0$. If $a_i a_j^+ = 0$, we get $\lambda a_i = a_i a_j a_j^+ + \beta a_i a_j^+ a_j = 0$, thus $B = 0$. This is the same if $a_j^+ a_i = 0$.

$\underline{(1,1)_b \cup (1,-1)_a}$:

- A_2, $A_2 \cup B_1$, $A_2 \cup B_2$: If $r + s \neq 0$, we find $B = 0$ (see the finite case). If $r + s = 0$, we define $\delta : B \to C := K[(x_i)_{i \in \mathcal{I}_\alpha}, (y_i)_{i \in \mathcal{I}_\alpha}]/\langle x_i y_i - 1\rangle$ by $a_i \mapsto x_i$ and $a_i^+ \mapsto y_i$. Thus (P_3) cannot hold, by lemma 5.
- E : According to the finite case, we must have $r + s = 0$.
  - If $\beta + 1 = 0$: We find a bosonic algebra A_α.
  - If $\beta + 1 \neq 0$: We can non-trivially map B onto $K[(x_i)_{i \in \mathcal{I}_\alpha}]/\langle x_i^2 - 1/(1+\beta)\rangle$ by $a_i, a_i^+ \mapsto x_i$. Thus (P_3) cannot hold.
- $E \cup A_1 \cup B_1$: Of course we must also have $r + s = 0$. If $\beta + 1 = 0$, we have a quotient of a bosonic (Weyl) algebra, which is simple. Thus $B = 0$. If $\beta + 1 \neq 0$, one shows that $Z(B) = B^0$ (see the finite case). Thus (P_3) cannot hold.

$\underline{(1,1)_a \cup (1,-1)_a}$:

- A_2, etc...: If $r = 0$ we have $a_i a_i^+ a_j = a_j = 0 \Rightarrow B = 0$. So $r \neq 0$, and we have :
$$a_j a_i a_i^+ = -q a_i a_i^+ a_j$$
$$\Rightarrow a_j = -q a_j$$
Thus $q = -1$ or else $B = 0$. Now, we can define $\phi : B \to C$, with $C = K\langle (x_i)_{i \in \mathcal{I}_\alpha}\rangle/\langle x_i x_j + x_j x_i - 2\delta_{ij}\rangle$, by $a_i, a_i^+ \mapsto x_i$, and we conclude by lemma 7.

- E : $r = 0 \Rightarrow a_j = a_i a_i^+ a_j + \beta a_i^+ a_i a_j = 0 - \beta a_i^+ a_j a_i = 0 \Rightarrow B = 0$.
  We can then assume $r \neq 0$. We have :
  $$a_j(a_i a_i^+ + \beta a_i^+ a_i) = a_j$$
  $$\Rightarrow -q(a_i a_i^+ + \beta a_i^+ a_i)a_j = a_j$$
  $$\Rightarrow (1+q)a_j = 0$$
  Therefore $q = -1$.
  - $\beta \neq -1$: We define $\phi : B \to K\langle (x_i)_{i\in\mathcal{I}_\alpha}\rangle / \langle x_i x_j + x_j x_i - \frac{2}{1+\beta}\delta_{ij}\rangle$. We conclude as in case A_2.
  - $\beta = -1$: We have a pseudo-bosonic algebra $\hat{A}_\alpha$.
- $E \cup A_1 \cup B_1$: One can easily show that $r = s$ and $1 + \beta \neq 0$. We can then map B onto $C = K\langle x_i | i \in \mathcal{I}_\alpha\rangle / \langle x_i x_j + x_j x_i - 2\delta_{ij}\rangle$ and use lemma 7.

$(1,1)_c \cup (2,0)$: The only case to study is E. In this case $B = 0$ (see $(1,1)_c \cup (1,-1)_b \cup (2,0)$).

$(1,1)_{a \text{ or } b} \cup (2,0)$:

- E : As we said at the beginning of this section, we have $\beta = 1$. Therefore, we can define $\phi : B \to C_1 := K\langle a, a^+\rangle / \langle a^2, {a^+}^2, aa^+ + a^+a - 1\rangle$, by setting $a_i \mapsto a$, $a_i^+ \mapsto a^+$. We then use lemma 8.
- $E \cup A_1 \cup B_1$: $a_i a_j = a_i(a_i a_i^+ + \beta a_j^+ a_j)a_j = 0$, so we are back to the case $(1,1)_c$.

$(1,-1)_b \cup (2,0)$:

- E or $E \cup A_1 \cup B_1$: $\beta = 1$, and if $i \neq k \neq j$, $a_i(a_k a_k^+ + a_k^+ a_k)a_j = \lambda a_i a_j = 0$, and we are back to a previous case.

$(1,-1)_a \cup (2,0)$:

- E : $\beta = 1$. We can assume $r \neq 0$, the case $s \neq 0$ being symmetrical. $S = \{a_i a_i^+ \to 1 - a_i^+ a_i, a_i^2 \to 0, {a_i^+}^2 \to 0, a_i a_j^+ \to q a_j^+ a_i\}$ is a confluent reduction system, which is adapted to $<_n$. Let $\mathcal{J} = (j_1, \ldots, j_r)$ and $\mathcal{K} = (k_1, \ldots, k_s)$ be two tuples of indices. We write $a_{\mathcal{J}}^+ = a_{j_1}^+ \ldots a_{j_r}^+$ and

$a_{\mathcal{K}} = a_{k_1} \dots a_{k_s}$. By convention, $a_\emptyset = a_\emptyset^+ = 1$. With these notations, the basis of irreducible monomials is $T = \{a_{\mathcal{J}}^+ a_{\mathcal{K}}\}$, with $\mathcal{J}$, $\mathcal{K}$ running over all tuples such that $j_m \neq j_{m+1}$, $k_m \neq k_{m+1}$. The notation $k > \mathcal{J}$ shall mean that k is greater than all indices appearing in $\mathcal{J}$. So let $k > \mathcal{J}, \mathcal{K}$. We have :

$$\begin{aligned} [a_{\mathcal{J}}^+ a_{\mathcal{K}}, a_k] &= a_{\mathcal{J}}^+ a_{\mathcal{K}} a_k - a_k a_{\mathcal{J}}^+ a_{\mathcal{K}} \\ &= a_{\mathcal{J}}^+ a_{\mathcal{K}} a_k - q^{|\mathcal{J}|} a_{\mathcal{J}}^+ a_k a_{\mathcal{K}} \end{aligned}$$

Then if $\mathrm{lm}(N_i){=}a_{\mathcal{J}}^+ a_{\mathcal{K}}$, we see that for k big enough : $\mathrm{lm}([N_i, a_k]) = a_{\mathcal{J}}^+ a_k a_{\mathcal{K}}$, which is different from $-a_k$ and from 0. Thus (P_3) cannot hold.

- $E \cup A_1 \cup B_1$: We have :

$$\lambda a_i a_j = a_i(a_0 a_0^+ + \beta a_0^+ a_0) a_j$$
$$\Rightarrow \lambda a_i a_j = a_i a_i a_i^+ a_j + \beta a_i a_j^+ a_j a_j$$
$$\Rightarrow a_i a_j = 0$$

<u>$(1, -1)_b$</u> :

- E,$E \cup A_1 \cup B_1$: With $k \neq i \neq l$, we have $a_k(a_i a_i^+ + \beta a_i^+ a_i) a_l = 0$.

<u>$(1, -1)_a$</u> :

- A_2 : If $r = 0$ then $B = 0$ (see $(1, -1)_a \cup (1, 1)_a$). If $r \neq 0$ then $S = \{a_i a_i^+ {\to} 1, a_i a_j^+ {\to} q a_j^+ a_i\}$ is confluent and adapted to $<_n$. $T = \{a_{\mathcal{J}}^+ a_{\mathcal{K}}\}$, where $\mathcal{J}$ and $\mathcal{K}$ run over all tuples of indices, is the corresponding basis, and if $k > \mathcal{J}$, $\mathcal{K}$ we have : $[a_{\mathcal{J}}^+ a_{\mathcal{K}}, a_k] = a_{\mathcal{J}}^+ a_{\mathcal{K}} a_k - q^{|\mathcal{J}|} a_{\mathcal{J}}^+ a_k a_{\mathcal{K}}$. Thus, we see that (P_3) does not hold, as in case $(1, -1)_a \cup (2, 0) \cup E$.

- $A_2 \cup B_1$, $A_2 \cup B_2$: $r \neq 0$ by the above. We have, with $i \neq j$:

$$a_j a_j^+ a_j a_j^+ = 1$$
$$\Rightarrow a_j(a_i^+ a_i) a_j^+ = q^2 a_i^+ a_j a_j^+ a_i = q^2 a_i^+ a_i = 1$$

therefore $q^2 a_i a_i^+ a_i = q^2 a_i = a_i$, thus $q^2 = 1$ or else $B = 0$.

  - If $q = 1$: We define $\phi : B \to K[x, y]/\langle xy - 1\rangle$, by $\phi(a_i) = x$, $\phi(a_i^+) = y$. We then use lemma 5.

- If $q=-1$: We define $\phi : B \to K\langle (x_i)_{i\in\mathcal{I}_\alpha}\rangle/\langle x_ix_j+x_jx_i-2\delta_{ij}\rangle$, and we use lemma 7.

- E : We can assume $r\neq 0$ (the case $s\neq 0$ is symmetrical). $S=\{a_ia_i^+\to 1-\beta a_i^+a_i, a_ia_j^+\to qa_j^+a_i\}$ is confluent and adapted to $<_n$, and $T=\{a_{\mathcal{J}}^+a_{\mathcal{K}}\}$. We find $[a_{\mathcal{J}}^+a_{\mathcal{K}}, a_k^+]=q^{|\mathcal{K}|}a_{\mathcal{J}}^+a_k^+a_{\mathcal{K}}-a_k^+a_{\mathcal{J}}^+a_{\mathcal{K}}$, and we can do as above.

- $E\cup A_1\cup B_1$: Let us assume $r\neq 0$, the case $s=0$ being symmetrical. We have :

$$\begin{aligned}(a_ia_i^+)a_i &= a_i(a_i^+a_i)\\ \Rightarrow a_i-\beta a_0^+a_0a_i &= a_ia_0^+a_0 = qa_0^+a_ia_0\\ \Rightarrow a_ia_i^+-\beta a_0^+a_0a_ia_i^+ &= qa_0^+a_ia_0a_i^+\\ \Rightarrow 1-\beta a_0^+a_0-\beta a_0^+a_0(1-\beta a_0^+a_0) &= q^2a_0^+(1-\beta a_0^+a_0)a_0\\ \Rightarrow 1+(-2\beta-q^2)a_0^+a_0+\beta^2a_0^+(1-\beta a_0^+a_0)a_0 &= -\beta q^2 {a_0^+}^2{a_0}^2\\ \Rightarrow 1+(\beta^2-2\beta-q^2)a_0^+a_0+(-\beta^3+\beta q^2){a_0^+}^2{a_0}^2 &= 0 \qquad (17)\end{aligned}$$

Now we can define an homomorphism ϕ from the Weyl algebra $A_1 := K\langle a,a^+\rangle/\langle aa^+-a^+a-1\rangle$ to B by $\phi(a)=a_0$, and $\phi(a^+)=a_0^+$. Since A_1 is simple, either $\phi=0$ or ϕ is injective. But from (17), $\mathrm{Ker}(\phi)\neq 0$. Then $\phi=0$, therefore $B=0$.

<u>$(1,1)_c$</u> :

- E, $E\cup A_1\cup B_1$: With $j\neq i$, we have $a_j(a_ia_i^++\beta a_i^+a_i)a_j={a_j}^2=0$, and we find a case already studied.

<u>$(1,1)_b$</u> :

- A_2, etc... : We define $\phi : B\to K[x,y]/\langle xy-1\rangle$ by $\phi(a_i)=x$, $\phi(a_i^+)=y$, and we use lemma 5.

- E or $E\cup A_1\cup B_1$:
  - $\beta\neq -1$: We define $\phi : B\to K[x,y]/\langle xy-1/(1+\beta)\rangle$
  - $\beta=-1$: We define $\psi : B\to A_1$, by $\psi(a_i)=a$, $\psi(a_i^+)=a^+$, and we use lemma 8.

<u>$(1,1)_a$</u> :

- A_2, etc... : We define $\phi : B \to K\langle (x_i)_{i\in\mathcal{I}_\alpha}\rangle / \langle x_ix_j + x_jx_i - 2\delta_{ij}\rangle$, as in case $(1,1)_a \cup (1,-1)_a$.
- E :
  - $\beta \neq -1$: We define ϕ as in case $(1,1)_a \cup (1,-1)_a$.
  - $\beta = -1$: We quotient out B by the ideal generated by $a_ia_i^+ - a_0a_0^+$ and $a_i^+a_i - a_0^+a_0$, we thus obtain the algebra B' of case $E \cup A_1 \cup B_1$. Now this algebra is non-zero and does not contain number operators (see below), so neither does B.
- $E \cup A_1 \cup B_1$: If $\beta \neq -1$ we do as above. If $\beta = 1$ we see that the reduction system $S_0 = \{a_ia_i^+ \to 1 + a_0^+a_0, a_j^+a_j \to a_0^+a_0, a_ja_i \to -a_ia_j, a_j^+a_i^+ \to -a_i^+a_j^+ | i, j \in \mathcal{I}_\alpha, i < j\}$, adapted to $<_n$ is not confluent. Let us call S the confluent reduction system that we get from S_0 by the non-commutative Buchberger algorithm (i.e. we inductively reduce every ambiguity. See [Berg], [Ufn] or [Bes1] for details on this algorithm). After three iterations (this can be calculated by hand, or preferably with a computer program, such as *bergman*, available at http://www.matematik.su.se/research/bergman/, or the one in [Bes1], available at http://perso.wanadoo.fr/fabien.besnard/), one sees that $\forall i, j, k$, $a_ia_ja_k^+$ and $a_ia_j^+a_k^+$ are reducible with respect to S (all the details are in [Bes1]).

  Now let T be the basis of irreducible monomials corresponding to S, and let x belong to $B^{(0,\dots,0)} := \{x \in B | \forall i \in \mathcal{I}_\alpha, [N_i, x] = 0\}$. According to what we have just said, x is of the form :

  - (1) $a_{i_1}^+ \dots a_{i_k}^+ a_{j_1} a_{i_{k+1}}^+ \dots a_{j_r} a_{i_{k+r}}^+ a_{j_{r+1}} \dots a_{j_{r+k}}$, or
  - (2) $a_{i_1} a_{j_1}^+ \dots a_{i_k} a_{j_k}^+$, or
  - (3) $a_{i_1}^+ a_{j_1} \dots a_{j_k}^+ a_{i_k}$

  In each case, the tuple of indices i and the tuple of indices j are equal up to the order.

  Since for all i, j, k, $a_ia_ja_k^+$ and $a_ia_j^+a_k^+$ are reducible, one can see that the three cases reduce to : $x = {a_0^+}^k {a_0}^k$. Thus :

$$B^{(0,\dots,0)} = \{{a_0^+}^k {a_0}^k | k \in \mathbf{N}\}$$

As a consequence, we find that $\forall i,j,\ N_i \in B^{(0,\dots,0)}$ is stable under the action of the transposition automorphism τ^*_{ij}. Now :

$$\tau^*_{ij}[N_i, a_i^+] = [\tau^*_{ij}N_i, \tau^*_{ij}a_i^+]$$
$$\Leftrightarrow \tau^*_{ij}a_i^+ = [N_i, a_j^+]$$
$$\Leftrightarrow a_j^+ = 0$$

Thus (P_3) cannot hold. We must now prove that $B \neq 0$.

We first note that the algebra A, generated by the ξ_i's and ξ_i^+'s, satisfying $\xi_i\xi_j + \xi_j\xi_i = \xi_i^+\xi_j^+ + \xi_j^+\xi_i^+ = 0$, for $i \neq j$, and $\xi_i\xi_i^+ = \xi_i^+\xi_i = 1$, is non-zero (we can for instance quotient it by the ideal generated by the $\xi_i - \xi_i^+$, thus obtaining a Clifford algebra which is clearly non-zero). Let us then consider $\bar{B}_1 := A \bigotimes A_1$, and set $b_i := \xi_i \otimes a$, $b_i^+ := \xi_i^+ \otimes a^+$. We have, for $i \neq j$:

$$b_ib_j + b_jb_i = \xi_i\xi_j \otimes a^2 + \xi_j\xi_i \otimes a^2 = (\xi_i\xi_j + \xi_j\xi_i) \otimes a^2 = 0$$

$$b_i^+b_j^+ + b_j^+b_i^+ = \xi_i^+\xi_j^+ \otimes a^{+2} + \xi_j^+\xi_i^+ \otimes a^{+2} = (\xi_i^+\xi_j^+ + \xi_j^+\xi_i^+) \otimes a^{+2} = 0$$
$$b_i^+b_i - b_j^+b_j = (\xi_i^+\xi_i) \otimes a^+a - (\xi_j^+\xi_j) \otimes a^+a = 1 \otimes a^+a - 1 \otimes a^+a = 0$$
$$b_ib_i^+ - b_jb_j^+ = (\xi_i\xi_i^+) \otimes aa^+ - (\xi_j\xi_j^+) \otimes aa^+ = 1 \otimes aa^+ - 1 \otimes aa^+ = 0$$

and :

$$b_ib_i^+ - b_i^+b_i = (\xi_i\xi_i^+) \otimes aa^+ - (\xi_i^+\xi_i) \otimes a^+a = 1 \otimes (aa^+ - a^+a) = 1 \otimes 1 = 1$$

We thus see that B can be non-trivially mapped to $\bar{B}_1$ by $\psi(a_i) := b_i$ and $\psi(a_i^+) = b_i^+$, which proves that $B \neq 0$.

$\underline{(2,0)}$:

- E or $E \cup A_1 \cup B_1$: We already know that $\beta = 1$. We can thus define ϕ as in case $(1,1)_b \cup (2,0)$.

$\underline{\emptyset}$: In each sub-case, we can define an homomorphism ϕ as in one of the cases above.

- A_2, etc... : $B \to K[x,y]/\langle xy - 1\rangle$

- E, $E \cup A_1 \cup B_1$
  - $\beta \neq -1$: $B \to K[x,y]/\langle xy - 1/(1+\beta)\rangle$
  - $\beta = -1$: $B \to A_1$

So we see that the only possibilities were (pseudo)-fermions and (pseudo)-bosons, that we have respectively found in cases $(1,1)_a$ or $_b \cup (1,-1)_a \cup (2,0) \cup E$, and $(1,1)_a$ or $_b \cup (1,-1)_a \cup E$.

C End of Proof of Theorem 3

Let B be an algebra fulfilling the hypotheses of theorem 3. This algebra will also fulfill the hypotheses of theorem 2, except (P_3), which is replaced by $(\tilde{P}_3)$ which is the same as (P_3) except that the N_i are allowed to belong to $\tilde{B}$ instead of B. One can verify that proposition 4 still holds in this context, since it only depends on the gradation of B by the i-numbers. Thus all we have to do is to re-examine the cases of appendix B which have been eliminated by the hypothesis (P_3). We will have then to verify whether they fulfill (H_1), (H_2), and $(\tilde{P}_3)$.

First of all, let us note that in the cases containing B_2 we have ${a_i^+}^n {a_i}^n = 1$, thus (H_1) cannot be true.

Let us look at the remaining cases.

$\underline{(1,1)_b \cup (1,-1)_a \cup (2,0)}$:

- E, $\beta = 1$: We have the pseudo-fermionic algebra $B = C_\alpha$. The hypotheses of lemma 14 are fulfilled, (H_1) and (H_2) are easily seen to be satisfied. The elements of $\tilde{B}$ can be written

  $$\sum_{l=0}^{\infty} \sum_{\substack{i_1<\ldots<i_k \\ j_1<\ldots<j_l}} \lambda_{i_1,\ldots,i_k}^{j_1,\ldots,j_l} a_{i_1}^+ \ldots a_{i_k}^+ a_{j_1} \ldots a_{j_l}$$

  the second sum being finite for each l.

  If $x \in Z(\tilde{B})$, $x = \lim_k x_k$, with $x_k \in B^0$, by lemma 15, so that we must have $\forall i \in \mathcal{I}_\alpha$, $\lim [x_k, a_i] = 0$.

  Thus $\forall n$, $\exists p$, $k \geq p \Rightarrow [x_k, a_i] \in V_n$. Now $[a_{i_1}^+ \ldots a_{i_k}^+ a_{j_1} \ldots a_{j_k}, a_i]$ equals $a_{i_1}^+ \ldots a_{i_{s-1}}^+ a_{i_{s+1}}^+ \ldots a_{i_k}^+ a_{j_1} \ldots a_{j_k}$, if $i = i_s$, and 0 if $i \notin \{i_1, \ldots, i_k\}$.

  We thus have $x_k \in K + V_n + V_{\mathcal{I}\setminus\{i\}}$, where $V_{\mathcal{I}\setminus\{i\}}$ is the left ideal generated by $\{a_j^+ | j \neq i\}$. So we see that $x \in K + V_n + V_{\mathcal{I}\setminus\{i\}}$, for all n and for all i. This shows that $x \in K$.

$\underline{(1,1)_a \cup (1,-1)_a \cup (2,0)}$: This case is similar to the previous one, except that for E, $\beta = 1$, we get $B = \hat{C}_{\mathcal{I}}$.

$(1,1)_b \cup (1,-1)_a$:

- A_2 : See E, with $\beta = 0$.

- $A_2 \cup B_1$: We must have $r+s=0$. As in the finite case one shows that $a_i^+ a_i = 1$. Thus (H_1) cannot be fulfilled.

- E : $r+s=0$. We are in the bosonic and q-bosonic ($q=-\beta$) cases. We can use lemma 14 with $T = \{a_{\mathcal{I}}^+ a_{\mathcal{J}} | i_1 \leq \ldots \leq i_k,\ j_1 \leq \ldots \leq j_l\}$. (H_2) is clearly true. One shows that $\tilde{B}$ is central exactly as we did in case $(1,1)_b \cup (1,-1)_a \cup (2,0)$. For $q \neq \pm 1$, one sees that the number operators are given by

$$N_i = \sum_{k=0}^{\infty} \frac{(1-q)^k}{1-q^k} a_i^{+k} a_i^{\,k} + \lambda_i.1$$

  (this had been shown in [CS]).

- $E \cup A_1 \cup B_1$: $r+s=0$. As we have already seen, we have $Z(B) = B^0$. Then if (H_1), (H_2) and $(\tilde{P}_3)$ were true, we would have $Z(\tilde{B}) = \tilde{B}^0$, which entails that $(\tilde{P}_3)$ cannot be true, a contradiction.

$(1,1)_a \cup (1,-1)_a$: This case is similar to the previous one. We find the pseudo-q-bosons in case E.

$(1,1)_c \cup (2,0)$: Since $\forall k \geq 2$, $V_k = \{0\}$, we have $\tilde{B} \simeq B$.

$(1,1)_{a\ or\ b} \cup (2,0)$

- E: $\beta = 1$. We define $\phi : B \to C_1$ as in appendix A. Suppose that there exist number operators $\tilde{N}_i \in \tilde{B}$. Then let $u_n \in B$ be such that $N_i = \lim_n u_n$. Take $j \neq i$. We must have $\lim_n([u_n, a_i] + a_i) = 0$ and $\lim_n [u_n, a_j] = 0$. Thus $\exists n$ such that $[u_n, a_i] + a_i \in V_2$ and $[u_n, a_j] \in V_2$. But $\phi(V_2) = 0$. Thus $[\phi(u_n), a] + a = 0$ and $[\phi(u_n), a] = 0$, which is absurd.

- $E \cup A_1 \cup B_1$: same as above.

$(1,-1)_a \cup (2,0)$

- E: $\beta = 1$.

- $r \neq 0$: The reduction system $S = \{a_i a_i^+ \to 1 - a_i^+ a_i, {a_i}^2 \to 0, {a_i^+}^2 \to 0, a_i a_j^+ \to q a_j^+ a_i | i \neq j\}$ is confluent. The basis of irreducible monomials is $T = \{a_{\mathcal{I}}^+ a_{\mathcal{J}} | i_r \neq i_{r+1} \text{ and } j_s \neq j_{s+1}\}$. Let us define the degree of a monomial in B to be the degree of its normal form. Then $d^\circ(a_{\mathcal{J}}^+) = k$, and $a_i a_{\mathcal{J}}^+$ is a linear combination of monomials of degree $\geq k$. More precisely, they are of degree $k+1$ whenever $i \notin \mathcal{J}$.
  Suppose there exists a number operator N_i. Then $N_i = \sum_k N_i^k$, with $d^\circ(N_i^k) = 2k$. Now let j be such that $j \neq i$ and j does not appear in any monomial of the support of N_i^1 or N_i^2 (there is a finite number of such monomials). Then $[N_i, a_j] = 0 \Rightarrow [N_i^1, a_j] = 0$. Now $N_i^1 = \sum_k \lambda_k a_k^+ a_k$ (finite sum). Thus $[N_i^1, a_j] = \sum_k \lambda_k (a_k^+ a_k a_j - q a_k^+ a_j a_k) = 0$, so $\lambda_k = 0$ and $N_i^1 = 0$.
  Now only $[N_i^1, a_i]$ can provide terms of degree 1 in $[N_i, a_i]$. Consequently, N_i cannot exist.
- $r = 0$: Let us multiply $a_{i_1}^+$ to the right by $a_{i_2} a_{i_2}^+ + a_{i_2}^+ a_{i_2}$, with $i_1 \neq i_2$. We get : $a_{i_1}^+ = a_{i_1}^+ a_{i_2}^+ a_{i_2}$. In the same way we have $a_{i_2}^+ = a_{i_2}^+ a_{i_3}^+ a_{i_3}$, with $i_3 \neq i_2$, thus $a_{i_1}^+ = a_{i_1}^+ a_{i_2}^+ a_{i_3}^+ a_{i_3} a_{i_2}$. By induction we see that $a_{i_1}^+ \in \bigcap_{k \in \mathbf{N}^*} V_k$, so (H_1) cannot be fulfilled.

$\underline{(1,-1)_a}$:

- A_2, $r = 0$: We do as in the case $(1,-1)_a \cup (2,0) \cup E$.
- $A_2 \cup B_1$, $r \neq 0$: we find $a_i^+ a_i = 1$.
- E :
  - $r \neq 0$: We have a confluent reduction system and we can use the same argument as in case $(1,-1)_a \cup (2,0)$.
  - $r = 0$: Since $a_j^+ a_i^+ a_i = a_j^+$, we can do as in case $(1,-1)_a \cup (2,0)$.

$\underline{(1,1)_b}$:

- A_2 : See E, $\beta = 0$.

- $A_2 \cup B_1$: The following relations hold in B :

$$\begin{array}{rcll} a_0^+ a_0 a_j^+ &=& a_j^+, & \forall j \neq 0 \\ a_j^+ a_k a_i^+ &=& a_i^+ a_k a_j^+, & \forall j > i,\ k \neq i,j \\ a_i a_j a_i^+ &=& a_j, & \forall i < j \\ a_i a_j a_k^+ &=& a_i a_k^+ a_j, & \forall i \leq j, k \neq i,j \\ a_k a_i^+ a_j^+ &=& a_i^+ a_k a_j^+, & \forall i \leq j, k \neq i,j \\ a_j a_i^+ a_j^+ &=& a_i^+, & \forall i < j \\ a_j a_k^+ a_i &=& a_i a_k^+ a_j, & \forall j > i,\ k \neq i,j \\ a_j a_0^+ a_0 &=& a_j, & \forall j \neq 0 \end{array}$$

  To see it, one should use the non-commutative Buchberger algorithm on the initial reduction system. This can be done with the computer programs already cited. See also [Bes1] for a detailed account of the calculations. So, as in appendix B, $(1,1)_a \cup E \cup A_1 \cup B_1$, we can prove that $B^0 = \text{Span}\{a_0^{+k} a_0{}^k | k \in \mathbf{N}\}$. Now take i and j, two distinct indices, and set $N_i = \lim_n x_n$. We have : $\forall n \in \mathbf{N}$, $\exists k$ such that $[x_k, a_i] + a_i \in V_n$ and $[x_k, a_j] \in V_n$. Now if $\tau^*_{i,j}$ is the automorphism induced by the (i,j)-transposition, we have : $\tau^*_{i,j} x_k = x_k$, and $\tau^*_{i,j} V_n = V_n$, so that $\tau^*_{i,j}[x_k, a_j] = [x_k, a_i] \in V_n$, thus $a_i \in V_n$, and this is true for all n. So if (H_1) is satisfied, $(\tilde{P}_3)$ is not.

- E or $E \cup A_1 \cup B_1$:

  - $\beta = -1$: We can send B onto A_1 by ψ, as in the finite case. Suppose that N_i is a number operator, and take $j \neq i$. If u_n is a sequence in B that converges towards N_i, we must have for n large enough : $[u_n, a_i] + a_i \in V_1$ and $[u_n, a_j] \in V_1$, so that $[\phi(u_n), a] + a \in a^+ A_1$ and $[\phi(u_n), a] \in a^+ A_1$. Now $(-a + a^+ A_1) \cap a^+ A_1 = \emptyset$, a contradiction.
  - $\beta \neq -1$: We can use the same method if we send B onto the q-bosonic algebra $(q = -\beta)$ A_1^q. Since $(-a + a^+ A_1^q) \cap A_1^q = \emptyset$, we arrive at the same conclusion.

$\underline{(1,1)_a}$:

- A_2 : Let us prove the following relations by induction :

$$a_i a_{i_1} \dots a_{i_j} a_i^+ = (-1)^j a_{i_1} \dots a_{i_j}$$

$$a_k a_{i_1}^+ \dots a_{i_j}^+ a_k^+ = (-1)^j a_{i_1}^+ \dots a_{i_j}^+$$

for all $i < i_1 < \dots < i_j < k$. They are true for $j = 0$, so let us suppose they are true for j. It suffices to multiply on the left by $a_{i_{j+1}}$ in the first relation, and on the right by $a_{i_{j+1}}^+$ in the second one, to see that they are still true for $j+1$. It is then easy to see that the reduction system formed by $a_i a_{i_1} \dots a_{i_j} a_i^+ \rightarrow (-1)^j a_{i_1} \dots a_{i_j}$, $a_k a_{i_1}^+ \dots a_{i_j}^+ a_k^+ \rightarrow (-1)^j a_{i_1}^+ \dots a_{i_j}^+$, and $a_j a_i \rightarrow - a_i a_j$, $a_j^+ a_i^+ \rightarrow - a_i^+ a_j^+$ for all $j > i$, is confluent. So (H_2) is not satisfied, indeed : $\forall i_1 \leq \dots \leq i_k$ and $i \notin \{i_1, \dots, i_k\}$, $a_i a_{i_1}^+ \dots a_{i_k}^+$ is irreducible and $V_n = \mathrm{Span}\{a_{\mathcal{J}}^+ x | \, |\mathcal{J}| = n$ and $a_{\mathcal{J}}^+ x$ irreducible$\}$.

- $A_2 \cup B_1$: We can use the same method as in $(1,1)_b$ (see [Bes1] for the computer calculations).

- E : Let us call p the quotient map $p : B \to B' := B/\langle\{a_i a_i^+ - a_0 a_0^+, a_i^+ a_i - a_0^+ a_0 | i \in \mathcal{I}\}\rangle$. Suppose B satisfies $(\tilde{P}_3)$ and set $N_i = \lim_n x_n$, $i \neq 0$, with $x_n \in B^0$. Then $p(x_n) \in B'^0$ but we know that B'^0 is generated by the ${a_0^+}^k {a_0}^k$'s. Thus $p(x_n)$ is invariant by any transposition τ_{ij}^*. For all k there exists an n such that $[x_n, a_i] + a_i \in V_k$, and $[x_n, a_j] \in V_k$. Thus $[p(x_n), p(a_i)] + p(a_i) \in p(V_k)$ and $[p(x_n), p(a_j)] \in p(V_k)$. Using τ_{ij}^* on the second equation, and subtracting from the first, we find $p(a_i) \in p(V_k)$. Then, using $\psi : B' \to \bar{B}_\beta$, defined in the appendix B :
$$a \otimes \xi_i \in (a^+ \otimes 1)\bar{B}_\beta$$
but this is false, as we can see by using the basis $\{{a^+}^r a^s \otimes \xi_t\}$, where $\{\xi_t\}$ is a basis of A containing ξ_i.

- $E \cup A_1 \cup B_1$: We know that $B^0 = \mathrm{Span}\{{a_0^+}^k {a_0}^k\}$, thus we can do as in case $(1,1)_b \cup A_2 \cup B_1$.

References

[Berg] G. Bergman, *The Diamond Lemma for Ring Theory*, Adv. in Math. **29** (1978), 178-218

[Bes1] F. Besnard, *Ph. D. Thesis*, Université Paris 7, 2000. http://www.math.jussieu.fr/˜besnard/these.ps (in French)

[Bes2] F. Besnard, *Number Operator Algebras and Deformations of ϵ-Algebras* (2000), preprint math-ph/0006012

[Bes3] F. Besnard, *Classification des algèbres à opérateurs de nombre de présentation quadratique*, C.R. Acad. Sci. Paris, Série I, t. 331 (2000), 17-20 (in French)

[CS] S. Chaturvedi, V. Srinivasan, *Aspects of q-oscillator Quantum Mechanics*, Phys. Rev. A, **44** (1991), No. 12, 8020-8023

[Sch] M. Scheunert, *Generalized Lie Algebras*, J. Math. Phys. **20**, 712 (1979)

[Ufn] V. Ufnarovski, *Introduction to Non-commutative Gröbner bases theory*, Gröbner bases and applications, London Math. Soc. lecture notes series 251, Cambridge University Press, 1998