M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 6, 2000

Paper 2
Received: Nov 19, 1999 Revised: Feb 10, 2000 Accepted:Feb 15, 2000
Editor: C.E. Wayne

Families of whiskered tori for a-priori stable/unstable Hamiltonian systems and construction of unstable orbits

Enrico Valdinoci

Department of Mathematics, The University of Texas at Austin, TX 78712-1082 (USA)
enrico@math.utexas.edu

Abstract

We give a detailed statement of a KAM theorem about the conservation of partially hyperbolic tori on a fixed energy level for an analytic Hamiltonian $H(I, \varphi, p, q) = h(I, pq; \mu) + \mu f(I, \varphi, p, q; \mu)$, where φ is a $(d-1)-$dimensional angle, I is in a domain of $\mathbb{R}^{d-1}$, p and q are real in a neighborhood 0, and μ is a small parameter. We show that invariant whiskered tori covering a large measure exist for sufficiently small perturbations. The associated stable and unstable manifolds also cover a large measure. Moreover, we show that there is a geometric organization to these tori. Roughly, the whiskered tori we construct are organized in smooth families, indexed by a Cantor parameter. The whole set of tori as well as their stable and unstable manifolds is smoothly interpolated. In particular, we emphasize the following items: sharp estimates on the relative measure of the surviving tori on the energy level, analyticity properties, including dependence upon parameters, geometric structures.

We apply these results to both "a-priori unstable" and "a-priori stable" systems. We also show how to use the information obtained in the KAM Theorem we prove to construct unstable orbits.

Contents

* * * * *

Acknowledgments

This work is part of my "tesi di laurea" [39] at Università di Roma 3 and the final version has been written at UT. Partially supported by CNR, INdAM, TARP. I would like to thank Profs. L. Chierchia and R. de la Llave for their help and encouragement. Comments of anonymous referees were useful. I am also indebted with L. Biasco for many helpful discussions. This paper is dedicated to my grandma, nonna Tina.

1 Introduction

Chains of whiskered invariant tori are the building blocks to prove *instability* of some Hamiltonian systems by a mechanism suggested in a celebrated example of Arnol'd [2]. Even if several precise formulations of Arnol'd diffusion have been proposed in the literature, all of them share that there are chains of invariant tori with hyperbolic directions so that the stable and unstable directions intersect. Then, orbits that follow these transition chains experience "large" changes in the actions.

In this paper, we want to make a detailed study of the survival of whiskered tori and prove results that we hope can be eventually used in the program outlined above.

In particular, we pay special attention to how the tori fit together. We not only prove measure theoretical statements about their abundance (which we show are optimal), but we also show that these tori are organized in a geometric manner. All the whiskered tori that we prove are invariant can be interpolated by a smooth family of whiskered tori (not all the members of the family are invariant). Moreover, we show that the set of invariant tori is organized in smooth families. This interpolation is constructed along the proof somewhat explicitly. We also construct a similar interpolation for the stable and unstable manifolds.

We also pay attention to the analyticity properties of the tori and of the transformation involved, including regularity with respect to the parameter of perturbation.

We will consider an unperturbed Hamiltonian $H(I,\varphi,p,q) = h(I,pq)$ with I in a domain[1] of $\mathbb{R}^{d-1}$, $\varphi \in \mathbb{T}^{d-1}$, p and q real in a neighborhood of the origin. Moreover, we will endow the phase space $\mathbb{R}^{d-1} \times \mathbb{T}^{d-1} \times \mathbb{R} \times \mathbb{R}$ with the standard symplectic structure $\omega = \sum_i dI_i \wedge d\varphi_i + dp \wedge dq$ so that I_i and φ_i are conjugate variables and so are p and q. We will refer to I as the action variables, φ as the angle variables and p, q as the hyperbolic variables.
The equations of motion of this unperturbed Hamiltonian are

$$\begin{aligned} \dot{I} &= -\partial_\varphi H \\ \dot{\varphi} &= \partial_I H \\ \dot{p} &= -\partial_q H \\ \dot{q} &= \partial_p H \end{aligned}$$

where the dot stands for the derivative with respect to time. We will denote by Φ_t the flow of these equations.
Note that fixing any I_0 in the domain of definition, the $d-1$ dimensional torus $\mathcal{T}_{I_0} = \{(I_0,\varphi,0,0),\ \varphi \in \mathbb{T}^{d-1}\}$ is invariant under the equations of motion. Moreover, the torus $\mathcal{T}_{I_0}$ is contained in the invariant manifold $\mathcal{S}_{I_0} = \{(I_0,\varphi,p,q)\}$, on which the motion is simply

$$\Phi^t(I_0,\varphi,p,q) = \left(I_0, \varphi + \omega t, pe^{-\lambda t}, qe^{\lambda t}\right)$$

with λ and ω depending only on the product of the hyperbolic variables p and q. If we call $\zeta = pq$, we have that ζ is an invariant of the motion and $\lambda(\zeta) = \partial_\zeta h(I_0,\zeta)$, $\omega(\zeta) = \partial_I h(I_0,\zeta)$.
The torus $\mathcal{T}_{I_0}$ has the following local stable and unstable manifolds

$$\begin{aligned} W^s_{I_0} &\equiv \{(I_0,\varphi,p,0)\,,\ \varphi \in \mathbb{T}^{d-1},\ |p| \le R\} \\ W^u_{I_0} &\equiv \{(I_0,\varphi,0,q)\,,\ \varphi \in \mathbb{T}^{d-1},\ |q| \le R\} \end{aligned}$$

and we will call them *whiskers*, following [2]. A formal definition of whiskered torus will be given in Definition 2.8 below.

We will prove that most of the above tori, together with their associated invariant manifolds, are preserved under perturbations. Roughly speaking, Theorem 3.1 below asserts that:

If the Hamiltonian

$$H(I,\varphi,p,q) = h(I,pq;\mu) + \mu f(I,\varphi,p,q;\mu) \tag{1.1}$$

is real analytic and "isoenergetically non-degenerate"[2] *and $|\mu|$ is sufficiently small, then there exists a smooth canonical transformation, close to the identity, analytic in the parameter μ, in the angles and in the hyperbolic variables, such that, on a suitable set, the new Hamiltonian depends only on the actions and on the product of the hyperbolic variables.*
In this way, $(d-1)-$dimensional invariant tori and smoothly interpolated $d-$dimensional whiskers are obtained. Such whiskered tori fill the space with density at least $1 - O(\sqrt{|\mu|})$.

Here and in the sequel, when we refer to a KAM-type transformation as "smooth", we mean "C^∞ in the sense of Whitney" (see, for example, [40], [26], [32] and [7]). Anyhow, following [9], a direct and fully constructive extension is possible, making use of elementary "bump functions".

The above result applies directly to the so called "a-priori unstable" systems, in the terminology of [11]. Such systems are obtained perturbing Hamiltonians which possess separatrices: see Definition 2.2 below. A detailed result for whiskered tori in a-priori unstable systems is given in Theorem 5.2 below.

[1] In this paper, we use the word "domain" to denote the closure of an open, bounded, connected set.

[2] Which means that the matrix

$$\begin{pmatrix} \partial_I^2 h & \partial_I h \\ \partial_I h & 0 \end{pmatrix}$$

is nonsingular on the energy level. See Section 2.2.

We also apply our KAM Theorem to "a-priori stable" systems (see Definition 2.1 below), showing that such systems have the above mentioned geometry near simple resonances. Theorem 6.5 provides a detailed statement for whiskered tori in a-priori stable systems. Also, in Theorem 6.6, we prove that near the Diophantine resonances, that the density of the "holes" is *exponentially small* in ε, i.e. the persisting tori fill a suitable domain of the energy level with density at least $1 - O(e^{-O(1/\varepsilon^c)})$, where c is a suitable constant.

Theorem 3.1 is similar to Lemma 1 of [11]. The main differences between our Theorem 3.1 and [11] are listed below:

(i) We obtain a relation between the Diophantine constant γ and the size of the perturbation μ similar to the one obtained in the classic KAM Theorem (i.e. $\gamma = O(\sqrt{|\mu|})$: see, for example [29] and [32]). We also obtain some characterizations of the set of validity of the Theorem. As a consequence, we obtain estimates on the measure of the space covered by the surviving tori, proving that the "holes" on which the Theorem fails are not wider than $O(\sqrt{|\mu|})$. We will also exhibit a simple example which proves that these estimates are sharp.

(ii) We prove the analytical dependence upon the parameter μ for fixed energy rather than for fixed frequencies, as proved in [11].

(iii) We study the geometry of the interpolation between tori, observing a structure of "filaments".

The first KAM results about persistence of whiskered tori go back to [22] and [44], and normal forms for whiskered tori can be found in [18], [21], [14], [30]. The recent paper [35] also considers a KAM theory about partially hyperbolic tori; they obtain, with a different method, the relation between γ and μ mentioned in (i). Moreover, the statement in [35] is obtained for fixed frequencies instead of fixed energy. The proof in [35] follows the scheme in [32] and deals with the frequency–angle variables instead of the action–angles coordinates used in our paper. Actually, for our purposes, we need a slightly more detailed statement than the one in [35]: here (as well as in [12]) we need a "good" characterization for the set in which the KAM Theorem holds [namely, the relations of the type of (3.6)], in order to construct the diffusion [i.e., the unstable orbits built in Section 4].

The construction of the KAM tori and of the unstable orbits carried out in our paper are robust enough to apply to the so called "anisochronous" cases[3], in which the KAM tori are separated by gaps.

A KAM theory for systems with a degenerate integrable part is developed in [42], for the hyperbolic case, and in [41], for the elliptic case.

The question of the persistence of KAM tori of codimension one (or bigger) is also addressed in [28], Chapter V, Section 4. Results for whiskered tori of higher codimension can be found in [8]. See also [43], which constructs a KAM algorithm for elliptic low dimensional tori under a mild non-degeneracy condition on the small divisors, extending the results of [17], [33], [6].

Also, [25], [24] and [38] consider the creation of low dimensional whiskered tori in perturbation of integrable systems.

Some of the papers quoted above take into account the case in which the system is not analytic. With regard to the problem of the conservation of the invariant tori and their hyperbolic manifolds for smooth Hamiltonians, see also the very recent paper [23].

The persistence of KAM tori for reversible (instead of Hamiltonian) systems is discussed in [36].

With respect to the papers quoted above, the target of Theorem 3.1 here is to provide a more detailed description of the partially hyperbolic tori of codimension one and investigate the structure of their

[3]See [11] for the nomenclature. Also, the "gap-bridging" procedure that inspires our paper was introduced in [11]. The isochronous systems (see [20]), on the contrary, are systems with a fixed Diophantine frequency: such systems do not present gaps between the tori, and this fact makes the construction of the unstable orbits easier.

whiskers, in order to apply directly these results to Arnol'd diffusion. We pay special attention to the smooth interpolation of such manifolds and provide a very strong normal form, in order to describe *exactly* the motion of a $(d+1)-$dimensional neighborhood of any invariant torus, even if it is not possible to determine *all* the motions nearby. The existence of such a normal form is crucial in the construction of the unstable orbits presented here in Section 4.

The scheme of the present paper is the following. In Section 2 we recall common definitions such as the ones for "a-priori stable" and "a-priori unstable" systems, isoenergetic non-degeneracy and chain of whiskered tori. We also discuss a formula relating the determinant of the matrix of isoenergetic non-degeneracy with the function that implicitly defines the energy level, and we derive some characterizations of the isoenergetic non-degeneracy.
In Section 3 we state the KAM Theorem about the preservation of whiskered tori and we discuss an example showing the optimality of our estimate on the density of the preserved tori. We also briefly emphasize the geometric structure of "filaments" related to these tori.
In Section 4 we show that the whiskered tori built in Section 3 can be used to construct "unstable" orbits (i.e. trajectories that exhibit an excursion of order 1 in the action variables), provided that there exist a chain of such whiskered tori, in which the unstable whisker of each torus intersects transversally the stable whisker of the next one. We also remark that this procedure also allow to construct orbits "drifting towards infinity".
In Section 5 and 6 we apply the KAM Theorems of Section 3 to the a-priori unstable and stable systems, respectively.
Section 7 contains a detailed proof of the KAM Theorem which makes use of a Newton scheme. The Appendix collects some elementary Lemmas.

2 Preliminaries

2.1 A-priori stable and unstable systems

We recall here the terminology of [11].

Definition 2.1 *The Hamiltonian system $H(I,\varphi) = h(I) + \varepsilon f(I,\varphi;\varepsilon)$, in which h and f are real analytic for I in a domain of $\mathbb{R}^d$, $\varphi \in \mathbb{T}^d$, and ε is a small parameter, is called* **a-priori stable**.

Such a-priori stable systems are often called *nearly-integrable* since they are perturbations of completely integrable systems written in action-angle coordinates.

Definition 2.2 *The Hamiltonian system*

$$H(I,\varphi,p,q) = \mathcal{R}(I;\mu) + \mathcal{P}(I,p,q;\mu) + \mu f(I,\varphi,p,q;\mu) \tag{2.1}$$

in which $\mathcal{R}$, $\mathcal{P}$ and f are real analytic for I in a domain of $\mathbb{R}^{d-1}$, $\varphi \in \mathbb{T}^{d-1}$, p and q are real in the neighborhood of the origin, and μ is a small parameter, is called **a-priori unstable** *if*

$$\begin{aligned} &\partial_p\mathcal{P}(I,0,0;\mu) = \partial_p(\mathcal{P}+\mathcal{R})(I,0,0;\mu) = 0 = \partial_q\mathcal{P}(I,0,0;\mu) = \partial_q(\mathcal{P}+\mathcal{R})(I,0,0;\mu) \\ \text{and}\quad &\det\partial^2_{(p,q)}\mathcal{P} = \det\partial^2_{(p,q)}(\mathcal{R}+\mathcal{P}) \le -C < 0 \end{aligned} \tag{2.2}$$

when I, p and q vary in their own set of definition, and C is a positive constant, independent of μ.

Some authors refer to the a-priori unstable systems as "initially hyperbolic". Condition (2.2) means that $p = 0 = q$ is a hyperbolic equilibrium. An example of a-priori unstable system is obtained choosing $\mathcal{R}$ as free rotators

$$\mathcal{R} = \frac{1}{2}(I_1^2 + \ldots + I_{d-1}^2) \tag{2.3}$$

and $\mathcal{P}$ as a pendulum

$$\mathcal{P} = \frac{1}{2}p^2 + g^2(\cos q - 1) \tag{2.4}$$

where g is a constant.
As it turns out, a-priori stable systems also have partially hyperbolic orbits near simple resonances. In the distinction between a-priori stable and a priori unstable systems, a crucial role is played by the size of the Lyapunov exponent near hyperbolic equilibria. This exponent is of order one in the case of an a-priori unstable system because of (2.2), while it is of order $\sqrt{\varepsilon}$ near the simple resonances of a generic a-priori stable system. This will be clarified in Section 6.
To better understand the previous remark, the reader may check that the following example [to be compared with the previous (2.3)-(2.4)] is a-priori stable:

$$H(I,\varphi) = \frac{1}{2}(I_1^2 + \ldots + I_{d-1}^2) + \frac{1}{2}I_d^2 + \varepsilon(\cos\varphi_d - 1)\,,$$

where $\varepsilon > 0$ is a small parameter. The procedure of making use of an "independent" parameter in a singular-perturbation problem was already used in [31]. The use of such a procedure in our paper will be clarified in Section 6.

2.2 The isoenergetic non-degeneracy

In this subsection, we will denote by $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$, and we will consider neighborhoods in $\mathbb{F}$. This is done to deal with both the real and the complex case at the same time.

Definition 2.3 *The (smooth) Hamiltonian $h(I)$, with I in a domain of $\mathbb{F}^d$, is called* **isoenergetically non-degenerate** *on the energy level $h = E$ if*

$$\det\begin{pmatrix} h'' & h' \\ (h')^T & 0 \end{pmatrix} \neq 0 \tag{2.5}$$

for any I in the domain of h such that $h(I) = E$.

Notational Remarks. In the rest of this paper, we will use the same notation for both column and row vectors: for instance, following [4] page 409, we write $\begin{pmatrix} h'' & h' \\ h' & 0 \end{pmatrix}$ instead of $\begin{pmatrix} h'' & h' \\ (h')^T & 0 \end{pmatrix}$. The only place in which we denote row vectors with the symbol of transposition "$\cdot^T$" is Lemma A.1, in order to avoid confusion between $v \cdot w$ [i.e. the scalar product between $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$] and vw^T [i.e. the matrix whose $(i,j)-$th entry is $v_i w_j$].
We will also denote $\omega \equiv h'$ and use the symbol "tilde" for the first $d-1$ components of a $d-$dimensional vector.
Also, we write $|\cdot|$ to denote a norm for real or complex vectors, and the sum of the absolute values of the components for integer vectors. We will not compute explicitly the constants appearing in the KAM proof, hence we do not fix explicitly the norm used in the finite-dimensional vector spaces, since they are, for our purposes, equivalent. On the other hand, a careful choice of the norms is necessary for a concrete and effective implementation of the scheme: see, for example, [10].

Proposition 2.4 *Consider the (smooth) Hamiltonian $h(I)$, with I in a domain of $\mathbb{F}^d$ and $d \geq 2$. Assume $\omega_d \neq 0$ on the energy level $h = E$. Denote by $I_d(\tilde{I})$ the function implicitly defined by $h(\tilde{I}, I_d(\tilde{I})) = E$. Then,*

$$\det \partial^2_{\tilde{I}} I_d = (-1)^d \omega_d^{-d-1} \det\begin{pmatrix} h'' & \omega \\ \omega & 0 \end{pmatrix}\,. \tag{2.6}$$

Proof. It follows from

$$\partial^2_{\tilde{I}} I_d = \frac{(\omega_d \partial^2_{\tilde{I}I_d} h - \partial^2_{I_d} h \tilde{\omega})\tilde{\omega}^T - \omega_d(\omega_d \partial^2_{\tilde{I}} h - \tilde{\omega}(\partial^2_{\tilde{I}I_d} h)^T)}{\omega_d^3}\,,$$

making use of Lemma A.1. □

Proposition 2.5 *Let $d \geq 2$. The following conditions are equivalent:*

(i) *h is isoenergetically non-degenerate on the energy level $h = E$ for I in a suitable domain of $\mathbb{F}^d$.*

(ii) *$\omega \neq 0$ on the energy level $h = E$ and, assuming for example $\omega_d \neq 0$ and denoting by $I_d(\tilde{I})$ the function implicitly defined by $h(\tilde{I}, I_d(\tilde{I})) = E$ and*

$$\alpha(\tilde{I}) \equiv \frac{\tilde{\omega}(\tilde{I}, I_d(\tilde{I}))}{\omega_d(\tilde{I}, I_d(\tilde{I}))}, \tag{2.7}$$

we have

$$\det \alpha' \neq 0 . \tag{2.8}$$

(iii) *The following function G is a local diffeomorphism near $\sigma = 1$:*

$$\begin{array}{rcl} G : \mathbb{F}^d \times \mathbb{F} & \longrightarrow & \mathbb{F}^d \times \mathbb{F} \\ (I;\sigma) & \longmapsto & (\sigma\omega(I),\, h(I)) . \end{array}$$

Proof. (i) and (iii) are equivalent because of the Implicit Function Theorem. The equivalence between (i) and (ii) follows from (2.6) and from $\partial_{\tilde{I}} I_d = -\alpha$. $\square$

Writing the details in the proof of the previous Proposition, it is easy to obtain the following

Corollary 2.6 *Consider the (smooth) Hamiltonian $h(I)$, with I in a domain of $\mathbb{F}^d$ and $d \geq 2$. Assume $\omega_d \neq 0$ on the energy level $h = E$, and define $\alpha(\tilde{I})$ as in (2.7). Then,*

$$\det \alpha' = -(\omega_d)^{-d-1} \cdot \det \begin{pmatrix} h'' & \omega \\ \omega & 0 \end{pmatrix} . \tag{2.9}$$

Similar definitions and results hold for a "partially hyperbolic" Hamiltonian $h(I, pq)$, since the variable $\zeta = pq$ plays in this case only the role of a parameter. In particular, the condition of isoenergetic non-degeneracy becomes

$$\det \begin{pmatrix} \partial_I^2 h & \partial_I h \\ \partial_I h & 0 \end{pmatrix} \neq 0 , \tag{2.10}$$

for any I in a domain of $\mathbb{F}^d$ and p, q in a neighborhood of 0. With a slight abuse of notation, we will refer to both (2.5) and (2.10) with the term of "isoenergetic non-degeneracy". All through the paper, the isoenergetic non-degeneracy condition concerns only the derivatives with respect to the actions I and never the derivatives with respect to the action $\zeta \equiv pq$.

Proposition 2.5 can also be slightly modified to include hyperbolic variables as follows:

Proposition 2.7 *Let $d \geq 2$. The following conditions are equivalent:*

(i) *$h(I, \zeta)$ is isoenergetically non-degenerate on the energy level $h = E$ for I in a domain of $\mathbb{F}^d$ and p, q in a neighborhood of 0.*

(ii) *$\omega \equiv \partial_I h \neq 0$ on the energy level $h = E$ and, assuming for example $\omega_d \neq 0$ and denoting by $I_d(\tilde{I}, \zeta)$ the function implicitly defined by $h(\tilde{I}, I_d(\tilde{I}, \zeta), \zeta) = E$ and*

$$\alpha(\tilde{I}, \zeta) \equiv \frac{\tilde{\omega}(\tilde{I}, I_d(\tilde{I}, \zeta), \zeta)}{\omega_d(\tilde{I}, I_d(\tilde{I}, \zeta), \zeta)},$$

we have

$$\det \partial_I \alpha \neq 0 .$$

(iii) *The following function G is a local diffeomorphism near $\sigma = 1$:*

$$\begin{array}{rcl} G : \mathbb{F}^d \times \mathbb{F} \times \mathbb{F} & \longrightarrow & \mathbb{F}^d \times \mathbb{F} \times \mathbb{F} \\ (I;\sigma;\zeta) & \longmapsto & (\sigma\omega(I,\zeta),\, h(I,\zeta), \zeta) . \end{array}$$

2.3 Whiskered tori and transversality

Following [2], we give the following

Definition 2.8 *A torus $\mathcal{T}$ is called a* **whiskered torus** *for the flow Φ^t if it is a connected component of the intersection of two manifolds W^s and W^u invariant under Φ^t, such that $\forall \zeta^s \in W^s$, $\lim_{t\to\infty} \operatorname{dist}\big(\Phi^t(\zeta^s), \mathcal{T}\big) = 0$ and $\forall \zeta^u \in W^u$, $\lim_{t\to-\infty} \operatorname{dist}\big(\Phi^t(\zeta^u), \mathcal{T}\big) = 0$.*

In the KAM setting, the motion on these tori will be conjugated to an irrational (and, in fact, Diophantine) rotation, the trajectories on W^s will converge exponentially fast to $\mathcal{T}$ and the trajectories on W^u will diverge exponentially fast from $\mathcal{T}$. Such a motion will be described in detail in (3.15) below.

We call W^s and W^u the stable and unstable whisker, respectively. We will denote by $T_p\mathcal{M}$ the tangent space at p of the manifold $\mathcal{M}$.
If $V = \operatorname{span}\{v_1, \dots, v_i\}$ and $W = \operatorname{span}\{w_1, \dots, w_j\}$ are vector spaces, we set

$$V + W \equiv \operatorname{span}\{v_1, \dots, v_i, w_1, \dots, w_j\} \ .$$

Definition 2.9 *Let $\mathcal{M}$ and $\mathcal{N}$ be submanifolds of the manifold $\mathcal{X}$. We say that $\mathcal{M}$ and $\mathcal{N}$ are* **transverse** *in the point p with respect to the ambient space $\mathcal{X}$ if $p \in \mathcal{M} \cap \mathcal{N}$ and $T_p\mathcal{M} + T_p\mathcal{N} = T_p\mathcal{X}$.*

In this paper, we will consider only transversality of whiskers with respect to a common, fixed energy level $h = E$. We now recall the standard definition of Diophantine vector:

Definition 2.10 *A vector $\omega \in \mathbb{R}^n$ is called $(\gamma, \tau)-$Diophantine if $|\omega \cdot n| \geq \gamma/|n|^\tau$ for any $n \in \mathbb{Z}^n - \{0\}$.*

We will use later the elementary fact that, if $\tau > n - 1$, the $(\gamma, \tau)-$Diophantine vectors fill the $n-$dimensional space with density $1 - O(\gamma)$. In the sequel, we will consider Diophantine vectors in the $(d-1)-$dimensional frequency space, so that in the rest of the paper $\tau > d - 2$ will be a fixed parameter.

3 A KAM Theorem about preservation of partially hyperbolic tori

In the sequel we will restrict to Hamiltonian systems whose number of degree of freedom is $d \geq 3$. This is motivated by the very well known fact that, in our setting, Arnol'd diffusion does not occur for autonomous Hamiltonian systems with less than 3 degrees of freedom (see for instance [1], [3]). Anyhow, it is easy to see that some of the results of this paper remain valid even in the case $d = 2$. But in the case $d = 2$ it is not possible to have transverse intersections between the unstable whisker of a torus in the chain with the stable whisker of the next torus, so that the procedure given in Section 4 can not work.

Theorem 3.1 *Fix $I^* \in \mathbb{R}^{d-1}$, $E \in \mathbb{R}$. Consider the Hamiltonian*

$$H(I, \varphi, p, q) = h(I, pq; \mu) + \mu f(I, \varphi, p, q; \mu) \tag{3.1}$$

with h and f real analytic in

$$\mathcal{O}_{\rho,\xi,R,\bar{\mu}} = \{(I, \varphi, p, q, \mu) \in \mathbb{C}^{(d-1)+(d-1)+1+1+1} \text{ s.t. } |I - I^*| \leq \rho,\ |\Im\varphi| \leq \xi,\ |p| \leq R,\ |q| \leq R,\ |\mu| \leq \bar{\mu}\}$$

and periodic in the angles φ. Denote $\zeta \equiv pq$ and assume that

$$\lambda_0 \equiv \inf_{|I-I^*|\leq\rho,\, |\zeta|\leq R^2,\, |\mu|\leq\bar{\mu}} |\partial_\zeta h| > 0$$

and that h is isoenergetically non-degenerate on the energy level $h(I, pq; \mu) = E$ when the variables vary in $\mathcal{O}_{\rho,\xi,R,\bar{\mu}}$.
Then, there exist R_∞, $0 < R_\infty \le R$, and a constant $\kappa_\star$ (eventually depending on d, τ, and the sizes of h and f), such that, for $|\mu| \le \mu_0 \equiv \kappa_\star \lambda_0^2 \le \bar{\mu}$, there exist:

(e1) a smooth canonical transformation Φ, close to the identity and real analytic (for a fixed action) in the angles, in the hyperbolic variables and in the parameter μ,

(e2) a function $h_\infty : \mathbb{R}^{(d-1)} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ with the same smoothness as Φ,

(e3) a set $\Omega_\mu \subseteq \mathbb{R}^{(d-1)+1+1}$, with density at least $1 - \kappa_\star\sqrt{\mu_0}$,

such that, for $|\mu| \le \mu_0$,

$$\begin{aligned}&\partial^n\left(H \circ \Phi(I', \varphi', p', q')\right) = \partial^n h_\infty(I', p'q'; \mu)\,,\ \ \forall (I', p', q') \in \Omega_\mu,\ \varphi' \in \mathbb{T}^{d-1},\ n \in \mathbb{N}^{2d}\\ &h_\infty(I', p'q'; \mu) = E\,,\ \ \forall (I', p', q') \in \Omega_\mu\,.\end{aligned} \tag{3.2}$$

Moreover, setting $\zeta' \equiv p'q'$,

$$\begin{aligned}&\forall (I', p', q') \in \Omega_\mu,\ \ |\partial_{\zeta'} h_\infty| \ge \lambda_0/2\,,\\ &\forall (I', p', q') \in \Omega_\mu,\ \ \varphi' \in \mathbb{T}^{d-1},\ \ q' \neq 0,\ \ |\partial_{p'}(H \circ \Phi)| = |q'\,\partial_{\zeta'} h_\infty| > 0\,,\\ &\forall (I', p', q') \in \Omega_\mu,\ \ \varphi' \in \mathbb{T}^{d-1},\ \ p' \neq 0,\ \ |\partial_{q'}(H \circ \Phi)| = |p'\,\partial_{\zeta'} h_\infty| > 0\,.\end{aligned} \tag{3.3}$$

More precisely, if we denote

$$\mathcal{D}_\tau \equiv \{I \in \mathbb{R}^{d-1} \text{ s.t. } |I - I^*| \le \rho' \text{ and } \partial_I h(I, 0; 0) \text{ is } (\gamma_0, \tau) - \text{Diophantine}\} \tag{3.4}$$

with a suitable $0 < \rho' < \rho$, and a suitable γ_0 depending on μ_0 [i.e. $\gamma_0 = \kappa_\star\sqrt{\mu_0}$], then there exist:

(E1) a function $\mathcal{I}^I_\infty(\zeta; \mu)$, with range in the action space, which is smooth in I, ζ, μ, and (for a fixed I) real analytic in ζ and μ, for $|\zeta| \le R_\infty^2$, $|\mu| \le \mu_0$, verifying $\mathcal{I}^I_\infty(0; 0) = I$,

(E2) a function $\alpha^I_\infty(\zeta; \mu)$, with the same regularity as $\mathcal{I}^I_\infty$, that verifies

$$\alpha^I_\infty(0; 0) = 0 \qquad \textit{and} \qquad \sup_{|I - I^*| \le \rho',\, |\zeta| \le R_\infty^2,\, |\mu| \le \mu_0} |\alpha_\infty| \le c\rho \le \frac{1}{2}\,, \tag{3.5}$$

where c is a constant with the dimensions of the inverse of an action, such that:

(P1) $\partial_{I'} h_\infty(\mathcal{I}^{\bar{I}}_\infty(\zeta; \mu), \zeta; \mu) = \partial_I h(\bar{I}, 0; 0) \cdot (1 + \alpha^{\bar{I}}_\infty(\zeta; \mu))$, $\forall \bar{I} \in \mathcal{D}_\tau$.

(P2) The set Ω_μ in (e3) can be described in the following two ways:

$$\begin{aligned}\Omega_\mu \;&=\; \{(\mathcal{I}^{\bar{I}}_\infty(p'q'; \mu), p', q'),\ \bar{I} \in \mathcal{D}_\tau,\ |p'| \le R_\infty,\ |q'| \le R_\infty\} =\\ &=\; \{(I', p', q') \text{ s.t. } |p'| \le R_\infty,\ |q'| \le R_\infty,\ \ h_\infty(I', p'q'; \mu) = E\\ &\qquad \text{and } \exists \bar{I} \in \mathcal{D}_\tau \text{ s.t. } \partial_{I'} h_\infty(I', p'q'; \mu) = \partial_I h(\bar{I}, 0; 0) \cdot (1 + \alpha^{\bar{I}}_\infty(p'q'; \mu))\}\,.\end{aligned} \tag{3.6}$$

Furthermore,

$$\begin{aligned}\Omega_\mu \;\subseteq\;& \{(I', p', q')\,,\ I' \text{ in a neighborhood of } I^*,\\ & p' \text{ and } q' \text{ in a neighborhood of } 0 \text{ s.t. } h_\infty(I', p'q'; \mu) = E\\ & \text{and } \partial_I h_\infty(I', p'q'; \mu) \text{ is } (\gamma_0/2, \tau) - \text{Diophantine}\}\end{aligned} \tag{3.7}$$

$$\begin{aligned}\Omega_\mu \;\supseteq\;& \{(I', p', q')\,,\ I' \text{ in a neighborhood of } I^*,\\ & p' \text{ and } q' \text{ in a neighborhood of } 0 \text{ s.t. } h_\infty(I', p'q'; \mu) = E\\ & \text{and } \partial_I h_\infty(I', p'q'; \mu) \text{ is } (2\gamma_0, \tau) - \text{Diophantine}\}\,.\end{aligned} \tag{3.8}$$

(P3) Denoting by Dens^E *the* $(2d-1)-$*dimensional restriction of the Lebesgue density on the energy level* $\{(I', \varphi', p', q') \text{ s.t. } h_\infty(I', p'q'; \mu) = E\}$ *and by* dens^E *the* $d-$*dimensional restriction in the space of the actions and the hyperbolic variables of the Lebesgue density on the surface defined by the energy relation* $\{(I', p', q') \text{ s.t. } h_\infty(I', p'q'; \mu) = E\}$*, we have*

$$\mathrm{dens}^E \Omega_\mu \ge 1 - \kappa_\star \sqrt{\mu_0}\,, \qquad \mathrm{Dens}^E \Phi(\Omega_\mu \times \mathbb{T}^{d-1}) \ge 1 - \kappa_\star \sqrt{\mu_0}\,. \tag{3.9}$$

(P4) We have the following equality of sets:

$$\begin{aligned} \Omega_\mu^0 &\equiv \{\mathcal{I}_\infty^{\bar I}(0;\mu),\ \bar I \in \mathcal{D}_\tau\} = \\ &= \{I \quad \text{s.t.} \quad h_\infty(I, 0; \mu) = E \text{ and} \\ &\qquad \exists \bar I \in \mathcal{D}_\tau \text{ s.t. } \partial_{I'} h_\infty(I, 0; \mu) = \partial_I h(\bar I, 0; 0) \cdot (1 + \alpha_\infty^{\bar I}(0;\mu))\}. \end{aligned} \tag{3.10}$$

Furthermore,

$$\begin{aligned} \Omega_\mu^0 &\subseteq \{I\,, \quad \text{in a neighborhood of } I^* \text{ s.t.} \\ &\qquad h_\infty(I, 0; \mu) = E \text{ and } \partial_I h_\infty(I, 0; \mu) \text{ is } (\gamma_0/2, \tau)\text{−Diophantine}\} \end{aligned} \tag{3.11}$$

$$\begin{aligned} \Omega_\mu^0 &\supseteq \{I\,, \quad \text{in a neighborhood of } I^*, \text{ s.t.} \\ &\qquad h_\infty(I, 0; \mu) = E \text{ and } \partial_I h_\infty(I, 0; \mu) \text{ is } (2\gamma_0, \tau)\text{−Diophantine}\}\,. \end{aligned} \tag{3.12}$$

(P5) Denoting by dens_E *the* $(d-2)-$*dimensional restriction of the Lebesgue density to the manifold defined by the energy relation* $h_\infty(I', 0; \mu) = E$*, we have*

$$\mathrm{dens}_E \Omega_\mu^0 \ge 1 - \kappa_\star \sqrt{\mu_0}\,. \tag{3.13}$$

We term the variables (I', φ', p', q') obtained with this procedure "normal coordinates", and refer to the Hamiltonian h_∞ as a "normal form", since the motion in the variables (I', φ', p', q') according to the Hamiltonian h_∞ is particularly simple. As a matter of fact, from (3.2) and (3.6), it follows that the tori (written in normal coordinates)

$$\mathcal{T}_{\bar I} = \{(\mathcal{I}_\infty^{\bar I}(0;\mu), \varphi', 0, 0),\ \varphi' \in \mathbb{T}^{d-1}\} \tag{3.14}$$

are invariant under the Hamiltonian flow of h_∞.
Moreover, it follows that $\mathcal{T}_{\bar I}$ is contained in the manifold with boundary

$$\mathcal{S}_{\bar I} = \{(\mathcal{I}_\infty^{\bar I}(p'q';\mu), \varphi', p', q'),\ \varphi' \in \mathbb{T}^{d-1},\ |p'| \le R_\infty,\ |q'| \le R_\infty\}\,,$$

which is locally invariant and on which the motion is simply:

$$\Phi_{h_\infty}^t(\mathcal{I}_\infty^{\bar I}(p'q';\mu), \varphi', p', q') = (\mathcal{I}_\infty^{\bar I}(p'q';\mu), \varphi' + \omega_\infty t, p' e^{-\lambda_\infty t}, q' e^{\lambda_\infty t})\,, \tag{3.15}$$

provided that $|p' e^{-\lambda_\infty(I', p'q')t}|$, $|q' e^{\lambda_\infty(I', p'q')t}| \le R_\infty$, where $\omega_\infty \equiv \partial_{I'} h_\infty$ and $\lambda_\infty \equiv \partial_{\xi'} h_\infty$ depend only on $\xi' \equiv p'q'$, $\bar I$ and μ. In particular, the whiskers are (locally) parameterized as

$$\begin{aligned} W_{\bar I}^s &= \{(\mathcal{I}_\infty^{\bar I}(0;\mu), \varphi', p', 0)\,,\ \varphi' \in \mathbb{T}^{d-1}\ |p'| \le R_\infty\} \quad \text{and} \\ W_{\bar I}^u &= \{(\mathcal{I}_\infty^{\bar I}(0;\mu), \varphi', 0, q')\,,\ \varphi' \in \mathbb{T}^{d-1}\ |q'| \le R_\infty\}\,. \end{aligned} \tag{3.16}$$

We propose the name of *fan* to call sets of the type $\Omega_\mu \times \mathbb{T}^{d-1}$, which collects the tori, their whiskers, and their normal hyperbolic trajectories.

The method of proof used here yields a very strong normal form, since (3.15) describes exactly the motion of a $(d+1)-$dimensional neighborhood of the torus (even if it does not determine *all* the motions near the torus). This normal form is at the basis of the construction of the unstable orbits presented here in Section 4 (as well as in [11] and in [12]). Another fundamental ingredient in the construction of such unstable trajectories will be the fact that Diophantine properties (or, more generally, rationally independence) of the "old frequency" $\partial_I h$ are preserved for the "new frequency" $\partial_{I'} h_\infty$, according to $(P1)$ of Theorem 3.1.

We remark the fact that the hypotheses of [12] can be readily derived[4] from the conclusions of our KAM Theorem. Namely, hypothesis (ii) of [12] follows from (3.2), (3.3), $(P1)$ and (3.15); hypothesis (iii) of [12] follows from (3.14) and (3.16). We also remind that for "isochronous" systems a much stronger normal form holds. See [20].

We also remark that, leaving out the hyperbolic variables p and q, our proof also establishes the classic KAM Theorem for Lagrangian tori in isoenergetically non-degenerate systems. Moreover, the same result as Theorem 3.1 holds for Hamiltonians depending on several small parameters $\mu^{(1)}, \ldots, \mu^{(n)}$: the proof would remain the same, denoting $\mu \equiv (\mu^{(1)}, \ldots, \mu^{(n)})$ and considering it as a vector.

The proof of Theorem 3.1 is deferred to Section 7.

We now derive from Theorem 3.1 a KAM result for Hamiltonians depending on two parameters ε and μ, in which the parameter ε plays the role of a fixed singular-perturbation parameter, while the dependence on μ will be uniform. We will apply the following Corollary 3.2 in the a-priori stable setting, in which the Lyapunov exponent is not bounded from zero uniformly in the parameter. In reference to this, see Lemma 6.3 below.

Corollary 3.2 *Fix $I^* \in \mathbb{R}^{d-1}$, $E \in \mathbb{R}$. Consider the Hamiltonian*

$$H(I, \varphi, p, q) = h(I, pq; \varepsilon) + f(I, \varphi, p, q; \varepsilon, \mu) \tag{3.17}$$

with h and f real for any real value of $(I, \varphi, p, q, \varepsilon, \mu)$, analytic, for any fixed ε, $|\varepsilon| \le \bar{\varepsilon}$, in

$$\begin{aligned} \mathcal{O}_{\rho,\xi,R,\bar{\mu}} \quad \equiv \quad & \{(I, \varphi, p, q, \mu) \in \mathbb{C}^{(d-1)+(d-1)+1+1+1} \text{ s.t. } |I - I^*| \le \rho,\ |\Im\varphi| \le \xi, \\ & |p| \le R,\ |q| \le R,\ |\mu| \le \bar{\mu}\}\,, \end{aligned}$$

and periodic in the angles φ. Fix $\varepsilon \in \mathbb{R}$, $|\varepsilon| \le \bar{\varepsilon}$, and denote $\zeta \equiv pq$. Assume that there exists a constant $C > 0$ such that

$$\sup_{\mathcal{O}_{\rho,\xi,R,\bar{\mu}} \times \{|\varepsilon| \le \bar{\varepsilon}\}} |h| \le C \qquad \text{and} \qquad \sup_{\substack{|I-I^*| \le \rho,\ |\Im\varphi| \le \xi \\ |p| \le R,\ |q| \le R,\ |\varepsilon| \le \bar{\varepsilon}}} |f| \le C\mu\,, \qquad \forall |\mu| \le \bar{\mu}$$

and that

$$\lambda_0 = \lambda_0(\varepsilon) \equiv \inf_{|I-I^*| \le \rho,\ |\zeta| \le R^2} |\partial_\zeta h| > 0\,.$$

Assume also that h is isoenergetically non-degenerate on the energy level $h(I, pq; \varepsilon) = E$ when the variables vary in $\mathcal{O}_{\rho,\xi,R,\bar{\mu}} \times \{|\varepsilon| \le \bar{\varepsilon}\}$.
Then the analogous statement of Theorem 3.1 holds, with the constant $\kappa_\star$ eventually depending on d, τ, C, but independent of ε.

Now we will show that the estimates (3.9) and (3.13) are optimal, using an example which is an extension of one of [29]. Consider

$$H(I_1, I_2, \varphi_1, \varphi_2, p, q; \mu) \equiv h(I_1, I_2, pq) + \mu(\cos\varphi_1 - 1)$$

[4]Beware of some slight changes in the notation with respect to [12]: for instance, the transformation Φ is called $\mathcal{C}$ in [12], and the normal variables (I', φ', p', q') here correspond to $(\vec{A}', \vec{\psi}, p, q)$ in [12]. For a comparison with the notations in [11], see footnote 4 of [12].

$$\text{with} \qquad h(I_1, I_2, pq) \equiv \frac{I_1^2}{2} + I_2 + pq\,, \tag{3.18}$$

where $\mu > 0$ is a small parameter. If $(I_1, I_2) \in \mathbb{R}^2$ is sufficiently close to the origin, h is isoenergetically non-degenerate. The unperturbed system has the invariant tori $\mathcal{T}_{(I_1,I_2)} = \{(I_1, I_2, \varphi_1, \varphi_2),\ (\varphi_1, \varphi_2) \in \mathbb{T}^2\}$. We will show that the tori destroyed by the perturbation have measure $\approx \sqrt{\mu}$: actually, if $\mathcal{X}$ is the set of (I_1, φ_1) enclosed inside the separatrices of the pendulum $I_1^2/2 + \mu(\cos\varphi_1 - 1)$, we have that there is no surviving KAM torus in $\mathcal{X}$, hence the measure of the holes is

$$\begin{aligned}
& \text{meas}_E\{(I_1, \varphi_1) \in \mathcal{X},\ \varphi_2 \in \mathrm{S}^1,\ h(I_1, I_2, 0) = E\} = \\
= \ & \text{meas}_E\{(I_1, \varphi_1) \in \mathcal{X},\ \varphi_2 \in \mathrm{S}^1,\ I_2 = E - I_1^2/2\} = \\
= \ & \int_{(I_1,\varphi_1)\in\mathcal{X},\ \varphi_2\in\mathrm{S}^1} \sqrt{1 + I_1^2}\, dI_1\, d\varphi_1\, d\varphi_2 \geq \int_{(I_1,\varphi_1)\in\mathcal{X},\ \varphi_2\in\mathrm{S}^1} dI_1\, d\varphi_1\, d\varphi_2 = 2\pi\, \text{meas}\, \mathcal{X} \geq 4\pi^2 \sqrt{\mu}\,.
\end{aligned}$$

In the KAM settings, the measure of the surviving tori is usually large for small values of the perturbation, but a surprising exception can be found in [37].

3.1 The filaments

We will observe that the interpolation between the tori preserved in Theorem 3.1 presents a structure of “filaments”. As stated in (3.14), these tori are interpolated by the smooth function $\mathcal{I}^I_\infty(\zeta;\mu)$, i.e. the preserved invariant tori correspond to $\mathcal{I}^{\bar I}_\infty(0;\mu)$ with $\bar I$ in the Diophantine set $\mathcal{D}_\tau$. Propositions 3.3 and 3.4 will prove that, fixing μ and letting $\bar I$ vary in $\mathcal{D}_\tau$, the curves $\mathcal{I}^{\bar I}_\infty(\zeta;\mu)$ obtained in this way do not have self-intersections, and different curves do not intersect, so they can be seen as filaments, side by side:

Proposition 3.3 *Fixed μ, R_∞ sufficiently small, we have:*

$$\mathcal{I}^I_\infty(\zeta;\mu) = \mathcal{I}^I_\infty(\zeta';\mu)\,,\ \ |\zeta|, |\zeta'| \leq R_\infty \iff \zeta = \zeta'\,.$$

Proof. First notice that $\partial_\zeta \mathcal{I}^I_\infty(\zeta;\mu) \neq 0$. If it were zero, differentiating $h_\infty\big(\mathcal{I}^I_\infty(\zeta;\mu), \zeta;\mu\big) = E$, one would get that the Lyapunov exponent is zero, in contradiction with our assumption. Then apply the Inverse Function Theorem. □

Proposition 3.4 *Fixed μ sufficiently small, let $\sigma : \mathbb{R} \longrightarrow \mathbb{R}^{d-1}$ be a smooth curve in the action space with $\sigma'(0) \neq 0$. Then, $\Psi(s,\zeta) \equiv \mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu)$ is injective near $s = 0 = \zeta$.*

Proof. We will show that $\text{Ran}\, \partial_{(s,\zeta)} \mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu)\Big|_{s=0,\zeta=0,\mu=0} = 2$. Then apply the Inverse Function Theorem. By contradiction, if there were $(a,b) \in \mathbb{R}^2 - \{0\}$ such that

$$0 = a\partial_s \mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu) + b\partial_\zeta \mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu)\Big|_{s=0,\zeta=0,\mu=0}\,. \tag{3.19}$$

Differentiating $h_\infty(\mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu), \zeta;\mu) = E$ we have:

$$\begin{aligned}
(\partial_I h_\infty) \circ (\mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu), \zeta;\mu) \cdot \partial_\zeta \mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu) + (\partial_\zeta h_\infty) \circ (\mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu), \zeta;\mu) &= 0 \\
(\partial_I h_\infty) \circ (\mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu), \zeta;\mu) \cdot \partial_s \mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu) &= 0
\end{aligned}$$

so that, multiplying (3.19) by $(\partial_I h_\infty) \circ (\mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu), \zeta;\mu)$, we get:

$$b\,(\partial_\zeta h_\infty) \circ (\mathcal{I}^{\sigma(0)}_\infty(0;0), 0;0) = 0$$

that, for the non-vanishing of the Lyapunov exponent, implies $b = 0$. Then, in order that $(a,b) \neq (0,0)$, it must be $a \neq 0$; so from (3.19) and $\mathcal{I}^I_\infty(0;0) = I$, we get $0 = \partial_s \mathcal{I}^{\sigma(s)}_\infty(\zeta;\mu)\Big|_{s=0,\zeta=0,\mu=0} = \sigma'(0)$, contradicting the hypothesis. □

4 Existence of unstable orbits

Following [12], we show now that the invariant partially hyperbolic tori, whose existence is ensured by Theorem 3.1, can be easily used to construct orbits with an excursion of order one in the actions, provided that the unstable whisker of each torus intersects *transversally* the stable whisker of the next one. The construction of such an unstable orbit can be seen as a generalization of the one in [2] and §23 of [5]. We also extend this procedure, via an elementary argument of point-set topology, showing the existence of an orbit "drifting towards infinity", as stated in (4.1); see also [16]. Inclination Lemmas and diffusion paths are also considered in Section 4 of [19].

The proof presented here is essentially "topological", in the sense that it makes use only of the continuous dependence on the initial data, so that it covers also cases in which the Hamiltonian has less smoothness.

The problem of showing the validity of the hypothesis of transverse intersection of the whiskers is not addressed in this paper. This assumption is established in [11] for a-priori unstable systems, under suitable regularity conditions and non-degeneracy of the perturbation.

Theorem 4.1 *Consider a chain of whiskered tori* $\{\mathcal{T}_{\bar{I}_j}\}_{\substack{j\in\mathbb{N}\\1\le j\le N}}$ *as in* (3.14)*, with whiskers* $\{W^s_{\bar{I}_j}\}_{\substack{j\in\mathbb{N}\\1\le j\le N}}$ *and* $\{W^u_{\bar{I}_j}\}_{\substack{j\in\mathbb{N}\\1\le j\le N}}$ *as in* (3.16)*, with flow in local coordinates as in* (3.15)*. If* $W^u_{\bar{I}_j}$ *intersects transversally* $W^s_{\bar{I}_{j+1}}$ *with respect to the* $E-$*energy level for* $j=1,\dots,N-1$*, then there exists an open set of points of the phase space arbitrarily close to the first torus* $\mathcal{T}_{\bar{I}_1}$ *that evolves under the flow arbitrarily close to the* N*th torus* $\mathcal{T}_{\bar{I}_N}$ *in a finite time.*
In particular, if there exists a constant c *independent of* μ *such that* $|\bar{I}_1-\bar{I}_N|\ge c$*, then the orbit constructed here exhibits an instability of order one in the action variables.*

Also, if the system admits a sequence of whiskered tori $\{\mathcal{T}_{\bar{I}_j}\}_{j\in\mathbb{N}}$*, verifying the same assumptions as above, with the property that*

$$\lim_{j\longrightarrow\infty}|\bar{I}_j|=\infty\,,$$

then, there exists an orbit $(I(t),\varphi(t),p(t),q(t))$ *such that*

$$\limsup_{t\longrightarrow\infty}|I(t)-I(0)|=\infty\,. \tag{4.1}$$

Proof. Consider a neighborhood U_i of the i-th torus in which the normal form above holds; the condition of transversality assures the existence of a piece of the stable manifold of the $(i+1)-$th torus lying in U_i. It is not difficult to see that this piece of manifold contains a curve at constant $q'=q'_0$

$$\Gamma_i(p)=\{(I'_i(p'),\varphi'_i(p'),p',q'_0)\,,\ |p'|\le r^*\}$$

such that $\Gamma_i(0)\in W^u_i\cap W^s_{i+1}\cap U_i$ and the evolution at time t (in local coordinates) of the point $(I'_i(p'),\varphi'_i(p'),p',q'_0)$ is simply given by

$$(I'_i(p'),\ \varphi'_i(p')+\omega(p')t,\ p'e^{-\lambda_i(p')t},\ q_0e^{\lambda_i(p')t})\ ,$$

where

$$\omega_i(p')\equiv\partial_{I'}h_\infty(I',p'q';\mu)\Big|_{\{I'=I'_i(0),q'=q_0\}}\,,\qquad \lambda_i(p')\equiv\partial_{\zeta'}h_\infty(I',p'q';\mu)\Big|_{\{I'=I'_i(0),q'=q_0\}}$$

and $\zeta'\equiv p'q'$. Hence, using also the irrationality of $\omega_i(p')$, one can see that, given any neighborhood B_i of a point in $W^s_i\cap U_i$, there exists p^*_i, $0<p^*_i<r^*$, and a finite time t^*_i such that the backward evolution of $\Gamma_i(p^*_i)$ at time t^*_i lies inside B_i. By continuity, there exists a small neighborhood B^*_i of $\Gamma_i(p^*_i)$ whose backward evolution at time t^*_i is contained in B_i.
This process can be iterated torus after torus, choosing B_{i+1} as the evolution of B^*_i in the neighborhood U_{i+1} of the $(i+1)-$th torus, leading to an unstable orbit. This proves the first claim of this result.

For the drift towards infinity, notice that, as below, one can construct a sequence of closed ball $\{B_j\}_{j\in\mathbb{N}}$ and a sequence of times $\{t_j\}_{j\in\mathbb{N}}$ such that

(i) each B_j is in a small neighborhood of $\mathcal{T}_{\bar{I}_j}$
(ii) $\Phi^{-t_{j-1}}(B_j) \subseteq B_{j-1}$, $\forall j \in \mathbb{N}$.

Hence, defining $\mathcal{B}_j \equiv \Phi^{-t_1-\ldots-t_{j-1}}(B_j)$ for any $j \in \mathbb{N}$, $j \geq 2$, we have that $\mathcal{B}_j \subseteq \mathcal{B}_{j-1} \subseteq B_1$. A well known argument shows that the intersection of all the $\mathcal{B}_j$'s is not empty. So if $\eta \in \bigcap \mathcal{B}_j$, the orbit $(I(t), \varphi(t), p(t), q(t)) = \Phi^t(\eta)$ has the desired property (4.1).

For further details about the construction of such unstable orbits see [12]. $\square$

5 Whiskered tori for a-priori unstable systems

The following Lemma provides a good "normal form" for the unperturbed part of an a-priori unstable system:

Lemma 5.1 *Let $H(I, \varphi, p, q) = \mathcal{R}(I; \mu) + \mathcal{P}(I, p, q; \mu)$ be analytic for I in a domain of $\mathbb{R}^{d-1}$, p and q in a neighborhood of the origin. Fixed $\mu \in \mathbb{R}$, assume that*

$$\begin{aligned} &\partial_p \mathcal{P}(I, 0, 0; \mu) = 0 = \partial_q \mathcal{P}(I, 0, 0; \mu) \\ &\det \partial^2_{(p,q)} \mathcal{P}(I, 0, 0; \mu) < 0\,. \end{aligned}$$

Then, there exists a canonical transformation $(I, \varphi, p, q) \longleftrightarrow (I', \varphi', p', q')$, real analytic for I' in a suitable domain of $\mathbb{R}^{d-1}$, $\varphi' \in \mathbb{T}^{d-1}$, p' and q' in a suitable neighborhood of the origin, sending H into the new Hamiltonian $h^\star(I', p'q')$, depending only on the actions I' and on the product of the hyperbolic variables $\zeta' \equiv p'q'$. This transformation does not affect the action variables, i.e. $I'(I, \varphi, p, q) = I$.
Of course, this transformation preserves the Lyapunov exponent, which in our case implies

$$\partial_\zeta h^\star(I', 0) = \sqrt{-\det \partial^2_{(p,q)} \mathcal{P}(I', 0, 0)}\,. \tag{5.1}$$

Furthermore, $\forall n \in \mathbb{N}^{d-1}$,

$$\partial^n_{I'} h^\star|_{\zeta=0} = \partial^n_{I'} H|_{p=q=0}\,. \tag{5.2}$$

Also, if there exist $\bar{\mu} > 0$ such that, for any $|\mu| \leq \bar{\mu}$,

$$\det \partial^2_{(p,q)} \mathcal{P}(I, 0, 0; \mu) \leq -C\,, \tag{5.3}$$

for a suitable positive constant C, independent of μ, then there exists $\mu_0 > 0$ such that the above transformation depends analytically on μ, for $|\mu| \leq \mu_0$.

Proof. See [27], where the convergence of the Birkhoff series is shown, or Appendix A3 of [11], in which a KAM algorithm is used. See also [13] for a more general approach. $\square$

Remark. In case condition (5.3) is not fulfilled, the above transformation may experience a very drastic loss of regularity in the parameter. This can be understood looking at equations (A3.9) and (A3.42) of [11], or just considering the following example. We claim that there is no canonical transformation $p = p(p', q'; \mu)$, $q = q(p', q'; \mu)$, *continuously* depending on the parameter μ, that sends the Hamiltonian $p^2/2 + \mu^2(\cos q - 1)$ into $\mu p'q' + (p'q')^2 G(p'q'; \mu)$, with G depending continuously on μ. Arguing by contradiction, we would obtain

$$\frac{1}{2}\Big(p(p', q'; 0)\Big)^2 = (p'q')^2 G(p'q'; 0)\,,$$

which implies $|p(p', q'; 0)| = |p'q'|\sqrt{2\,G(p'q'; 0)}$. Then,

$$\partial_{p'} p(0, 0; 0) = \lim_{p' \to 0} \frac{p(p', 0; 0) - p(0, 0; 0)}{p'} = 0\,,$$

and analogously $\partial_{q'} p(0,0;0) = 0$. Hence

$$\partial_{p'} p(0,0;0)\partial_{q'} q(0,0;0) - \partial_{q'} p(0,0;0)\partial_{p'} q(0,0;0) = 0\,,$$

which contradicts that the transformation is symplectic. $\square$

Here is the application of the KAM Theorem to the a-priori unstable systems:

Theorem 5.2 *Let*

$$H(I,\varphi,p,q) = \mathcal{R}(I;\mu) + \mathcal{P}(I,p,q;\mu) + \mu f(I,\varphi,p,q;\mu)$$

be an a-priori unstable Hamiltonian according to Definition 2.2. Assume that $H_0 \equiv \mathcal{R}+\mathcal{P}$ is isoenergetically non-degenerate on the level surface $H_0 = E$. Then $\exists \mu_0 > 0$ such that, if $|\mu| \le \mu_0$, the energy level is filled by whiskered tori with density at least $1 - O(\sqrt{\mu_0})$. More precisely:

There exist μ_0 and R_∞, $0 < \mu_0 \le \bar\mu$ and $0 < R_\infty \le R$, such that, for $|\mu| \le \mu_0$, there exist:

(e1) a smooth canonical transformation Φ, close to the identity and real analytic (for a fixed action) in the angles, in the hyperbolic variables and in the parameter μ,

(e2) a function $h_\infty : \mathbb{R}^{(d-1)} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ with the same smoothness as Φ,

(e3) a set $\Omega_\mu \subseteq \mathbb{R}^{(d-1)+1+1}$, with density at least $1 - O(\sqrt{\mu_0})$,

such that, fixed $|\mu| \le \mu_0$,

$$\begin{array}{l} \partial^n \left(H \circ \Phi(I',\varphi',p',q')\right) = \partial^n h_\infty(I',p'q';\mu)\,,\ \ \forall (I',p',q') \in \Omega_\mu,\ \varphi' \in \mathbb{T}^{d-1},\ n \in \mathbb{N}^{2d} \\ h_\infty(I',p'q';\mu) = E\,,\ \ \forall (I',p',q') \in \Omega_\mu\,. \end{array} \tag{5.4}$$

Moreover, setting $\zeta' \equiv p'q'$,

$$\begin{array}{l} \forall (I',p',q') \in \Omega_\mu,\ \varphi' \in \mathbb{T}^{d-1},\ q' \ne 0,\ \ |\partial_{p'}(H\circ\Phi)| = |q'\,\partial_{\zeta'} h_\infty| > 0\,, \\ \forall (I',p',q') \in \Omega_\mu,\ \varphi' \in \mathbb{T}^{d-1},\ p' \ne 0,\ \ |\partial_{q'}(H\circ\Phi)| = |p'\,\partial_{\zeta'} h_\infty| > 0\,. \end{array}$$

More precisely: one can find a ball $B \subset \mathbb{R}^{d-1}$ in the actions and a set $\mathcal{D}_\tau$ of the form

$$\mathcal{D}_\tau \equiv \{I \in B \text{ and } \partial_I H_0(I,0;0) \text{ is } (\gamma_0,\tau) - \text{diophantine}\}$$

with a suitable γ_0 [i.e. $\gamma_0 = O(\sqrt{\mu_0})$], such that there exist
(E1) a function $\mathcal{I}^{\bar I}_\infty(\zeta;\mu)$, with range in the action space, which is smooth in I, ζ, μ, and (for a fixed I) real analytic in ζ and μ, for $|\zeta| \le R^2_\infty$, $|\mu| \le \mu_0$, verifying $\mathcal{I}^I_\infty(0;0) = I$,

(E2) a function $\alpha^I_\infty(\zeta;\mu)$ with the same regularity as $\mathcal{I}^I_\infty$, that verifies $\alpha^I_\infty(0;0) = 0$,

such that:

(P1) $\partial_{I'} h_\infty(\mathcal{I}^{\bar I}_\infty(\zeta;\mu),\zeta;\mu) = \partial_I H_0(\bar I,0;0)\cdot(1+\alpha^{\bar I}_\infty(\zeta;\mu))$, $\forall \bar I \in \mathcal{D}_\tau$.

(P2) The set Ω_μ in (e3) can be described in the following two ways:

$$\begin{array}{rcl} \Omega_\mu &=& \{(\mathcal{I}^{\bar I}_\infty(p'q';\mu),p',q'),\ \bar I \in \mathcal{D}_\tau,\ |p'| \le R_\infty,\ |q'| \le R_\infty\} = \\ &=& \{(I',p',q') \text{ s.t. } |p'| \le R_\infty,\ |q'| \le R_\infty,\ h_\infty(I,p'q';\mu) = E \\ && \text{and } \exists \bar I \in \mathcal{D}_\tau \text{ s.t. } \partial_{I'} h_\infty(I,p'q';\mu) = \partial_I H_0(\bar I,0;0)\cdot(1+\alpha^{\bar I}_\infty(p'q';\mu))\}\,. \end{array} \tag{5.5}$$

(P3) Denoting by Dens^E [resp., by dens^E] the $(2d-1)-$dimensional restriction of the Lebesgue density on the energy level $\{(I',p',q') \text{ s.t. } h_\infty(I',p'q';\mu) = E\}$ [resp., the $d-$dimensional restriction in the space of the actions of the Lebesgue density on the energy level], we have

$$\mathrm{dens}^E \Omega_\mu \ge 1 - O(\sqrt{\mu_0})$$

$$\mathrm{Dens}\,^{E}\Phi(\Omega_\mu\times\mathbb{T}^{d-1})\geq 1-O(\sqrt{\mu_0}) \tag{5.6}$$

(P4) We have the following equality of sets:

$$\begin{aligned}\Omega^0_\mu &\equiv \{\mathcal{I}^{\bar I}_\infty(0;\mu),\ \bar I\in\mathcal{D}_\tau\} = \\ &= \{I \text{ s.t. } h_\infty(I,0;\mu)=E \\ &\quad\ \text{and } \exists\bar I\in\mathcal{D}_\tau \text{ s.t. } \partial_{I'}h_\infty(I,0;\mu)=\partial_I H\Big|_{\substack{I=\bar I,\mu=0\\ p=0=q}}\cdot(1+\alpha^{\bar I}_\infty(0;\mu))\}\,.\end{aligned} \tag{5.7}$$

(P5) Denoting by $\mathrm{dens}\,_E$ *the* $(d-2)-$*dimensional restriction of the Lebesgue density to the manifold defined by the energy relation* $h_\infty(I',0;\mu)=E$*,*

$$\mathrm{dens}\,_E\Omega^0_\mu\geq 1-O(\sqrt{\mu_0})\,. \tag{5.8}$$

Finally, one can take $\mu_0=O\Big(\inf(-\det\partial^2_{(p,q)}\mathcal{P})\Big)$.

Proof. Using Lemma 5.1, we obtain the new Hamiltonian

$$\mathcal{H}(I',\varphi',p',q')=h^\star(I',p'q';\mu)+\mu f^\star(I',\varphi',p',q';\mu)\,.$$

Notice that by (5.2) the matrices of isoenergetic non-degeneracy of $h^\star$ and H_0 agree in the origin of the hyperbolic coordinates; and by (5.1) $\partial_{\zeta'}h^\star>0$, where $\zeta'\equiv p'q'$. Therefore, Theorem 3.1 can be applied. □

6 Whiskered tori for a-priori stable systems

In this section, ε will be a strictly positive, fixed, small parameter. Our target will be to look at an a-priori stable system near a simple resonance and recognize that these systems (under extremely mild conditions) are "hyperbolic in the first order". In this way we will be able to apply the previous results to the a-priori stable case too.

We note that this implies that the $d-$dimensional resonant tori break down for generic perturbations, creating $(d-1)-$dimensional whiskered tori. The mechanism of such a breakdown was considered, without measure estimates, in [38] and [25].

Lemma 6.1 *Consider the function*

$$h^{[0]}(I,p,q;\varepsilon)=h(I,p;\varepsilon)+\varepsilon f(I,p,q;\varepsilon) \tag{6.1}$$

with h *and* f *real analytic for* (I,p) *in a domain of* $\mathbb{R}^{d-1}\times\mathbb{R}$ *and* $q\in\mathrm{S}^1$*. Assume that there exists* $(\bar I,\bar p)\in\mathbb{R}^{d-1}\times\mathbb{R}$*, verifying* $\partial_p h(\bar I,\bar p;0)=0$ *and* $\partial^2_p h(\bar I,\bar p;0)\neq 0$*. Assume that the function* $\bar f(q)\equiv f(\bar I,\bar p,q;0)$ *has a non singular critical point, i.e. there exists* $\bar q$ *such that* $\partial_q\bar f(\bar q)=0$ *and* $\partial^2_q\bar f(\bar q)\neq 0$*. Then, two functions exist* $p(I;\varepsilon)$ *and* $q(I;\varepsilon)$*, real analytic for* I *near* $\bar I$ *and* ε *small, with* $p(\bar I;0)=\bar p$*,* $q(\bar I;0)=\bar q$*, such that*

$$\partial_p h^{[0]}(I,\,p(I;\varepsilon),\,q(I;\varepsilon);\varepsilon)=0=\partial_q h^{[0]}(I,\,p(I;\varepsilon),\,q(I;\varepsilon);\varepsilon)\,. \tag{6.2}$$

Moreover, if $\bar f$ *has a non singular maximum and a nonsingular minimum, we can make the previous choice of* $\bar q$ *in order to verify*

$$\partial^2_p h(\bar I,\bar p;0)\,\partial^2_q f(\bar I,\bar p,\bar q;0)<0\,. \tag{6.3}$$

Proof. Apply the Implicit Function Theorem to

$$\mathcal{L}(p,q,I,\varepsilon) \equiv \Big(\partial_p h^{[0]}(I,p,q;\varepsilon),\ \partial_q f(I,p,q;\varepsilon)\Big)$$

near $I = \bar{I}$, $p = \bar{p}$, $q = \bar{q}$ and $\varepsilon = 0$. $\square$

The following Lemma sets the equilibria found in (6.2) in the origin:

Lemma 6.2 *Consider the Hamiltonian system* (6.1)*, under the same assumptions as the previous Lemma. Let $\varphi \in \mathbb{T}^{d-1}$ denote the angles conjugated to the actions I. Then, the canonical transformation associated to the generating function*

$$\mathcal{G}(I^{[1]},p^{[1]},\varphi,q) = \Big(q - q(I^{[1]};\varepsilon)\Big)\ p^{[1]} + p(I^{[1]};\varepsilon)\ \sin\Big(q - q(I^{[1]};\varepsilon)\Big) + \varphi\cdot I^{[1]}$$

sends the Hamiltonian (6.1) *into a new Hamiltonian*

$$h^{[1]}(I^{[1]},p^{[1]},q^{[1]};\varepsilon)\ , \quad \textit{verifying} \quad \partial_{p^{[1]}} h^{[1]}(I^{[1]},0,0;\varepsilon) = 0 = \partial_{q^{[1]}} h^{[1]}(I^{[1]},0,0;\varepsilon)\,. \tag{6.4}$$

Proof. Straightforward check. $\square$

We now inspect the hyperbolic structure of the above $h^{[1]}$ near $I = \bar{I}$ and $p = 0 = q$, showing that, for ε small enough, $h^{[1]}$ inherits such a hyperbolic structure from the one of $h^{[0]}$ stated in (6.3). In detail:

Lemma 6.3 *Let $h^{[1]}$ be the Hamiltonian obtained from $h^{[0]}$ in the previous Lemma. Define*

$$\lambda(I,p,q;\varepsilon) \equiv \sqrt{-\det\partial^2_{(p^{[1]},q^{[1]})} h^{[1]}(I,p,q;\varepsilon)}\,. \tag{6.5}$$

Then,

$$\Big(\lambda(I,0,0;\varepsilon)\Big)^2 = -\varepsilon(\partial_p^2 h\,\partial_q^2 f) - \varepsilon^2\det\partial^2_{(p,q)} f\,,$$

with the functions on the right hand side evaluated in $p = p(I;\varepsilon)$ and $q = q(I;\varepsilon)$. In particular, if ε is small enough, $\lambda(\bar{I},0,0;\varepsilon)$ is real and positive, and $|\Re\lambda(I,p,q;\varepsilon)| \geq c_*\sqrt{\varepsilon}$, for a suitable constant c_*, for any I in a suitable neighborhood of $\bar{I}$ and p and q near 0.

Proof. Straightforward check. $\square$

Lemma 6.4 *Consider the system* (6.1)*. Assume that $(\bar{I},\bar{p}) \in \mathbb{R}^{d-1}\times\mathbb{R}$ verifies $\partial_p h(\bar{I},\bar{p};0) = 0$ and $\partial_p^2 h(\bar{I},\bar{p};0) \neq 0$. Assume that the function $\bar{f}(q) \equiv f(\bar{I},\bar{p},q;0)$ has a non singular maximum and a non singular minimum. Then, there exists a canonical transformation $(I,\varphi,p,q) \longleftrightarrow (I^{[2]},\varphi^{[2]},p^{[2]},q^{[2]})$, defined for $p^{[2]}$ and $q^{[2]}$ in a neighborhood of* 0*, $I^{[2]}$ in a neighborhood of $\bar{I}$ and $\varphi^{[2]} \in \mathbb{T}^{d-1}$, with new Hamiltonian $h^{[2]}(I^{[2]},\zeta^{[2]};\varepsilon)$ verifying*

$$|\partial_{\zeta^{[2]}} h^{[2]}(I^{[2]},p^{[2]}q^{[2]};\varepsilon)| = |\lambda(I^{[2]},p^{[2]},q^{[2]};\varepsilon)| \geq c_*\sqrt{\varepsilon} \tag{6.6}$$

for a suitable constant c_*, for any $I^{[2]}$ in a suitable neighborhood of $\bar{I}$, $p^{[2]}$ and $q^{[2]}$ near 0*, where we defined $\zeta^{[2]} \equiv p^{[2]}q^{[2]}$ and $\lambda(I,p,q;\varepsilon)$ is defined in* (6.5)*. Furthermore,*

$$\partial^n_{I^{[2]}} h^{[2]}(I^{[2]},0;\varepsilon) = \partial^n_I(h+\varepsilon f)(I^{[2]},\ p(I^{[2]};\varepsilon),\ q(I^{[2]};\varepsilon);\varepsilon)\,,\ \ \forall n \in \mathbb{N}^{d-1}\,. \tag{6.7}$$

Proof. First apply Lemma 6.2 to obtain a Hamiltonian like (6.4), and recall also Lemma 6.3. Then apply Lemma 5.1. $\square$

The next theorem will show the existence of whiskered tori near simple resonances for a-priori stable systems. It will follow via Corollary 3.2, applying the previous Lemmas, where $(J_1,\dots,J_d)$ and $(\psi_1,\dots,\psi_d)$ in the next statement will correspond respectively to $(I_1,\dots,I_{d-1},p)$ and $(\varphi_1,\dots,\varphi_{d-1},q)$ of the Lemmas above. This is done making use of a classical result in perturbation theory, namely the Averaging Theorem (see, for instance, §5 of [3] and §52 of [4]).

Theorem 6.5 *Fix $\nu \in \mathbb{N}$, $\nu \geq 2$. Consider the system $H(J,\psi) = h(J) + \varepsilon f(J,\psi;\varepsilon)$, with h and f real analytic for J in a domain of $\mathbb{R}^d$ and $\psi \in \mathbb{T}^d$. Assume that h is isoenergetically non-degenerate on the energy level $h = E$ with respect to the first $(d-1)$ action variables. Let $\bar{J}$ be such that $\partial_{J_d} h(\bar{J}) = 0$, $\partial^2_{J_d} h(\bar{J}) \neq 0$ and let $\partial_{(J_1,\ldots J_{d-1})} h(\bar{J})$ be rationally independent. Set*

$$\mathcal{F}_{\bar{J}}(x) \equiv \frac{1}{\operatorname{meas} \mathbb{T}^{d-1}} \int_{\mathbb{T}^{d-1}} f(\bar{J},\psi_1,\ldots,\psi_{d-1},x;0)\, d\psi_1 \ldots d\psi_{d-1}\,.$$

Assume that $\mathcal{F}_{\bar{J}}$ has nonsingular maximum and minimum. Then, a suitable subset (depending on ν) of the energy level near $\bar{J}$, is filled by whiskered invariant tori with density at least $1 - O(\varepsilon^{\nu/2})$, provided that ε is small enough; more precisely, the tori [resp., the fan[5]] fill the space, near $\bar{J}$, with $(2d-3)-$dimensional density [resp., $(2d-1)-$dimensional density] at least $1 - O(\varepsilon^{\nu/2})$.
More precisely: there exist

(i) a smooth canonical transformation $(J,\psi) = \Phi(I',\varphi',p',q')$, with $I' \in \mathbb{R}^d$, $p',q' \in \mathbb{R}$, $\varphi' \in \mathbb{T}^{d-1}$,

(ii) a smooth function $h_\infty : \mathbb{R}^{d-1} \times \mathbb{R} \longmapsto \mathbb{R}$,
(iii) a set $\Omega_{\varepsilon,\nu} \subset \mathbb{R}^{(d-1)+1+1}$, with density at least $1 - O(\varepsilon^{\nu/2})$,

such that:

$$\begin{aligned}&\partial^n (H \circ \Phi(I',\varphi',p',q')) = \partial^n h_\infty(I',p'q';\varepsilon)\,,\ \forall (I',p',q') \in \Omega_{\varepsilon,\nu},\ \varphi' \in \mathbb{T}^{d-1}, n \in \mathbb{N}^{2d}\\ &h_\infty(I',p'q';\varepsilon) = E\,,\ \forall (I',p',q') \in \Omega_{\varepsilon,\nu}\,.\end{aligned}$$

In the coordinates (I',φ',p',q'), the above mentioned tori are given by

$$\mathcal{T}(I') = \{(I',\varphi',0,0)\,,\ \varphi' \in \mathbb{T}^{d-1}\}\,,$$

for I' in a suitable set $\Omega^0_{\varepsilon,\nu}$, whose density is at least $1 - O(\varepsilon^{\nu/2})$. The corresponding (local) whiskers are

$$\begin{aligned}W^s(I') &= \{(I',\varphi',p',0)\,,\ \varphi' \in \mathbb{T}^{d-1}\ \ |p'| \leq R_\infty\}\\ W^u(I') &= \{(I',\varphi',0,q')\,,\ \varphi' \in \mathbb{T}^{d-1}\ \ |q'| \leq R_\infty\}\,,\end{aligned}$$

for a suitable $R_\infty > 0$.
Furthermore: for any $I' \in \Omega^0_{\varepsilon,\nu}$ there exists a smooth function $\mathcal{I}_{I',\varepsilon,\nu} : \mathbb{R} \longrightarrow \mathbb{R}^{d-1}$ such that $\mathcal{I}_{I',\varepsilon,\nu}(0) = I'$ and

$$\Omega_{\varepsilon,\nu} = \{(\mathcal{I}_{I',\varepsilon,\nu}(p'q'),p',q')\ \ I' \in \Omega^0_{\varepsilon,\nu}\ \ |p'| \leq R_\infty\,,\ |q'| \leq R_\infty\}\,.$$

Moreover, setting $\zeta' \equiv p'q'$ and $\lambda_\infty \equiv \partial_{\zeta'} h_\infty$, we have that $|\lambda_\infty| \geq c^* \sqrt{\varepsilon}$, for a suitable constant $c^* > 0$ and for any $(I',p',q') \in \Omega_{\varepsilon,\nu}$. Also, $\omega_\infty \equiv \partial_{I'} h_\infty$ is a $(\gamma,\tau)-$Diophantine vector with $\gamma = O(\varepsilon^{\nu/2})$ for any $(I',p',q') \in \Omega_{\varepsilon,\nu}$.
Finally, for any $(I',p',q') \in \Omega_{\varepsilon,\nu}$ and for any $\varphi' \in \mathbb{T}^{d-1}$,

$$\Phi^t_{h_\infty}(I',\varphi',p',q') = (I',\ \varphi' + \omega_\infty(I',p'q')\,t,\ p' e^{-\lambda_\infty(I',p'q')t},\ q' e^{\lambda_\infty(I',p'q')t})\,,$$

provided that $|p' e^{-\lambda_\infty(I',p'q')t}|$, $|q' e^{\lambda_\infty(I',p'q')t}| \leq R_\infty$.

Proof. Making use of the Averaging Theorem, we can find a canonical transformation, close to the identity for small ε, sending the Hamiltonian $H(J,\psi)$ of the hypothesis into $H^\flat(I,\varphi,p,q) = h(I,p) + \varepsilon f^\flat(I,p,q;\varepsilon) + O(\varepsilon^\nu)$. Such a transformation is defined in a suitable neighborhood of $\bar{J}$ (which is small if ν is big). Moreover

$$f^\flat(I,p,q;0) = \frac{1}{\operatorname{meas} \mathbb{T}^{d-1}} \int_{\mathbb{T}^{d-1}} f(I,p,\psi_1,\ldots,\psi_{d-1},q;0)\, d\psi_1 \ldots d\psi_{d-1}\,.$$

[5] Recall the notation of the fan at page 10.

Then, use Lemma 6.4 and Corollary 3.2 with $\mu \equiv \varepsilon^\nu$. Notice also that it is important that Theorem 3.1 contains a quantitative estimate on how small μ_0 is. In particular, it must be smaller than $\kappa_* \lambda_0^2$, and this estimate is satisfied if $\lambda \approx \sqrt{\epsilon}$, and $\mu \approx \epsilon^\nu$, with $\nu \geq 2$. □

The statement of the previous Theorem can be sharpened considering Diophantine simple resonances and optimizing the choice of ν as done in the Nekhoroshev theory:

Theorem 6.6 *Consider the system $H(J, \psi) = h(J) + \varepsilon f(J, \psi; \varepsilon)$, under the same assumptions as Theorem 6.5. Assume also that $\partial_{(J_1,\ldots J_{d-1})} h(\bar{J})$ is $(\gamma, \tau)-$Diophantine. Then, if ε is small enough, a neighborhood of $\bar{J}$ in the energy level is filled by whiskered invariant tori with density at least $1 - O(e^{-O(1/\varepsilon^c)})$, where $c > 0$ is a suitable constant. More precisely, the tori [resp., the fan.] fill the space, near $\bar{J}$, with $(2d-3)-$dimensional density [resp., $(2d-1)-$dimensional density] at least $1 - O(e^{-O(1/\varepsilon^c)})$*

Proof. Following the notations of [34], we set $\Lambda \equiv \{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \text{ s.t. } n_1 = \ldots = n_{d-1} = 0\}$, and

$$K \equiv c_1 \left(\frac{\gamma^2}{\varepsilon}\right)^{1/(2\tau+2)}, \qquad \alpha \equiv \frac{\gamma}{2K^\tau}, \qquad r \equiv c_2 \frac{\alpha}{K \sup |h''|}, \tag{6.8}$$

where the c_i's are suitable constants, chosen so that the hypotheses of the "Normal Form Lemma" of [34], page 192, are verified. Applying it to $H(J, \psi)$, it leads to the new Hamiltonian $H^\flat(I, \varphi, p, q) = h(I, p) + \varepsilon f^\flat(I, p, q; \varepsilon) + f_*(I, \varphi, p, q)$, with

$$f^\flat(I, p, q; 0) = \frac{1}{\text{meas}\, \mathbb{T}^{d-1}} \int_{\mathbb{T}^{d-1}} f(I, p, \psi_1, \ldots, \psi_{d-1}, q; 0)\, d\psi_1 \ldots d\psi_{d-1}\,,$$

and the size of f_* is controlled by $\varepsilon\, e^{-O(1/\varepsilon^c)}$. Then, as in the proof of the previous Theorem, apply Lemma 6.4 and Theorem 4.1. □

Notice that, in the proof of the previous Theorem, an explicit dependence of the constants with respect to the size of the domain of analyticity can be easily carried out. Namely, if the strip of analyticity in the angles ψ has width ξ, then the "Normal Form Lemma" bounds the size of f_* by $\varepsilon \exp(-K\xi/6)$, where K is defined in (6.8). Related measure estimates for elliptic equilibria can be found in [15] and in Section 4.1.5 of [7].

7 Proof of the KAM Theorem about partially hyperbolic tori

Proof of Theorem 3.1. The proof presented here makes use of a Newton-type algorithm, that will provide a sequence of canonical transformations converging on a suitable Cantor set. The general step of the algorithm can be summarized as follows:

Defining recursively suitable quantities as in (7.27)–(7.35), and assuming condition (7.36) [which is fulfilled by $\gamma_0 = O(\sqrt{\mu_0})$], there exists a sequence of canonical changes of variables Φ_j, converging in a suitable Cantor set, transforming the Hamiltonian (3.1) into $H_j = h_j + f_j$, with h_j depending only on the actions and on the product of the hyperbolic variables, and $\sup_{V_j} |f_j| \leq \theta_j$, where V_j is a sequence of sets, converging to a Cantor set, and θ_j converges to zero super-exponentially fast.
Also, the set V_j can be written as follows:

$$\begin{aligned} V_j \;=\; & \Big\{(I, \varphi, p, q; \mu) \in \mathbb{C}^{(d-1)+(d-1)+1+1+1} \text{ s.t. } |p| \leq R_j,\, |q| \leq R_j,\, |\Im\varphi| \leq \xi_j,\, |\mu| \leq \mu_0, \\ & \textit{and there exists } \bar{I} \in \mathcal{D}_\tau \textit{ st } |I - \mathcal{I}_j^{\bar{I}}(pq; \mu)| \leq \tilde{\rho}_j \Big\}, \end{aligned}$$

where $\mathcal{D}_\tau$ is defined in (3.4), the quantities R_j, $\tilde{\rho}_j$ and ξ_j are defined in (7.27)–(7.35), and $\mathcal{I}_j$ and α_j are functions defined via the Implicit Function Theorem by the relations

$$\begin{aligned} \partial_I h_j(\mathcal{I}_j^I(\zeta; \mu), \zeta; \mu) &= \partial_I h(I, 0; 0) \cdot (1 + \alpha_0^I(\zeta; \mu)) \cdot \ldots \cdot (1 + \alpha_j^I(\zeta; \mu)) \\ h_j(\mathcal{I}_j^I(\zeta; \mu), \zeta; \mu) &= E\,. \end{aligned}$$

The fact that the KAM tori are of codimension (not higher than) one, i.e. p and q are (at most) one-dimensional, is crucial, in this argument, for the estimate on the small divisors.

In order to have dimensional estimates, we introduce a constant c with the dimensions of the inverse of an action. This is done only to have "dimensional" estimates: the reader who does not find it useful may set $c = 1$ in the sequel. In this way the matrix of isoenergetic non-degeneracy becomes

$$\mathcal{U}_0 \equiv \begin{pmatrix} \partial_I^2 h & c\omega \\ c\omega & 0 \end{pmatrix}, \quad \text{where } \omega \equiv \partial_I h.$$

In the sequel, we will often make use of the following easy relation: for $\delta < 1$

$$\sum_{j\geq 0} e^{-j\delta} = \frac{e^\delta}{e^\delta - 1} \leq \frac{e^\delta}{\delta} \leq \frac{e}{\delta}\,. \tag{7.1}$$

Also, we will use that, if $a \geq 0$, $0 < \delta < 1$, then there exist two constants C and C' (depending only on d and a) such that

$$\begin{aligned} \sum_{n\in\mathbb{Z}^d} |n|^a\, e^{-|n|\delta} &\leq C\,\delta^{-(d+a)} \\ \sum_{\substack{n\in\mathbb{Z}^d\\ |n|\geq N}} |n|^a\, e^{-|n|\delta} &\leq C'\,\delta^{-(d+a)}\, e^{-N\delta/2}\,. \end{aligned} \tag{7.2}$$

Now we start the iterative process. The first step is slightly different from the other ones, since we need to build the first couple of functions $\mathcal{I}_0^I$ and α_0^I as follows.

THE FIRST STEP. Set $h_0(I, pq; \mu) \equiv h(I, pq; \mu)$, $f_0(I, \varphi, p, q; \mu) \equiv \mu f(I, \varphi, p, q; \mu)$. Consider $\rho_0 \leq \rho/4$, $R_0 \leq R/4$ and $\mu_0 \leq \bar{\mu}/4$ small enough: then, via the Implicit Function Theorem, thanks to the isoenergetic non-degeneracy, we can find two functions $\mathcal{I}_0^I(\zeta; \mu)$ and $\alpha_0^I(\zeta; \mu)$, real analytic in $|\zeta| \leq R_0^2$ and $|\mu| \leq \mu_0$, verifying

$$\begin{aligned} &\mathcal{I}_0^I(0;0) = I\,, && \partial_I h_0(\mathcal{I}_0^I(\zeta;\mu), \zeta;\mu) = \omega_0(I)\,(1 + \alpha_0^I(\zeta;\mu))\,, && h_0(\mathcal{I}_0^I(\zeta;\mu), \zeta;\mu) = E\,, \\ &|\mathcal{I}_0^I(\zeta;\mu) - I| \leq \rho_0 \quad \text{and} && |\alpha_0^I(\zeta;\mu)| \leq c\rho_0\,, && \end{aligned} \tag{7.3}$$

where $\omega_0(I) \equiv \partial_I h_0(I, 0; 0)$. Let θ_0, A_0, B_0 and L_0 be such that $\sup|f_0| \leq \theta_0$, $\sup|\partial_I^2 h_0| \leq A_0$, $\sup|\mathcal{U}_0^{-1}| \leq B_0$, and $\sup|\partial_I h_0| \leq L_0$, where the sup is done over $\mathcal{O}_{\rho,\xi,R,\mu_0}$. Obviously, we may choose $\theta_0 = O(\mu_0)$.
For any real analytic $F(I, \varphi, p, q; \mu)$ we will write the Taylor–Fourier expansion

$$F(I, \varphi, p, q; \mu) = \sum_{\substack{k,j\in\mathbb{N}\\ n\in\mathbb{Z}^{d-1}}} F_{kjn}(I;\mu)\, p^k q^j\, e^{in\cdot\varphi}\,.$$

Also, without loss of generality, we may assume that $\forall |I - I^*| \leq 2\rho_0$, $|p| \leq 2R_0$, $|q| \leq 2R_0$, $|\mu| \leq 2\mu_0$, we have that $|\Re\partial_\zeta h_0(I, pq; \mu)| \geq \lambda_0/2$.

THE ITERATIVE SCHEME. Fix N_0 suitably large (see (7.7) below) and $\tilde{\rho}_0$ suitably small (see (7.11) below). Also define $\xi_0 \equiv \xi/2$ and fix δ_0, $0 < \delta_0 < \min\{1, \xi_0/4\}$. Denote f_{kjn}^0 the Taylor–Fourier terms of f_0. Set

$$\chi_0(I', \varphi', p', q'; \mu) \equiv \sum_{\substack{k,j\in\mathbb{N},\, n\in\mathbb{Z}^{d-1}\\ |k-j|+|n|>0,\, |n|\leq N_0}} \frac{-f_{kjn}^0(I';\mu)}{(k-j)\partial_\zeta h_0(I', p'q';\mu) - i\partial_I h_0(I', p'q';\mu)\cdot n}\,(p')^k (q')^j\, e^{in\cdot\varphi'} \tag{7.4}$$

defined on the set

$$
\begin{aligned}
V^0_{\tilde\rho_0,R_0,\xi_0,\mu_0} \quad \equiv \quad & \Big\{(I,\varphi,p,q;\mu)\in\mathbb{C}^{(d-1)+(d-1)+1+1+1} \text{ s.t. } |p|\le R_0,\ |q|\le R_0,\ |\Im\varphi|\le\xi_0,\ |\mu|\le\mu_0, \\
& \text{and there exists } \bar I\in\mathcal{D}_\tau \text{ st } |I-\mathcal{I}_0^{\bar I}(pq;\mu)|\le\tilde\rho_0\Big\}\,.
\end{aligned}
$$

In the definition of $V^0_{\tilde\rho_0,R_0,\xi_0,\mu_0}$, the index "0" high above refers to the index "0" of $\mathcal{I}_0^I$. We now consider the Lie transform $(I,\varphi,p,q)\equiv\Phi^1_{\chi_0}(I',\varphi',p',q')$. From (7.4):

$$
\begin{aligned}
\{h_0,\chi_0\} \quad = \quad & -f_0(I',\varphi',p',q';\mu)+ \\
& + \sum_{\substack{k,j\in\mathbb{N} \\ n\in\mathbb{Z}^{d-1},\,|n|>N_0}} f^0_{kjn}(I';\mu)\,(p')^k(q')^j\,e^{in\cdot\varphi'}+\sum_{k\in\mathbb{N}} f^0_{kk0}(I';\mu)\,(p'q')^k\,. \qquad (7.5)
\end{aligned}
$$

The next c_i's in this section stand for suitable constants (that can be explicitly determined by the algorithm). Set $\gamma_0^*\equiv\min\{\gamma_0,\lambda_0\}$. From (7.2):

$$
\begin{aligned}
\sup_{V^0_{\tilde\rho_0/2,R_0e^{-\delta_0},\xi_0-\delta_0,\mu_0}} \quad & |\textstyle\sum_{\substack{k,j\in\mathbb{N},\,n\in\mathbb{Z}^{d-1}\\ |n|>N_0}} f^0_{kjn}(I';\mu)\,(p')^k(q')^j\,e^{in\cdot\varphi'}|\le \\
& \le c_1\theta_0\delta_0^{-(d+2)}\,e^{-N_0\delta_0/2}=\theta_0^2\delta_0^{-(d+2)}\,Ec^2\,(\gamma_0^*)^{-2}\,, \qquad (7.6)
\end{aligned}
$$

where we have chosen

$$
N_0\equiv\frac{2}{\delta_0}\,\log\frac{c_1(\gamma_0^*)^2}{c^2E\theta_0}\,. \qquad (7.7)
$$

Furthermore, from (7.5), (7.1) and (7.2):

$$
\begin{aligned}
\sup_{V^0_{\tilde\rho_0/2,R_0e^{-\delta_0},\xi_0-\delta_0,\mu_0}} |\{h_0,\chi_0\}| \quad & = \quad \sup_{V^0_{\tilde\rho_0/2,R_0e^{-\delta_0},\xi_0-\delta_0,\mu_0}} |\sum_{\substack{k,j\in\mathbb{N},\,n\in\mathbb{Z}^{d-1}\\ |k-j|+|n|>0\\ |n|\le N_0}} f^0_{kjn}(I';\mu)\,(p')^k(q')^j\,e^{in\cdot\varphi}|\le \\
& \le \quad c_1\,\theta_0\,\delta_0^{-d-2}\,. \qquad (7.8)
\end{aligned}
$$

Estimates on the small divisors. Assume that

$$
\rho_0\le\min\left\{\frac{1}{3c},\,\frac{\lambda_0}{8cL_0N_0}\right\}\,. \qquad (7.9)
$$

Define[6] $A_0^*\equiv A_0$. Assume also that

$$
\gamma_0\le 2A_0^*N_0^{\tau+1}\min\{\rho_0/2,\,R_0^2\}\,. \qquad (7.10)
$$

Define

$$
\tilde\rho_0\equiv\frac{\gamma_0}{2A_0^*N_0^{\tau+1}}\le\min\left\{\frac{\rho_0}{2},\,R_0^2,\,\frac{\gamma_0}{2A_0N_0^{\tau+1}},\,\frac{\lambda_0}{8A_0N_0}\right\} \qquad (7.11)
$$

the inequality above following from (7.10). Now, $\forall\bar I\in\mathcal{D}_\tau$ and $n\in\mathbb{Z}^{d-1}-\{0\}$,

$$
|\partial_I h_0\left(\mathcal{I}_0^{\bar I}(\zeta;\mu),\zeta;\mu\right)\cdot n|=|(1+\alpha_0^{\bar I}(\zeta;\mu))\,\omega_0(\bar I)\cdot n|\ge(1-c\rho_0)\,|\omega_0(\bar I)\cdot n|\ge\frac{2}{3}\,\frac{\gamma_0}{|n|^\tau}\,. \qquad (7.12)
$$

Thus, if $|I-\mathcal{I}_0^{\bar I}(\zeta;\mu)|\le\tilde\rho_0$,

$$
|\partial_I h_0(I,\zeta;\mu)\cdot n| \quad \ge \quad |\partial_I h_0\left(\mathcal{I}_0^{\bar I}(\zeta;\mu),\zeta;\mu\right)\cdot n|-|\partial_I h_0\left(\mathcal{I}_0^{\bar I}(\zeta;\mu),\zeta;\mu\right)-\partial_I h_0(I,\zeta;\mu)|\,N_0\ge
$$

[6] One needs the dummy definition of A_0^* just to make the notation uniform with the $j-$th step of the algorithm, in which one will set $A_j^*\equiv\max\{A_0,A_j\}$.

$$\begin{aligned}
&\geq \frac{2\gamma_0}{3\,|n|^\tau} - A_0\tilde\rho_0 N_0 \geq \\
&\geq \frac{\gamma_0}{6\,|n|^\tau}\,, \qquad \forall n\in\mathbb{Z}^{d-1}-\{0\}.
\end{aligned} \tag{7.13}$$

Besides, since $\omega_0(\bar I)$ is real, if $k\neq j$ and $|I-\mathcal{I}_0^{\bar I}(\zeta;\mu)|\leq\tilde\rho_0$:

$$\begin{aligned}
& |i\partial_I h_0(I,\zeta;\mu)\cdot n+\partial_\zeta h_0(I,\zeta;\mu)\,(k-j)|\geq \\
\geq\; & |\Re[i\partial_I h_0\left(\mathcal{I}^{\bar I}(\zeta;\mu),\zeta;\mu\right)\cdot n+\partial_\zeta h_0(I,\zeta;\mu)\;(k-j)]|-A_0\tilde\rho_0 N_0\geq \\
\geq\; & |\Re[i\left(1+\alpha_0^{\bar I}(\zeta;\mu)\right)\,\omega_0(\bar I)\cdot n+\partial_\zeta h_0(I,\zeta;\mu)\;(k-j)]|-\lambda_0/8= \\
=\; & |\Re[i\alpha_0^{\bar I}(\zeta;\mu)\,\omega_0(\bar I)\cdot n+\partial_\zeta h_0(I,\zeta;\mu)\;(k-j)]|-\lambda_0/8\geq \\
\geq\; & |k-j|\,|\Re\,\partial_\zeta h_0(I,\zeta;\mu)|-|\Re[i\alpha_0^{\bar I}(\zeta;\mu)\,\omega_0(\bar I)\cdot n]|-\lambda_0/8\geq \\
\geq\; & |\Re\,\partial_\zeta h_0(I,\zeta;\mu)|-|i\alpha_0^{\bar I}(\zeta;\mu)\,\omega_0(\bar I)\cdot n|-\lambda_0/8\geq\lambda_0/4\,.
\end{aligned} \tag{7.14}$$

The estimate on the small divisors in χ_0 is thus given by (7.13) and (7.14). These inequalities also show the convergence of the series defining χ_0 on $V^0_{\tilde\rho_0,R_0,\xi_0,\mu_0}$.

Estimates on the Lie transform. From the estimates on the small denominators and (7.2), it follows that

$$\sup_{V^0_{\tilde\rho_0,R_0e^{-\delta_0/2},\xi_0-\delta_0/2,\mu_0}} |\chi_0|\leq c_2\,\frac{\theta_0}{\gamma_0^*}\,\delta_0^{-\kappa_0} \tag{7.15}$$

so that, by the Cauchy Estimate:

$$\begin{aligned}
\sup_{V^0_{\tilde\rho_0/2,R_0e^{-\delta_0},\xi_0-\delta_0,\mu_0}}\;\sup_{1\leq i\leq d-1}|\partial_{I'_i}\chi_0| &\leq c_3\,\frac{\theta_0}{\gamma_0^*\,\tilde\rho_0}\,\delta_0^{-\kappa_1} \\
\sup_{V^0_{\tilde\rho_0/2,R_0e^{-\delta_0},\xi_0-\delta_0,\mu_0}}\;\sup_{1\leq i\leq d-1}|\partial_{\varphi'_i}\chi_0| &\leq c_3\,\frac{\theta_0}{\gamma_0^*}\,\delta_0^{-\kappa_1} \\
\sup_{V^0_{\tilde\rho_0/2,R_0e^{-\delta_0},\xi_0-\delta_0,\mu_0}}|\partial_{p'}\chi_0| &\leq c_3\,\frac{\theta_0}{\gamma_0^*\,R_0}\,\delta_0^{-\kappa_1} \\
\sup_{V^0_{\tilde\rho_0/2,R_0e^{-\delta_0},\xi_0-\delta_0,\mu_0}}|\partial_{q'}\chi_0| &\leq c_3\,\frac{\theta_0}{\gamma_0^*\,R_0}\,\delta_0^{-\kappa_1}\,,
\end{aligned} \tag{7.16}$$

where κ_i's denote suitable constants (depending only on d and τ). Hence, using Lemma A.3, $\forall|t|\leq 3$,

$$\Phi^t_{\chi_0}(V^0_{\tilde\rho_0/4,R_0e^{-4\delta_0},\xi_0-4\delta_0,\mu_0})\subseteq V^0_{\tilde\rho_0/3,R_0e^{-3\delta_0},\xi_0-3\delta_0,\mu_0}\subseteq V^0_{\tilde\rho_0,R_0e^{-\delta_0},\xi_0-\delta_0,\mu_0} \tag{7.17}$$

provided that

$$c_5\,\frac{\theta_0}{\gamma_0^*\,\tilde\rho_0}\,\delta_0^{-\kappa_2}\leq 1\,. \tag{7.18}$$

Estimates on the new Hamiltonian. Define

$$\begin{aligned}
h_0^\dagger(I',\varphi',p',q';\mu) &\equiv \int_0^1(1-t)\,\{\{h_0,\chi_0\},\,\chi_0\}\circ\Phi^t_{\chi_0}(I',\varphi',p',q';\mu)\,dt \\
f_0^\dagger(I',\varphi',p',q';\mu) &\equiv \int_0^1\{f_0,\chi_0\}\circ\Phi^t_{\chi_0}(I',\varphi',p',q';\mu)\,dt \\
f_0^*(I',p'q';\mu) &\equiv \sum_{k\in\mathbb{N}} f^0_{kk0}(I';\mu)(p'q')^k \\
h_1(I',p'q';\mu) &\equiv h_0(I',p'q';\mu)+f_0^*(I',p'q';\mu) \\
f_1(I',\varphi',p',q';\mu) &\equiv h_0^\dagger(I',\varphi',p',q';\mu)+f_0^\dagger(I',\varphi',p',q';\mu)+\sum_{\substack{k,j\in\mathbb{N},\;n\in\mathbb{Z}^{d-1}\\ |n|>N_0}} f^0_{kjn}(I';\mu){p'}^k{q'}^j\,e^{in\cdot\varphi'} \\
H_1(I',\varphi',p',q') &\equiv H\circ\Phi^1_{\chi_0}(I',\varphi',p',q')
\end{aligned}$$

Using Lemma A.5 (at the first order for f_0 and at the second order for h_0) one has

$$\begin{aligned} h_0 \circ \Phi^1_{\chi_0} &= h_0 + \{h_0, \chi_0\} + h_0^\dagger \\ f_0 \circ \Phi^1_{\chi_0} &= f_0 + f_0^\dagger \ . \end{aligned}$$

This implies, by (7.5), that

$$H_1(I', \varphi', p', q') = h_1(I', p'q'; \mu) + f_1(I', \varphi', p', q'; \mu) \, . \tag{7.19}$$

By Lemma A.6, (7.15) and (7.8), making use of (7.17) to control the domains, we obtain:

$$\sup_{V^0_{\tilde\rho_0/4, R_0 e^{-4\delta_0}, \xi_0 - 4\delta_0, \mu_0}} |h_0^\dagger| \le \sup_{V^0_{\tilde\rho_0/3, R_0 e^{-3\delta_0}, \xi_0 - 3\delta_0, \mu_0}} |\{\{h_0, \chi_0\}, \chi_0\}| \le c_6 \frac{\theta_0^2}{\gamma_0^* \tilde\rho_0} \delta_0^{-\kappa_3} \tag{7.20}$$

$$\sup_{V^0_{\tilde\rho_0/4, R_0 e^{-4\delta_0}, \xi_0 - 4\delta_0, \mu_0}} |f_0^\dagger| \le c_6 \frac{\theta_0^2}{\gamma_0^* \tilde\rho_0} \delta_0^{-\kappa_3} \ . \tag{7.21}$$

Hence, by (7.6)

$$\sup_{V^0_{\tilde\rho_0/4, R_0 e^{-4\delta_0}, \xi_0 - 4\delta_0, \mu_0}} |f_1| \le c_7 \frac{\theta_0^2}{\gamma_0^* \tilde\rho_0} \delta_0^{-\kappa_4} \equiv \theta_1 \tag{7.22}$$

Then, setting $\rho_1 \equiv \tilde\rho_0/8$, $R_1 \equiv R_0 \, e^{-4\delta_0}$, $\xi_1 \equiv \xi_0 - 4\delta_0$, we obtain a new Hamiltonian like (7.19) with $\sup_{V^0_{2\tilde\rho_1, R_1, \xi_1, \mu_0}} |f_1| \le \theta_1$.

By the Implicit Function Theorem, we obtain two functions $\mathcal{I}_1^I(\zeta; \mu)$ and $\alpha_1^I(\zeta; \mu)$, real analytic for $|\zeta| \le R_1^2$ and $|\mu| \le \mu_0$ verifying

$$|\mathcal{I}_1^I(\zeta; \mu) - \mathcal{I}_0^I(\zeta; \mu)| \le \rho_1/2 \tag{7.23}$$

$$|\alpha_1^I(\zeta; \mu)| \le c\rho_1/2 \tag{7.24}$$

$$\begin{aligned} \partial_I h_1 \left(\mathcal{I}_1^I(\zeta; \mu), \zeta; \mu\right) &= \partial_I h_0 \left(\mathcal{I}_0^I(\zeta; \mu), \zeta; \mu\right) \cdot \left(1 + \alpha_1^I(\zeta; \mu)\right) = \\ &= \omega_0(I) \cdot \left(1 + \alpha_0^I(\zeta; \mu)\right) \cdot \left(1 + \alpha_1^I(\zeta; \mu)\right) \end{aligned} \tag{7.25}$$

$$h_1 \left(\mathcal{I}_1^I(\zeta; \mu), \zeta; \mu\right) = E \, . \tag{7.26}$$

By construction $V^1_{\rho_1, R_1, \xi_1, \mu_0} \subseteq V^0_{2\rho_1, R_1, \xi_1, \mu_0}$, so that $\sup_{V^1_{\rho_1, R_1, \xi_1, \mu_0}} |f_1| \le \theta_1$, and we can iterate the previous arguments (writing the appropriate index instead of the index 0), from (7.4) onwards.

ITERATION OF THE ALGORITHM. Set $\gamma^* \equiv \min\{\gamma_0, \lambda_0/2\}$. Fix a suitable $\ell > 1$, and define recursively:

$$\delta_j \equiv \frac{\delta_{j-1}}{\ell} = \frac{\delta_0}{\ell^j} \tag{7.27}$$

$$N_j \equiv \frac{2}{\delta_j} \log \frac{c_1 (\gamma_j^*)^2}{c^2 E \theta_j} \tag{7.28}$$

$$\tilde\rho_j \equiv \frac{\gamma_j^*}{2 A_j^* \, N_j^{\tau+1}} \tag{7.29}$$

$$\rho_{j+1} \equiv \frac{\tilde\rho_j}{8} \tag{7.30}$$

$$R_{j+1} \equiv R_j \, e^{-4\delta_j} = R_0 \, e^{-4 \sum_{i=0}^j \delta_i} \tag{7.31}$$

$$\xi_{j+1} \equiv \xi_j - 4\delta_j = \xi_0 - 4 \sum_{i=0}^{j} \delta_i \tag{7.32}$$

$$
\begin{aligned}
\theta_{j+1} &\equiv c_7 \frac{\theta_j^2}{\gamma_j^* \tilde{\rho}_j} \delta_j^{-\kappa_4} && (7.33)\\
\varepsilon_j &\equiv \frac{c^2 E\theta_j}{c_1(\gamma_j^*)^2} && (7.34)\\
\gamma_j^* &\equiv \min\{\gamma_{j-1}^*, \lambda_j\}\,. && (7.35)
\end{aligned}
$$

Obviously

$$
N_j = \frac{2}{\delta_j} \log \frac{1}{\varepsilon_j}\,.
$$

Iterating the scheme, one obtains

$$
H_i(I,\varphi,p,q) = h_i(I,pq;\mu) + f_i(I,\varphi,p,q;\mu)
$$

with

$$
\sup_{V_i} |f_i| \le \theta_i
$$

where

$$
V_i \equiv V^i_{\tilde{\rho}_i, R_i, \xi_i, \mu_0}\,.
$$

In order to apply recursively the algorithm above, one has to check that the following conditions are satisfied at the general $\bar{\jmath}-$th step of the scheme:

$(C1)$ The sup [resp., the inf] over $V_{\bar{\jmath}}$ of a quantity involving only $h_{\bar{\jmath}}$ (or its derivatives up to a suitable order) is less or equal than the double of the corresponding sup [resp., greater or equal than the half of the corresponding inf] over V_0 of the corresponding quantity with index 0 (e.g.: $\lambda_{\bar{\jmath}} \equiv \inf_{V_{\bar{\jmath}}} |\partial_\zeta h_{\bar{\jmath}}| \ge \lambda_0/2$, etc.),

$(C2)$ The matrix $\mathcal{U}_{\bar{\jmath}}$ is nonsingular on $V_{\bar{\jmath}}$,

$(C3)$ $\gamma_j \le 2A^*_{\bar{\jmath}} N_{\bar{\jmath}}^{\tau+1} \min\{\rho_{\bar{\jmath}}/2,\, R^2_{\bar{\jmath}}\}$ and $\rho_j \le \lambda_j/(8cL_jN_j)$,

$(C4)$ There exists a constant C^* such that $\varepsilon_{\bar{\jmath}} \le (C^*\, \Lambda_0^\tau\, \varepsilon_0)^{2^{\bar{\jmath}}}$.

To prove $(C1)$–$(C4)$, we will assume the following **main condition**:

$$
K_1 \left(\log \frac{c_1(\min\{\gamma_0,\lambda_0\})^2}{c^2 E\theta_0} \right)^{K_2} \frac{c^2 E\theta_0}{(\min\{\gamma_0,\lambda_0\})^2} \le 1 \tag{7.36}
$$

where K_1 and K_2 are suitable constants. We remark [see (7.42) below] that this condition is satisfied choosing $\gamma_0 = O(\sqrt{\theta_0})$, for θ_0 small enough, $\theta_0 \le O(\lambda_0^2)$.
The proof of $(C1)$–$(C4)$ is by induction, assuming them true for $i = 1, \ldots, \bar{\jmath}-1$. In these pages, k_i's will stand for suitable constants.
First notice that, by definition of $h_{\bar{\jmath}}$, the relation $\lambda_{\bar{\jmath}} \ge \lambda_{\bar{\jmath}-1}/2$ follows, and so $\gamma^*_{\bar{\jmath}} \ge \gamma^*_{\bar{\jmath}-1}/2$.
Notice also that $\varepsilon_i \ge \varepsilon_{i-1}^2$, $\forall 1 \le i \le \bar{\jmath}$, so

$$
\varepsilon_i \ge \varepsilon_{i-1}^2 \ge \ldots \ge \varepsilon_0^{2^i}\,,\ \forall 1 \le i \le \bar{\jmath}\,. \tag{7.37}
$$

Therefore, defining $\Lambda_0 \equiv \log(1/\varepsilon_0)$,

$$
N_i \le \frac{2^{i+1}\ell^i}{\delta_0} \log \frac{1}{\varepsilon_0} = \frac{2^{i+1}\ell^i}{\delta_0} \Lambda_0\,,\ \forall 1 \le i \le \bar{\jmath} \tag{7.38}
$$

Making use of the inductive hypothesis, this implies that

$$
\tilde{\rho}_i \ge \frac{\gamma_0}{k_1^i\, A_0^*\, \Lambda_0^{\tau+1}}\,,\ \forall 1 \le i \le \bar{\jmath}-1\,. \tag{7.39}
$$

Thus

$$\theta_{\bar{\jmath}} \le \frac{k_2^{\bar{\jmath}} \theta_{\bar{\jmath}-1}^2 \, A_0 \, \Lambda_0^{\tau+1}}{(\gamma_{\bar{\jmath}-1}^*)^2} \, .$$

Then,

$$\varepsilon_{\bar{\jmath}} \le k_3^{\bar{\jmath}} \, A_0 \Lambda_0^{\tau+1} \, c^{-2} E^{-1} \, \varepsilon_{\bar{\jmath}-1}^2 \, . \tag{7.40}$$

Iterating (7.40), one gets $(C4)$. Furthermore, it is easy to see that $\sup_{V_j} |\mathcal{U}_j - \mathcal{U}_{j-1}| \le B_0^{-1} 3^{-j}$, hence

$$\sup_{V_{\bar{\jmath}}} |\mathcal{U}_{\bar{\jmath}} - \mathcal{U}_0| \le \sum_{i=1}^{\bar{\jmath}} \sup_{V_{\bar{\jmath}}} |\mathcal{U}_i - \mathcal{U}_{i-1}| \le B_0^{-1} \sum_{i \ge 1} 3^{-i} = \frac{1}{2B_0} \, .$$

This implies $(C2)$, via Lemma A.2. Incidentally, we have also proved that $B_{\bar{\jmath}} \le 2B_0$. The other relations in $(C1)$ follow in the same way. Also, the already proved $(C4)$ implies that

$$N_{\bar{\jmath}} \ge \frac{2^{\bar{\jmath}+1} \ell^{\bar{\jmath}}}{\delta_0} \log \frac{1}{C^* \, \Lambda_0 \, \varepsilon_0} \, . \tag{7.41}$$

then, recalling (7.38), one obtains $(C3)$.

Passage to the limit. From (7.23):

$$\sup_{|\zeta| \le R_\infty^2, \, |\mu| \le \mu_0} |\mathcal{I}_{j+m}^I - \mathcal{I}_j^I| \le \sum_{i=j}^{j+m-1} \sup_{|\zeta| \le R_\infty^2, \, |\mu| \le \mu_0} |\mathcal{I}_{i+1}^I - \mathcal{I}_i^I| \le \sum_{i=j}^{j+m-1} \rho_i \le \sum_{i \ge j} \frac{\rho_0}{4^i} \, ,$$

showing the uniform convergence of $\mathcal{I}_j^I$ to a suitable $\mathcal{I}_\infty^I$ for $|\zeta| \le R_\infty^2$ and $|\mu| \le \mu_0$.
Also, if we set

$$\alpha_\infty^I(\zeta; \mu) \equiv \prod_{j=0}^{\infty} \big(1 + \alpha_j^I(\zeta; \mu)\big) - 1 \, , \;\; \forall |\zeta| \le R_\infty^2, \;\; |\mu| \le \mu_0 \, ,$$

using the fact that $|\alpha_j| \le c\rho_j \le c\rho_0/4^j$ it is easy to prove that the above product converges uniformly and that $|\alpha_\infty| \le c\rho$.

Via iteration of (7.15), the convergence of the transformation $\Phi_j \equiv \Phi_{\chi_j}^1 \circ \ldots \circ \Phi_{\chi_0}^1$ readily follows. Since the convergences are uniform for complex $|\mu| \le \mu_0$, $|p| \le R_\infty$, $|q| \le R_\infty$, $|\Im \varphi| \le \xi_\infty$, we obtain the claimed analyticity in the angles, in the hyperbolic variables and in the parameter μ.

CHARACTERIZATION OF THE SETS Ω_μ AND Ω_μ^0 OF VALIDITY OF THE THEOREM [see (3.6) and (3.10)] AND MEASURE OF THE PRESERVED TORI. Let $\bar{x}$ such that if $x \ge \bar{x}$ then $c_1 K_1 \, (\log x)^{K_2} \, x^{-1} \le 1$. Set

$$\gamma_0 \equiv \sqrt{\frac{c^2 \, E \, \bar{x} \, \theta_0}{c_1}} = O(\sqrt{\theta_0}) = O(\sqrt{|\mu_0|}) \, . \tag{7.42}$$

Then, the KAM condition (7.36) is satisfied.
Since h_∞ is isoenergetically non-degenerate for I and ζ sufficiently close to I^* and 0 respectively and for μ_0 small enough, without loss of generality, we may assume that $\inf_{I \in B, \, |\zeta| \le R_\infty^2, |\mu| \le \mu_0} |\partial_{I_{d-1}} h_\infty| > 0$, where B stands for a suitable $(d-1)-$dimensional domain.
Then, if we denote by the symbol "tilde" the projection onto the first $d-2$ components and by ι the function implicitly defined by $h_\infty(\tilde{I}, \iota(\tilde{I}), 0; \mu) = E$, Proposition 2.5 implies that

$$\mathcal{A}(\tilde{I}) \equiv \frac{\partial_{\tilde{I}} h_\infty(\tilde{I}, \iota(\tilde{I}), 0; \mu)}{\partial_{I_{d-1}} h_\infty(\tilde{I}, \iota(\tilde{I}), 0; \mu)} \tag{7.43}$$

is a diffeomorphism. This proves that

$$I = \mathcal{I}_\infty^{\bar I}(0;\mu) \iff h_\infty(I,0;\mu) = E \text{ and } \partial_I h_\infty(I,0;\mu) = \omega_0(\bar I)\cdot(1+\alpha_\infty^{\bar I}(0;\mu))\,. \tag{7.44}$$

And this implies (3.10) and (3.11).
We now prove that the function $\mathcal{I}_\infty$ is locally invertible on the $E-$energy level, i.e., there exist a suitable $\rho' > 0$ such that

$$\begin{aligned}&\text{for any } I,\ |I - I^*| \le \rho',\ h_\infty(I,0;\mu) = E, \text{ there exists a unique } \bar I,\ |\bar I - I^*| \le \rho\\ &\text{such that } h_0(\bar I, 0; 0) = E \text{ and } I = \mathcal{I}_\infty^{\bar I}(0;\mu).\end{aligned} \tag{7.45}$$

To show this, recall Proposition 2.5 and consider the local diffeomorphism

$$G(I,\sigma) \equiv (\sigma\partial_I h_0(I,0;0), h_0(I,0;0))\ . \tag{7.46}$$

Given I, let $\bar I$ be the component in the actions of $G^{-1}(\partial_I h_\infty(I,0;\mu),E)$. From (7.43) it follows that $\mathcal{A}(\tilde I) = \mathcal{A}(\tilde{\mathcal{I}}_\infty^{\bar I}(0;\mu))$, proving the existence in (7.45). The uniqueness follows from the fact that $\mathcal{I}_\infty^{\bar I}(0;\mu) = \mathcal{I}_\infty^{\bar y}(0;\mu)$ and $h_0(\bar I,0;0) = E = h_0(\bar y,0;0)$ imply $G(\bar I, 1+\alpha_\infty^{\bar I}(0;\mu)) = G(\bar y, 1+\alpha_\infty^{\bar y}(0;\mu))$. Then, (3.12) readily follows from (7.45). The other characterizations of the set of validity of the Theorem and the related estimates on the measure can be proved with similar arguments. □

* * * * *

Appendix

A Some technical Lemmas

A.1 Some linear algebra

Lemma A.1 *Consider $A \in \mathrm{Mat}(n\times n)$. Let a,b,c,d be $n-$dimensional column vectors and $\alpha,\beta,\gamma \in \mathbb{R}$, with $\beta \ne 0 \ne \gamma$. Then*

$$\det\begin{pmatrix} A & a & b\\ c^T & \alpha & \gamma\\ d^T & \beta & 0\end{pmatrix} = (-\gamma\beta)^{-n+1}\det\left((\gamma a - \alpha b)d^T - \beta(\gamma A - bc^T)\right)\ . \tag{A.1}$$

Proof. We have:

$$\begin{aligned}
\det\begin{pmatrix} A & a & b\\ c^T & \alpha & \gamma\\ d^T & \beta & 0\end{pmatrix} &= \gamma^{-n}\det\begin{pmatrix} \gamma A & \gamma a & \gamma b\\ c^T & \alpha & \gamma\\ d^T & \beta & 0\end{pmatrix} = \gamma^{-n}\det\begin{pmatrix} (\gamma A\ \ \gamma a\ \ \gamma b) - b(c^T\ \ \alpha\ \ \gamma)\\ c^T\ \ \alpha\ \ \gamma\\ d^T\ \ \beta\ \ 0\end{pmatrix} =\\
&= \gamma^{-n}\det\begin{pmatrix} \gamma A - bc^T & \gamma a - \alpha b & 0\\ c^T & \alpha & \gamma\\ d^T & \beta & 0\end{pmatrix} =\\
&= (\gamma\beta)^{-n}\det\begin{pmatrix} \beta(\gamma A - bc^T) & \gamma a - \alpha b & 0\\ \beta c^T & \alpha & \gamma\\ \beta d^T & \beta & 0\end{pmatrix} =\\
&= (\gamma\beta)^{-n}\det\begin{pmatrix} \begin{pmatrix}\beta(\gamma A - bc^T)\\ \beta c^T\\ \beta d^T\end{pmatrix} - \begin{pmatrix}\gamma a - \alpha b\\ \alpha\\ \beta\end{pmatrix} d^T & \begin{matrix}\gamma a - \alpha b\\ \alpha\\ \beta\end{matrix} & \begin{matrix}0\\ \gamma\\ 0\end{matrix}\end{pmatrix} =\\
&= (\gamma\beta)^{-n}\det\begin{pmatrix} \beta(\gamma A - bc^T) - (\gamma a - \alpha b)d^T & \gamma a - \alpha b & 0\\ \beta c^T - \alpha d^T & \alpha & \gamma\\ 0 & \beta & 0\end{pmatrix} =
\end{aligned}$$

$$= \quad (\gamma\beta)^{-n}\cdot(-1)^{2n+3}\gamma\det\begin{pmatrix}\beta(\gamma A-bc^T)-(\gamma a-\alpha b)d^T & \gamma a-\alpha b\\ 0 & \beta\end{pmatrix} =$$
$$= \quad (\gamma\beta)^{-n}\cdot(-1)^{2n+3}\gamma\cdot(-1)^{2n+2}\beta\det\left(\beta(\gamma A-bc^T)-(\gamma a-\alpha b)d^T\right).$$

proving (A.1). □

A.2 Perturbations of nonsingular matrices.

Lemma A.2 *Let M, N be square matrices of the same order. If M is nonsingular and $|N| < 1/|M^{-1}|$, then $(M+N)$ is nonsingular too. Moreover*

$$|(M+N)^{-1}| \le \frac{|M^{-1}|}{1-|M^{-1}|\,|N|}\,.$$

In particular, if M is nonsingular and $|N| \le 1/(2\,|M^{-1}|)$, then $(M+N)$ is nonsingular too and $|(M+N)^{-1}| \le 2\,|M^{-1}|$.

Proof. We have that $M^{-1}\sum_{k\ge0}(-1)^k(M^{-1}N)^k = (M+N)^{-1}$. □

A.3 Estimates on Lie transforms

Lemma A.3 *Let V be a domain of $\mathbb{C}^d\times\mathbb{C}^d$. Fixed $r=(r_1,\dots,r_{2d})\in\mathbb{R}^{2d}$, with $r_i\ge0$, we call*

$$V_r \equiv \{z=(z_1,\dots,z_{2d})\in\mathbb{C}^{2d} \text{ s.t. } \exists w=(w_1,\dots,w_{2d})\in V \text{ s.t. } |z_i-w_i|\le r_i\}\,.$$

Assume that $\chi(x,y)$ is real analytic on V_r. Fixed $r'\in\mathbb{R}^{2d}$, $0\le r'_i<r_i$, then $\Phi^t_\chi(V_{r'})\subseteq V_r$ provided that $|t|\le t_0$ with

$$t_0\max\left\{\frac{\sup_{V_r}|\partial_{y_i}\chi|}{r_i-r'_i}\,,\ \frac{\sup_{V_r}|\partial_{x_i}\chi|}{r_{i+d}-r'_{i+d}}\,,\ i=1,\dots,d\right\}\le 1\,. \tag{A.2}$$

Proof. If $(x,y)\in V_{r'}$, then $\exists(\bar x,\bar y)\in V$ such that $|x_i-\bar x_i|\le r'_i$ and $|y_i-\bar y_i|\le r'_{i+d}$, $1\le i\le d$. Set $(x(t),y(t))\equiv\Phi^t_\chi(x,y)$. If the thesis were false, there would exist $\bar t$, $|\bar t|<t_0$, which is the time of "first exit" from V_r. Explicitly, $(x(t),y(t))\in V_r$ for all $|t|\le|\bar t|$ and $(x(\bar t),y(\bar t))\in\partial V_r$. But

$$|x_i(\bar t)-x_i| = |x_i(\bar t)-x_i(0)| \le \sup_{|t|\le|\bar t|}|\dot x_i(t)|\,|\bar t| < \sup_{V_r}|\partial_{y_i}\chi|\,t_0 \le r_i-r'_i \tag{A.3}$$

that[7]shows $|x_i(\bar t)-\bar x_i| \le |x_i(\bar t)-x_i|+|x_i-\bar x_i| < r_i-r'_i+r'_i = r_i$. In the same way one sees $|y_i(\bar t)-\bar y_i|<r_{i+d}$. So $(x(\bar t),y(\bar t))\in\operatorname{Int}V_r$, in contrast with $(x(\bar t),y(\bar t))\in\partial V_r$. □

We denote by $L_\chi H\equiv\{H,\chi\}$ the Poisson operator and by L^j_χ the operator L_χ applied for j times.

Lemma A.4 *For all $m\in\mathbb{N}$*

$$\frac{d^m}{dt^m}\left(H\circ\Phi^t_\chi(x,y)\right) = (L^m_\chi H)\circ\Phi^t_\chi(x,y)\,.$$

Proof. Induction over m. □

Lemma A.5 *Assume the hypotheses over r, r', χ, t_0 in Lemma A.3. Fix $k\in\mathbb{N}$, $k\ge1$. Then*

$$H\circ\Phi^t_\chi(x,y) = \sum_{j=0}^{k-1}\frac{t^j}{j!}\,L^j_\chi H + t^k R_k(x,y;t)\,,$$

[7]In (A.3) we assumed $\sup_{V_r}|\partial_{y_i}\chi|\ne0$; if not, obviously $|x_i(\bar t_i)-x_i| = 0 < r_i-r'_i$, and the argument goes on in the same way.

with

$$
\begin{aligned}
R_k(x,y;t) &= \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!}\,\frac{d^k}{d\tau^k}\,\left(H\circ\Phi_\chi^\tau(x,y)\right)\big|_{\tau=ts}\,ds && \text{(A.4)}\\
\sup_{V_{r'}}|R_k| &\le \frac{\sup_{V_r}|H|}{(t_0-|t|)^k}\,,\quad \forall |t|<t_0\,. && \text{(A.5)}
\end{aligned}
$$

Proof. Set $\mathcal{F}(t)\equiv H\left(\Phi_\chi^t(x,y)\right)$. By the Taylor expansion one has

$$
\mathcal{F}(t)=\sum_{j=0}^{k-1}\frac{t^j}{j!}\,\frac{d^j\mathcal{F}}{dt^j}(0)+\int_0^1\frac{(1-s)^{k-1}}{(k-1)!}\,\frac{d^j\mathcal{F}}{dt^j}(ts)\,t^k\,ds
$$

so that (A.4) follows from Lemma A.4. Then, by the Cauchy Estimate and Lemma A.3,

$$
\begin{aligned}
\sup_{V_{r'}}|R_k| &\le \sup_{(x,y)\in V_{r'},\,|\tau|\le|t|}\left|\frac{d^k}{d\tau^k}\left(H\circ\Phi_\chi^\tau(x,y)\right)\right|\int_0^1\frac{(1-s)^{k-1}}{(k-1)!}\,ds\le\\
&\le \frac{1}{k!}\cdot k!\cdot\frac{\sup_{(x,y)\in V_{r'},\,|\tau|\le t_0}\left|H\circ\Phi_\chi^\tau(x,y)\right|}{(t_0-|t|)^k}\le\\
&\le \frac{\sup_{V_r}|H|}{(t_0-|t|)^k}\,,
\end{aligned}
$$

proving (A.5). □

Lemma A.6 *Assume the hypotheses over r, r' and χ in Lemma A.3. Assume also that χ is real analytic on V_R with $R_i>r_i$ and that H is real analytic in V_r. Then,*

$$
\sup_{V_{r'}}|L_\chi^j H|\le j!\,\sup_{V_r}|H|\,(\sup_{V_R}|\chi|)^j\left(\max\left\{\frac{1}{(r_i-r_i')(R_{i+d}-r_{i+d})},\ \frac{1}{(r_{i+d}-r_{i+d}')(R_i-r_i)}\right\}\right)^j. \tag{A.6}
$$

Proof. If we set

$$
t_0\equiv\frac{1}{\max\left\{\frac{\sup_{V_r}|\partial_{y_i}\chi|}{r_i-r_i'},\ \frac{\sup_{V_r}|\partial_{x_i}\chi|}{r_{i+d}-r_{i+d}'}\,,\ i=1,\ldots,d\right\}}
$$

we have that t_0 verifies (A.2). By Lemmas A.4 and A.3, one gets by the Cauchy Estimate

$$
\begin{aligned}
\sup_{V_{r'}}|L_\chi^j H| &= \sup_{V_{r'}}\left|\frac{d^j}{dt^j}\bigg|_{t=0}H\circ\Phi_\chi^t\right|\le\frac{j!}{t_0^j}\sup_{V_{r'},\,|t|\le t_0}|H\circ\Phi_\chi^t|\le\\
&\le \frac{j!}{t_0^j}\sup_{V_r}|H|=j!\,\sup_{V_r}|H|\left(\max\left\{\frac{\sup_{V_r}|\partial_{y_i}\chi|}{r_i-r_i'},\ \frac{\sup_{V_r}|\partial_{x_i}\chi|}{r_{i+d}-r_{i+d}'}\right\}\right)^j\le\\
&\le j!\,\sup_{V_r}|H|\,(\sup_{V_R}|\chi|)^j\left(\max\left\{\frac{1}{(r_i-r_i')(R_{i+d}-r_{i+d})},\ \frac{1}{(r_{i+d}-r_{i+d}')(R_i-r_i)}\right\}\right)^j,
\end{aligned}
$$

proving (A.6). □

References

[1] Arnol'd, V.I., *Small denominators and problems of stability of motion in classical and celestial mechanics*, in Russian Mathematical Survey 18(6), 1963, p.91.

[2] Arnol'd, V.I., *Instability of Dynamical Systems with several degrees of freedom*, in Soviet Mathematics – Doklady 5(3), 1964, p.581.

[3] Arnol'd, V.I. ed., *Dynamical Systems III – Encyclopaedia of Mathematical Sciences, Vol. 3*, New York, Springer–Verlag, 1985.

[4] Arnol'd, V.I., *Mathematical methods of Classical Mechanics* New York, Springer–Verlag, 1989.

[5] Arnol'd, V.I. – Avez, A., *Ergodic problems of classical mechanics*, New York, Benjamin, 1968.

[6] Bourgain, J., *On Melnikov's persistency problem*, in Math. Res. Lett., 4(4), 1997, p.445.

[7] Broer, H.W. – Huitema, G.B. – Sevryuk, M.B., *Quasi-Periodic Motions in Families of Dynamical Systems*, New York, Springer–Verlag, 1996.

[8] Cheng, C. – Wang, S., *The surviving of lower-dimensional tori from a resonant torus of Hamiltonian systems*, in Journal of Diff. Equations, 155(2), 1999, p.311.

[9] Chierchia, L., *A direct method for constructing solutions of Hamilton-Jacobi Equation*, in Meccanica, 25(4), 1990, p.246.

[10] Chierchia, L. – Celletti, A., *Construction of analytic KAM surfaces and effective stability bounds*, in Comm. Math. Phys., 118(1), 1988, p.119.

[11] Chierchia, L. – Gallavotti, G., *Drift and diffusion in phase space*, in Ann. Inst. H. Poincaré Phys. Théor., 60(1), 1994, p.1.

[12] Chierchia, L. – Valdinoci, E., *A note on the construction of Hamiltonian trajectories along heteroclinic chains*, in Forum Math., 12(2), 2000, p.247.

[13] DeLatte, D., *On normal forms in Hamiltonian dynamics, a new approach to some convergence questions*, in Ergodic Theory Dynam. Systems, 15(1), 1995, p.49.

[14] Delshams, A. – Gelfreich, V. – Jorba, A. – Seara, T.M., *Exponentially small splitting of separatrices under fast quasiperiodic forcing*, in Comm. Math. Phys., 189(1), 1997, p.35.

[15] Delshams, A. – Gutiérrez, P., *Estimates on invariant tori near an elliptic equilibrium point of a Hamiltonian system*, in Journal of Diff. Equations, 131(2), 1996, p.277.

[16] Delshams, A. – de la Llave, R. – Seara, T.M., *A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of* $\mathbb{T}^2$, Comm. Math. Phys., 209(2), 2000, p.353.

[17] Eliasson, L.H., *Perturbations of stable invariant tori for Hamiltonian systems*, in Ann. Scuola Norm. Sup. Pisa Cl. Sci. IV, 15(1), 1989, p.115.

[18] Eliasson, L.H., *Biasymptotic solutions of perturbed integrable Hamiltonian systems*, in Bol. Soc. Brasil. Mat. (N.S.), 25(1), 1994, p.56.

[19] Fontich E. – Martin P., *Arnold diffusion in perturbations of analytic integrable Hamiltonian systems*, mp_arc@math.utexas.edu, #98-319.

[20] Gallavotti, G., *Hamilton–Jacobi's equation and Arnold's diffusion near invariant tori in a priori unstable isochronous systems*, mp_arc@math.utexas.edu, #97-555.

[21] Gentile, G., *A proof of existence of whiskered tori with quasi-flat homoclinic intersections in a class of almost integrable Hamiltonian systems*, Forum Math. 7(6), 1995, p.709.

[22] Graff, S.M., *On the conservation of Hyperbolic Invariant Tori for Hamiltonian Systems*, in Journal of Diff. Equations, 15, 1974, p.1.

[23] Huang, D. – Zengrong, L., *On the persistence of lower-dimensional invariant hyperbolic tori for smooth Hamiltonian systems*, to appear in Nonlinearity.

[24] Jorba, A. – de la Llave, R. – Zou, M., *Lindstedt series for lower dimensional tori*, in Hamiltonian systems with three or more degrees of freedom, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 533, Dordrecht, Kluwer Acad. Publ., 1999, p.151.

[25] de la Llave, R. – Wayne, C.E., *Whiskered and low dimensional tori in nearly-integrable Hamiltonian systems*, preprint, 1996.

[26] Malgrange, B., *Ideals of differentiable functions*, London, Oxford University Press, 1967.

[27] Moser, J., *The Analytic Invariants of an Area-Preserving Mapping near a Hyperbolic Fixed Point*, in Comm. on Pure and Applied Mathematics, 9, 1956, p.673.

[28] Moser, J., *Stable and random motions in dynamical systems*, Annals of Mathematics Studies, 77, Princeton University Press, 1973.

[29] Neishtadt, A.I., *Estimates in the Kolmogorov Theorem on Conservation of Conditionally Periodic Motions*, in Journal of Applied Mathematics and Mechanics, 45(6), 1982, p.766.

[30] Niederman, L., *Dynamic around a chain of simple resonant tori in nearly integrable Hamiltonian systems*, preprint, mp_arc@math.utexas.edu, #97-142.

[31] Poincaré, H., *Les méthodes nouvelles de la mécanique céleste*, Gauthier Villars, Paris, 1899.

[32] Pöschel, J., *Integrability of Hamiltonian Systems on Cantor Sets*, in Comm. on Pure and Applied Mathematics, 35(5), 1982, p.653.

[33] Pöschel, J., *On elliptic lower-dimensional tori in Hamiltonian systems*, in Math. Zeitschrift, 202(4), p.559.

[34] Pöschel, J., *Nekhoroshev estimates for quasi-convex Hamiltonian systems*, in Math. Zeitschrift, 213(2), 1993, p.187.

[35] Rudnev, M. – Wiggins, S., *KAM Theory Near Multiplicity One Resonant Surfaces*, in Journal of Nonlinear Science, 7, 1997, p.177.

[36] Sevryuk, M.B., *Lower-dimensional tori in reversible systems*, in Chaos, 1(2), 1991, p.160.

[37] Sevryuk, M.B., *Invariant sets of degenerate Hamiltonian systems near equilibria*, mp_arc@math.utexas.edu, #98-684.

[38] Treshchev, D.V., *The mechanism of destruction of resonance tori of Hamiltonian Systems*, in Math. USSR Sbornik, 68(1), 1991, p.181.

[39] Valdinoci, E., *Tori di transizione nella teoria KAM*, tesi di laurea at Università di Roma 3, mp_arc@math.utexas.edu, #98-152.

[40] Whitney, H., *Analytical extension of differentiable functions defined on closed set*, in Trans. A.M.S., 36, 1934, p.63.

[41] Xu, J., *Persistence of elliptic lower-dimensional invariant tori for small perturbation of degenerate integrable Hamiltonian systems*, in J. Math. Anal. Appl., 208(2), 1997, p.372.

[42] You, J., *A KAM theorem for hyperbolic-type degenerate lower-dimensional tori in Hamiltonian systems*, in Comm. Math. Phys., 192(1), 1998, p.145.

[43] You, J., *Perturbations of lower-dimensional tori for Hamiltonian systems*, in Journal of Diff. Equations, 152(1), 1999, p.1.

[44] Zehnder, E., *Generalized implicit function theorems with applications to some small divisor problems II*, in Comm. Pure Appl. Math., 29(1), 1976, p.49.