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FORM PERTURBATIONS OF THE SECOND QUANTIZED DIRAC FIELD

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Abstract. We give a criterion on when a form perturbation of the free Dirac operator defines a form perturbation of the second quantized free Dirac field. Moreover, we show that the potentials allowing this quantization are regular in the sense of Klaus and Scharf. Furthermore we prove that only non-local potentials allow for this construction in three dimensions. In two dimensions, though, all local potentials with finite Dirichlet norm are allowed.

1. Introduction

The construction of Hamiltonians in quantum field theory, is one of the challenges of mathematical physics. Various elaborate procedures have been invented to modify a formally given expression in a physically meaningful way (renormalization) with the aim to end up with a mathematically meaningful object. However, the basic strategy of quantum mechanics for finitely many degrees of freedom (N-particle quantum mechanics), namely to define the Hamiltonian as a (form) perturbation of a positive operator, has attracted relatively low, in fact too low, attention for the quantum mechanics of infinitely many degrees of freedom (quantum field theory). The purpose of this paper is to investigate how far this straightforward strategy is bearing.

This problem is – in general – rather difficult for interacting quantum field theories. The simplest non-trivial case is the external field problem in quantum field theory to which we will restrict our attention. We will study the second quantized Dirac Hamiltonian $\mathbb{D}_0$ and perturbations of it.

In fact, even in this simple case there are only very few results known: Ruijsenaars [12, 13, 14, 15] and Thaller [17], Chapter 10, present the external field

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Hamiltonian in a concise way. (To enhance readability we will follow the notation of Thaller's book although we will introduce the basic notation in Section 3.) The condition for the existence of a perturbed Hamiltonian – without any renormalization – which can be found, e.g., in Thaller's book, namely that the one-particle perturbation is trace class (see (35)), is rather strong. Including the normal ordering with respect to the free field allows a weaker hypothesis. The presently known best results in this context were given by Carey and Ruijsenaars [1] under a certain Hilbert-Schmidt condition on the underlying one-particle operator (see the end of Subsection 2.5) and by Fredenhagen [3]. Our first result is a further weakening of the requirement on the one-particle potential (see Section 3, Theorems 1 and 2). In fact we will show that our condition is not only sufficient but also necessary for relative boundedness.

Naturally the question arises: What is the relation between the way of Section 3 to define the Hamiltonian of a quantum field and other approaches – in particular the second quantization in the Furry picture[1]? A particular class of Hamiltonians of interest in the Furry picture are those Hamiltonians that have a ground state – called the dressed vacuum – which is an element of the Fock space of the free field. External electro-magnetic potentials (and if local, the fields defined through them) that have this property are called *regular*. An early treatment of this problem was given by Moses [9, 10] (see also Friedrichs [4], Chapter 28). Later, Klaus and Scharf [7] showed that it is sufficient and necessary for a potential V to generate a regular external field that the difference of the projections onto the positive spectral subspace of the free Dirac operator and the Dirac operator perturbed by V is a Hilbert-Schmidt operator. They also show, that potentials with singularities of the type $1/|x|^\gamma$ with $0 \leq \gamma < 1$ are allowed. That $\gamma = 1$ does not yield a regular field was shown by Klaus [6]. This problem is addressed in Section 4. The first result of this section, Theorem 5, can be stated loosely speaking as follows: all external fields that allow for the constructions of external field Hamiltonians of Section 3 are indeed regular fields. (Note, that this is not a trivial statement, since the Hamiltonian defined from V through the procedure of Section 3 is – in general – not unitarily equivalent to the one constructed in the Furry picture.) The second more important result of this section is, that the weakened condition – as compared with Carey and Ruijsenaars [1] and Fredenhagen [3] – still does not allow for local potentials in dimension three. The situation is different, though, in two dimensions as will be shown in Subsection 4.3.

2. Some Basic Notation

In this section we fix some notation – following largely Thaller [17] – in order to formulate our results in a precise way. The reader familiar with the tenth chapter of Thaller's book can proceed immediately to the next section.

2.1. **The Dirac Operator.** We start from the free Dirac operator D_0

$$D_0 := c\alpha \cdot \frac{\hbar}{i}\nabla + mc^2\beta, \tag{1}$$

where the $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the four Dirac matrices, self-adjoint 4×4 matrices in standard representation (Thaller [17]), and c, the velocity of light, and $\hbar$, the rationalized Planck constant, are positive physical constants that we will take equal to one in the following. The operator D_0 can be self-adjointly realized in $\mathfrak{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$. Its domain is $H^1(\mathbb{R}^3) \otimes \mathbb{C}^4$.

[1]In the Furry picture the Hamiltonian is defined through a renormalization different from normal ordering with respect to free electron and positron subspace: the initial normal ordering with respect to the free field has been replaced by the one given by the perturbed Dirac operator. (See also the remarks below Equation (4))

The Dirac operator for an electron in the electric field $\nabla\varphi$ and the magnetic field $\nabla \times \mathfrak{A}$ is given by the general Dirac operator

$$D = c\alpha \cdot (\frac{\hbar}{i}\nabla + e\mathfrak{A}) + mc^2\beta - e\varphi \tag{2}$$

where e is the electronic charge. In the following, we will always assume, that D is self-adjoint. We will also use the notation $V := -e\varphi$. The operator D can be self-adjointly implemented in $\mathfrak{H}$, in fact, for not too singular potentials its domain agrees with the domain of the free Dirac operator.

2.2. **Relevant Hilbert Spaces.** We fix now a Dirac operator D. Its spectral subspace fixes the notion of electrons and positrons: The subspace

$$\mathfrak{F}_+^{(1)} := \mathfrak{H}_+ := P_+(\mathfrak{H}) \tag{3}$$

is the *electron* state space. Here $P_+ := \chi_{(0,\infty)}(D)$ is the orthogonal projection onto the positive spectral subspace of D.

To define the *positron* state space we denote by C the anti-unitary transformation

$$C\psi := i\beta\alpha_2\bar{\psi}.$$

Then

$$\mathfrak{F}_-^{(1)} := C\mathfrak{H}_- \tag{4}$$

is the positron state space where $\mathfrak{H}_- := P_-(\mathfrak{H}) := [\chi_{(-\infty,0)}(D)](\mathfrak{H})$.

In passing, we remark that in the special case that D is the free Dirac operator, one says that one describes the resulting quantum field in the "free picture". If normal ordering with respect to the given external potential is used, one says that one describes the resulting quantum field in the "Furry picture". Many other intermediate choices are possible. – Although the word "picture" suggests that these procedures describe the same physical situation, it should be noted at this point that it is not clear that Hamiltonians resulting from various analogous constructions in these pictures are unitarily equivalent. In fact, it might not be possible to define Hamiltonians for all choices. Which of these "pictures" is suitable to describe the physical situation at hand is not merely a mathematical problem but mainly a physical one. Some hints in this direction can be found in Sucher [16], Chapter 3.

We now continue our constructions of Hilbert spaces and turn to the definition of the multi-particle spaces. As usual, they are given as the antisymmetric tensor products of these spaces, i.e., the n-particle space for electrons is

$$\mathfrak{F}_+^{(n)} := \begin{cases} \mathbb{C} & n = 0 \\ \bigwedge_{\nu=1}^{n} \mathfrak{H}_+ & n \in \mathbb{N} \end{cases}$$

and the m-particle space for positrons is

$$\mathfrak{F}_-^{(m)} := \begin{cases} \mathbb{C} & m = 0 \\ \bigwedge_{\nu=1}^{m} C\mathfrak{H}_- & m \in \mathbb{N}. \end{cases}$$

The space $\mathfrak{F}^{(n,m)}$ of n electrons and m positrons is the tensor product of these two, i.e.,

$$\mathfrak{F}^{(n,m)} := \mathfrak{F}_+^n \otimes \mathfrak{F}_-^m.$$

Accordingly we have the electron-positron Fock space

$$\mathfrak{F} := \bigoplus_{n,m=0}^{\infty} \mathfrak{F}^{(n,m)}.$$

2.3. **Creation and Annihilation Operators.**

2.3.1. *Particles.* For any $f \in \mathfrak{H}$, we define in $\mathfrak{F}$ the "particle annihilation operator" $a(f)$, which maps each subspace $\mathfrak{F}^{(n+1,m)}$ into $\mathfrak{F}^{(n,m)}$,

$$
\begin{aligned}
(5)\quad & (a(f)\psi)^{(n,m)}(x_1,\ldots,x_n;y_1,\ldots,y_m) \\
&= \sqrt{n+1}\sum_{\sigma=1}^{4}\int_{\mathbb{R}^3} d^3\mathbf{x}\overline{(P_+f)(x)}\psi^{(n+1,m)}(x,x_1,\ldots,x_n;y_1,\ldots,y_m),
\end{aligned}
$$

where $x=(\mathbf{x},\sigma)$. We observe that the map $f\mapsto a(f)$ is anti-linear.

We also set – in slight abuse of notation – $a(p) := a(\mathbf{p},s) := a(e^{i\mathbf{p}\cdot}u_s(\mathbf{p})/(2\pi)^{3/2})$ for $s=1,2$. For fixed $\mathbf{p}$ the vectors $u_s(\mathbf{p})$ are defined as eigenvectors

$$(\alpha\cdot\mathbf{p}+m\beta)u_s(\mathbf{p}) = \pm E(\mathbf{p})u_s(\mathbf{p})$$

with the plus sign for $s=1,2$ and the minus sign for $s=3,4$, i.e., solutions of the Dirac equation in Fourier variables. These vectors can be picked as

$$
\begin{aligned}
&u_1(\mathbf{p}) := \sqrt{\tfrac{m+E(\mathbf{p})}{2E(\mathbf{p})}}\begin{pmatrix}1\\0\\ \frac{p_3}{m+E(\mathbf{p})}\\ \frac{p_1+ip_2}{m+E(\mathbf{p})}\end{pmatrix}, \quad u_2(\mathbf{p}) := \sqrt{\tfrac{m+E(\mathbf{p})}{2E(\mathbf{p})}}\begin{pmatrix}0\\1\\ \frac{p_1-ip_2}{m+E(\mathbf{p})}\\ \frac{-q_3}{m+E(\mathbf{p})}\end{pmatrix},\\
(6)\quad &\\
&u_3(\mathbf{p}) := \sqrt{\tfrac{m+E(\mathbf{p})}{2E(\mathbf{p})}}\begin{pmatrix}-\frac{p_3}{m+E(\mathbf{p})}\\ -\frac{p_1+ip_2}{m+E(\mathbf{p})}\\1\\0\end{pmatrix}, \quad u_4(\mathbf{p}) := \sqrt{\tfrac{m+E(\mathbf{p})}{2E(\mathbf{p})}}\begin{pmatrix}\frac{-p_1+ip_2}{m+E(\mathbf{p})}\\ \frac{q_3}{m+E(\mathbf{p})}\\0\\1\end{pmatrix}.
\end{aligned}
$$

Note that $a(\mathbf{p},s)$ and its formal adjoint $a^*(\mathbf{p},s)$ are not defined as operators into the Fock space. Loosely speaking we can view $a^*(\mathbf{p},s)$ as creation operator of a free electron with energy $E(\mathbf{p})$ and spin s.

Next we define the "particle creation operator" as the mapping $\mathfrak{F}^{(n-1,m)}$ into $\mathfrak{F}^{(n,m)}$ given by

(7)

$$(a^*(f)\psi)^{(n,m)} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(-1)^{j+1}P_+f(x_j)\psi^{(n-1,m)}(x_1,\ldots,\hat{x}_j,\ldots,x_n;y_1,\ldots,y_m).$$

As usual the hat indicates that the corresponding argument is omitted. The operator $a^*(f)$ is the adjoint of $a(f)$ and the map $f\mapsto a^*(f)$ is linear.

We then have the standard "**C**anonical **A**nti-Commutation **R**elations", or short CAR , where the braces are the anti-commutator

$$
\begin{aligned}
(8)\quad & \{a(f_1),a(f_2)\} &=& \{a^*(f_1),a^*(f_2)\} = 0,\\
(9)\quad & \{a(f_1),a^*(f_2)\} &=& (f_1,P_+f_2)\mathbf{1},
\end{aligned}
$$

where (f_1,f_2) is the scalar product in $L^2(\mathbb{R}^3)^4$. Moreover, $a^*(f)$ is – as the notation suggests – the adjoint of $a(f)$. Because we are in a fermionic situation, $a(f)$ and $a^*(f)$ are bounded operators on $\mathfrak{F}$, in fact we have

$$(10)\qquad \|a(f)\psi\|^2+\|a^*(f)\psi\|^2 = (\psi,\{a(f),a^*(f)\}\psi)_{\mathfrak{F}} = \|P_+f\|^2\|\psi\|^2.$$

For the vacuum state Ω, a unit vector spanning $\mathfrak{F}^{0,0}$, we obtain

$$(11)\qquad a(f)\Omega = 0\in\mathfrak{F}$$

and

$$(12)\qquad (a^*(f)\Omega)^{(n,m)} = \begin{cases} P_+f & (n,m)=(1,0)\\ 0 & \text{otherwise.}\end{cases}$$

2.3.2. *Anti-Particles.* We first define the "anti-particle annihilation operator" by

$$(b(g)\psi)^{(n,m)}(x_1,\dots,x_n;y_1,\dots,y_m) = (-1)^n\sqrt{m+1}\sum_{t=1}^{4}\int_{\mathbb{R}^3} d^3\mathbf{y}\overline{[CP_-g](y)}\psi^{(n,m+1)}(x_1,\dots,x_n;y,y_1,\dots,y_m), \tag{13}$$

and the "anti-particle creation operator"

$$(b^*(g)\psi)^{(n,m)}(x_1,\dots,x_n;y_1,\dots,y_m) = \frac{(-1)^n}{\sqrt{m}}\sum_{k=1}^{m}(-1)^{k+1}[CP_-g](y_k)\psi^{(n,m-1)}(x_1,\dots,x_n;y_1,\dots,\hat{y}_k,\dots,y_m). \tag{14}$$

We observe that, due to the anti-linearity of C, the map $g\mapsto b(g)$ is linear and the map $g\mapsto b^*(g)$ is anti-linear.

The "CAR" for the positron creation and annihilation operators are

$$\{b(g_1),b(g_2)\} = \{b^*(g_1),b^*(g_2)\}=0, \tag{15}$$

$$\{b(g_1),b^*(g_2)\} = (g_2,P_-g_1)\mathbf{1} \tag{16}$$

analogously to the one of the electron operators.

Let us denote by an operator with superscript $\sharp$ either the operator itself or its adjoint, i.e., $b^\sharp=b^\star$ or b and $a^\sharp=a^\star$ or a, we have

$$\{b^\sharp(g),a^\sharp(f)\}=0\ . \tag{17}$$

As before $b(g)$ and $b^*(g)$ are bounded operators on $\mathfrak{F}$. We note that b depends linearly and b^* anti-linearly on g. For the vacuum state Ω we obtain analogously to (11)

$$b(g)\Omega=0\in\mathfrak{F} \tag{18}$$

and

$$(b^*(g)\Omega)^{(n,m)}=\begin{cases} CP_-g & \text{for } (n,m)=(0,1)\\ 0 & \text{otherwise.}\end{cases} \tag{19}$$

We observe that the equations $a(f)\psi=0$, $b(g)\psi=0$ for all $f\in\mathfrak{H}_+$ and $g\in\mathfrak{H}_-$ imply $\psi=\alpha\Omega$ for some complex α. This gives the uniqueness of the vacuum up to a constant of modulus one.

In analogy with $a(p)$ and $a^*(p)$ one can also introduce the corresponding operators $b(p)$ and $b^*(p)$.

2.4. **The Field Algebra.** As in Subsection 2.2 assume that the Hilbert space can be written as a sum of two orthogonal subspaces, i.e., $\mathfrak{H}=\mathfrak{H}_+\oplus\mathfrak{H}_-$. This is actually the case for $\mathfrak{H}:=L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ with our choice made in (3) and (4), if zero is not in the spectrum of the Dirac operator D in (2), which we will assume in the following. Recall, that we denote the projections on $\mathfrak{H}_\pm$ by $P_\pm$. For any $f\in\mathfrak{H}$, we define the "field operator" $\Psi(f)$ on the Fock space $\mathfrak{F}$ by

$$\Psi(f)=a(f)+b^*(f). \tag{20}$$

This field operator is bounded on $\mathfrak{F}$ and depends anti-linearly on f. The adjoint operator $\Psi^*(f)$ is also bounded and depends linearly on f. We have

$$\Psi^*(f)=a^*(f)+b(f). \tag{21}$$

In terms of the fields operators, the CAR become

$$\{\Psi(f_1),\Psi(f_2)\} = \{\Psi^*(f_1),\Psi^*(f_2)\}=0, \tag{22}$$

$$\{\Psi(f_1),\Psi^*(f_2)\} = (f_1,f_2)\mathbf{1}, \tag{23}$$

for $f_1,f_2\in\mathfrak{H}$.

We have also

$$\|\Psi(f)\psi\|^2 + \|\Psi^*(f)\psi\|^2 = (\psi, \{\Psi(f), \Psi^*(f)\}\psi)_{\mathfrak{F}} = \|f\|^2\|\psi\|^2, \tag{24}$$

and hence (after some computation)

$$\|\Psi(f)\| = \|\Psi^*(f)\| = \|f\|. \tag{25}$$

We remark that Ω satisfies the equations

$$\Psi(f)\Omega = 0, \ \forall f \in \mathfrak{H}_+, \tag{26}$$

and

$$\Psi^*(f)\Omega = 0, \ \forall f \in \mathfrak{H}_-. \tag{27}$$

This also characterizes the vacuum state up to a constant of modulus one.

2.5. **Second Quantization of One-Particle Observables.** In this section we will "second quantize" linear self-adjoint operators A in the underlying one-particle Hilbert space $\mathfrak{H}$. We will assume that the form domain of the operator A is invariant under $P_\pm$. Given the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ as given through (3) and (4), we can identify A with the matrix

$$A := \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} \tag{28}$$

where $A_{++} = P_+AP_+$, $A_{+-} = P_+AP_-$, $A_{-+} = P_-AP_+$ and $A_{--} = P_-AP_-$ occur. We have $A_{+-}^* = A_{-+}$, because of the self-adjointness of A. In the case where the subspaces $\mathfrak{H}_\pm$ are left invariant by A, i.e., in the case that $A_{+-} = A_{-+} = 0$, this can be easily done and is very much the same as in the situation were no anti-particles are around. Our prime interest is thus directed to the situation where the off-diagonal terms do not vanish.

At this point we would like to begin to define the second quantization of A. We start with the formal expression

$$A\Psi^*\Psi = \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}(e_i, Ae_j)\Psi^*(e_i)\Psi(e_j) = \sum_{j=1}^{\infty}\Psi^*(Ae_j)\Psi(e_j), \tag{29}$$

where the $e_1, e_2, e_3, \dots$ form an orthonormal basis of $\mathfrak{H}$.

Writing (29) in terms of the operators a and b gives a sum of 4 terms

$$A\Psi^*\Psi = Aa^*a + Aa^*b^* + Aba + Abb^*, \tag{30}$$

where each term is defined in the following way, after choosing orthonormal bases $\{f_j\}$ and $\{g_k\}$ of $\mathfrak{H}_+$ and $\mathfrak{H}_-$,

$$Aa^*a \equiv \sum_{ij}(f_i, Af_j)a^*(f_i)a(f_j) \equiv A_{++}a^*a, \tag{31}$$

$$Aa^*b^* \equiv \sum_{ik}(f_i, Ag_k)a^*(f_i)b^*(g_k) \equiv A_{+-}a^*b^*, \tag{32}$$

$$Aba \equiv \sum_{kj}(g_k, Af_j)b(g_k)a(f_j) \equiv A_{-+}ba, \tag{33}$$

$$Abb^* \equiv \sum_{k\ell}(g_k, Ag_\ell)b(g_k)b^*(g_\ell) \equiv A_{--}bb^* \tag{34}$$

where the elements of the matrix representation (28) of A occur.

An elementary but essential question is, whether the so defined operator has the vacuum vector in its form domain. Computing $(\Omega, A\Psi^*\Psi\Omega)$ gives

$$(\Omega, A\Psi^*\Psi\Omega) = (\Omega, Ab^*b\Omega) = \operatorname{tr} A_{--}. \tag{35}$$

Hence, if A_{--} is not trace class, not even the vacuum belongs to the domain of the formally defined operator $A\Psi^*\Psi$ or to the form domain of $A\Psi^*\Psi$.

The latter expression represents, in the case of the Hamiltonian the energy of the electrons with negative energy. According to the idea of Dirac these should be filled and their energy should be discarded. This "rule" is implemented by what is called "normal ordering" or "Wick ordering":

- In each product all creation operators are moved to the left of the annihilation operators
- For each transposition which is necessary to perform the rearrangement the product is multiplied by a factor -1 (according to the CAR).

This procedure is denoted by double dots

$$:Abb^*: = -\sum_{kl}(g_k, Ag_l)b^*(g_l)b(g_k). \tag{36}$$

Using the canonical anti-commutation relations, we have $:Abb^*: = Abb^* - \operatorname{tr} A_{--}$ and the effect of this renormalization is that

$$(\Omega, :A\Psi^*\Psi:\Omega) = 0, \tag{37}$$

which shows that normal ordering really implements Dirac's idea. Thus one is motivated to define the second quantization of a self-adjoint operator on $\mathfrak{H}$ formally as

$$\mathbb{A} :=: A\Psi^*\Psi := A_{++}a^*a + A_{+-}a^*b^* + A_{-+}ba - A_{--}b^*b. \tag{38}$$

Our aim, which we will tackle in Section 3, will be, to give this formal expression a meaning in the case where A is a perturbation of the free Dirac operator, i.e., $A = D_0 + V$.

The diagonal elements can be treated as for Hamiltonians defined on the multi-particle spaces $\mathfrak{F}^{(n,0)}$ and $\mathfrak{F}^{(0,m)}$. The non-diagonal terms are more difficult but have also been previously treated. In this context the results of Carey and Ruijsenaars [1] are to be mentioned who show that A can be second quantized with the domain of the number operator $\sum_\nu (a(f_\nu)^* a(f_\nu) + b(g_\nu)^* b(g_\nu))$, if $A_{+-} \in \mathfrak{S}_2$. (Here and in the following we will denote the Hilbert-Schmidt operators over a Hilbert space $\mathfrak{H}$ by $\mathfrak{S}_2(\mathfrak{H})$. The compact operators will be denoted by $\mathfrak{S}_\infty(\mathfrak{H})$. [In case where there is no confusion possible, we will drop the reference to the Hilbert space.] The corresponding operator norms will be denoted by an index 2 and ∞.) Moreover, Fredenhagen's work has to be mentioned here, who starts off in the same spirit as we but restricts to bounded operators. His work is closest to what we will do.

3. Perturbation of the Free Second Quantized Dirac Hamiltonian

In this section we are interested in perturbations of the free second quantized and normal ordered Dirac operator $\mathbb{D}_0$. We assume that the one-particle operator A of Subsection 2.5 to be second quantized is a perturbation of the free Dirac operator, i.e., we make the choice $A := D := D_0 + V$. Throughout the rest of the paper, we shall always assume, that V is a symmetric quadratic form such that the form domain $\mathfrak{Q}(D_0)$ of D_0 is contained in the form domain of V and that the quadratic form

$$\begin{array}{rcl}\mathfrak{Q}(D_0) & \to & \mathbb{C}\\ \psi & \mapsto & (\psi, D_0\psi) + V[\psi,\psi]\end{array}$$

defines uniquely a self-adjoint operator D with $\mathfrak{Q}(D) := \mathfrak{Q}(D_0)$ such that for all $\psi \in \mathfrak{Q}(D_0)$ the identity $(\psi, D\psi) := (\psi, D_0\psi) + V[\psi,\psi]$ holds. We continue to write

P_+ and P_- for the spectral projections onto the positive and negative spectral subspaces of D.

Moreover, we specify the Dirac operator that defines the notion of electrons and positrons and the field algebra, i.e., the Dirac operator that is the basis of the constructions of Section 2 and in particular of Subsection 2.3, to be the free Dirac operator. To distinguish from the above operator, we will add a superscript 0, i.e., we write P^0_+ for the orthogonal projection onto the positive and P^0_- for the orthogonal projection onto the negative spectral subspace of the free operator and we write $\mathfrak{H}^0_\pm$ for the corresponding subspaces.

In other words, D is the operator to be second quantized, whereas D_0 defines the meaning of electrons and positrons.

We wish to generalize Carey's and Ruijsenaars' result in such a way that we no longer require $D_{+-} \in \mathfrak{S}_2$. As we will be able to show, our weakening of the conditions of Carey and Ruijsenaars will be as far as one go, if one wants to keep the property that we have a form perturbation of $\mathbb{D}_0$ (see Subsection 3.2). The price we are willing to pay for this is, to give up any control on the domain of the second quantized operator, i.e., we will be concerned only with the operators defined through the corresponding quadratic forms in the spirit of Friedrichs. This is actually in spirit similar to Fredenhagen's [3] approach. We carry it, however, out relax the requirements and proof a stronger statement.

3.1. **Condition on the One-Particle Potential.** We abbreviate

$$E(\mathbf{p}) := (c^2\mathbf{p}^2 + m^2c^4)^{1/2}.$$

By $\mathcal{F}$ we denote the Fourier transform

$$(\mathcal{F}f)(\mathbf{p}) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\mathbf{p}\cdot\mathbf{x}} f(\mathbf{x})d\mathbf{x}.$$

Our first result is

Theorem 1. *Let $D = D_0 + V$ be the Dirac operator with potential V. Pick the subspaces that define the field algebra as $\mathfrak{H}^0_+ := \chi_{(0,\infty)}(D_0)(\mathfrak{H})$ and $\mathfrak{H}^0_- := \chi_{(-\infty,0)}(D_0)(\mathfrak{H})$. Assume that V is such that the operator K with integral kernel*

$$K(p,p') := (E(\mathbf{p}) + E(\mathbf{p}'))^{-1/2}(\mathcal{F}^{-1}P^0_- V P^0_+ \mathcal{F})(p,p') \tag{39}$$

is a Hilbert-Schmidt operator. Then, for all $f \in \mathfrak{F}$ with finite kinetic energy, i.e., all f in the form domain of $\mathbb{D}_0$, the estimate

$$|(f, D_{+-}\Psi^*\Psi f)| \le 2^{1/2}\|K\|_2 (f, \mathbb{D}_0 f)^{\frac12}\|f\| \tag{40}$$

holds, where we have used the notation from (28).

Let us remark the following:

- The estimate (40) can be replaced by

$$|(f, D_{+-}\Psi^*\Psi f)| \le (f,\mathbb{D}_0 f)^{\frac12}\|E(-i\nabla)^{-\frac12}P^0_- V P^0_+\|_2\|f\|, \tag{41}$$

  if the asymmetric condition $E(-i\nabla)^{-\frac12}P^0_- V P^0_+ \in \mathfrak{S}_2$ is fulfilled. In fact the proof of this estimate is simpler; it essentially requires only the first part of the proof given below.
- Theorem 1 implies in particular, that the operator $D_{+-}\Psi^*\Psi$ is form bounded with respect to the kinetic energy $\mathbb{D}_0$ with arbitrarily small positive constant.
- In the case, that $\hat{V}$ is a measurable function the Hilbert-Schmidt hypothesis of the theorem means explicitly that

$$\int d\mathbf{p}d\mathbf{p}' \frac{|\hat{V}(\mathbf{p}-\mathbf{p}')|^2}{E(\mathbf{p}) + E(\mathbf{p}')}\left(1 - \frac{c^2\mathbf{p}\cdot\mathbf{p}' + m^2c^4}{E(\mathbf{p})E(\mathbf{p}')}\right) < \infty. \tag{42}$$

- Although we have $D_{+-} = P^0_+ V P^0_-$, we prefer to leave D_{+-} on the left hand side of (40) to emphasize that we are aiming for an estimate of the off-diagonal terms of the second quantized Hamiltonian.

Proof. We start from (29) for $A := D_{+-}$

$$D_{+-}\Psi^*\Psi = \sum_{\mu,\nu}(f_\mu, Dg_\nu)a^*(f_\mu)b^*(g_\nu) \tag{43}$$

where $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ are orthonormal basis of $\mathfrak{H}_+$ and $\mathfrak{H}_-$ respectively which will be picked suitably, later. We analyze this operator in Fourier representation. As already mentioned in the above remarks, we have $D_{+-} = P^0_+ V P^0_-$. By $V_{+-}(p,p')$ we denote distribution kernel of $\mathcal{F}^{-1}P^0_- V P^0_+ \mathcal{F}$, i.e., the kernel of $P^0_- V P^0_+$ in the Fourier representation. Recall that $p = (\mathbf{p}, \sigma)$. Now set

$$V_{+-,1}(p,p') := \begin{cases} V_{+-}(p,p') & |\mathbf{p}| > |\mathbf{p}'| \\ 0 & |\mathbf{p}| \leq |\mathbf{p}'| \end{cases}$$

and $V_{+-,2}(p,p') := V_{+-}(p,p') - V_{+-,1}(p,p')$. Correspondingly we decompose the second quantization of $D_{+-} = D_{+-,1} + D_{+-,2}$ with

$$(f, D_{+-,1}\Psi^*\Psi f) = \sum_\nu \int dp \int dp' V_{+-,1}(p,p') g_\nu(p')(a(p)f, b^*(g_\nu)f) \tag{44}$$

where $\int dp := \sum_{\nu=1}^4 \int_{\mathbb{R}^3} d\mathbf{p}$. Note that the integration for this part of the integral kernel can be restricted to $|\mathbf{p}| > |\mathbf{p}'|$ because of the support properties of $V_{+-,1}$.

We now rewrite (44) in the form

$$(f, D_{+-}\Psi^*\Psi f) = \sum_\nu \int dp \int dp' E(\mathbf{p})^{-\frac{1}{2}} V_{+-,1}(p,p') g_\nu(p')(E(\mathbf{p})^{\frac{1}{2}} a(p)f, b^*(g_\nu)f). \tag{45}$$

The hypothesis that the kernel of K – and therefore also the kernel $K_1(p,p') := E(\mathbf{p})^{-\frac{1}{2}} V_{+-,1}(p,p')$ – is a Hilbert-Schmidt kernel, implies that we can find a sequence of nonnegative numbers λ_ν (the singular values of K_1 which are square summable), an orthonormal basis g_ν of $\mathfrak{H}_+$, and a corresponding orthonormal basis f_ν in $\mathfrak{H}_-$ such that

$$\int dp' K_1(p,p') g_\nu(p') = \lambda_\nu f_\nu(p).$$

Inserting this in our expression (44) yields

$$\begin{aligned} (f, D_{+-,1}\Psi^*\Psi f) &= \sum_\nu \int dp \lambda_\nu f_\nu(p)(E(\mathbf{p})^{\frac{1}{2}} a(p)f, b^*(g_\nu)f) \\ &= \int dp (E(\mathbf{p})^{\frac{1}{2}} a(p)f, b^*(\sum_j \lambda_\nu \bar{f}_j(p) g_\nu)f). \end{aligned} \tag{46}$$

Thus we can estimate

$$|(f, D_{+-,1}\Psi^*\Psi f)| \leq \int dp \|E(\mathbf{p})^{\frac{1}{2}} a(p)f\| \, \|b^*(\sum_\nu \lambda_\nu \bar{f}_\nu(p) g_\nu)f\|. \tag{47}$$

The next step is to apply the Cauchy-Schwarz inequality with respect to the p-variable

$$|(f, D_{+-,1}\Psi^*\Psi f)| \leq \left(\int |E(\mathbf{p})| \|a(p)f\|^2 dp\right)^{\frac{1}{2}} \left(\int \|b^*(\sum_\nu \lambda_\nu \bar{f}_\nu(p) g_\nu)f\|^2 dp\right)^{\frac{1}{2}}. \tag{48}$$

Let us now estimate $\int \|b^*(\sum_\nu \lambda_\nu \bar{f}_\nu(p) g_\nu) f\|^2 dp$. Using that b^* has norm one, we get

$$\|b^*(\sum_\nu \lambda_\nu \bar{f}_\nu(p) g_\nu) f\|^2 \leq \|\sum_\nu \lambda_\nu \bar{f}_\nu(p) g_\nu\|^2 \|f\|^2 = \sum_\nu \lambda_\nu^2 |f_\nu(p)|^2 \|f\|^2. \tag{49}$$

Integration over p gives the estimate

$$|(f, D_{+-,1}\Psi^*\Psi f)| \leq T_+(f) \left(\sum_\nu \lambda_\nu^2\right)^{\frac{1}{2}} \|f\| \tag{50}$$

where

$$T_+(f) := \left(\int |E(\mathbf{p})| \|a(p) f\|^2 dp\right)^{\frac{1}{2}}$$

which is the kinetic energy of the electrons. This gives

$$|(f, D_{+-,1}\Psi^*\Psi f)| \leq T_+(f) \left(\sum_\nu \lambda_\nu^2\right)^{\frac{1}{2}} \|f\|. \tag{51}$$

Since $\sum_\nu |\lambda_\nu|^2 = \|V_{+-,1}\|_2^2 \leq 2\|K\|_2^2$ we get

$$|(f, D_{+-,1}\Psi^*\Psi f)| \leq 2^{1/2} \|K\|_2 \, T_+(f) \, \|f\|. \tag{52}$$

Now we treat the case that $|\mathbf{p}'| \geq |\mathbf{p}|$: the equivalent of (45) is

$$\begin{aligned} &(f, D_{+-,2}\Psi^*\Psi f) \\ &= \sum_\nu \int dp \int dp' E(\mathbf{p}')^{-\frac{1}{2}} V_{+-,2}(p,p') f_\nu'(p') (a(f_\nu') f, E(\mathbf{p}')^{\frac{1}{2}} b^*(p') f) \end{aligned} \tag{53}$$

where $f_1', f_2', \ldots$ form an orthonormal basis of $\mathfrak{H}_-$. The last factor in (53), the scalar product can also be written as

$$-(E(\mathbf{p}')^{1/2} b(p') f, a^*(f_\nu) f).$$

Then following the steps of the previous case with changing the role of p and p' and a and b and observing that the adjoint of a Hilbert-Schmidt operator is again a Hilbert-Schmidt operator with the same norm yields

$$|(f, D_{+-,2}\Psi^*\Psi f)| \leq 2^{1/2} \|K\|_2 \, T_-(f) \, \|f\| \tag{54}$$

where $T_-(f)$ is the kinetic energy of the positrons.

Adding (52) and (54) yields the desired result. □

As a corollary of this theorem we obtain

Theorem 2. *Under the hypothesis of Theorem 1, the assumption that the form domain of the symmetric quadratic form V contains $H^{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^4$ and that the negative part of $P_+^0 V P_+^0$ and the positive part $P_-^0 V P_-^0$ are relatively form bounded with respect to $|\nabla|$ with form bound less than one, there exists a uniquely determined self-adjoint operator $\mathbb{D}$ on the Fock space, such that its form domain contains the form domain of $\mathbb{D}_0$ and that for all vectors f in the form domain of $\mathbb{D}_0$ the identity $(f, \mathbb{D} f) := (f, : D\Psi^*\Psi : f)$ holds.*

Proof. The result follows immediately from Theorem 1, the remark thereafter (form boundedness of the off-diagonal elements with arbitrary small constant), and the fact that the diagonal elements leave each N-particle subspace of the electron and positron subspaces invariant. □

3.2. **Optimality of the Hilbert-Schmidt Condition.** We showed that, if $K \in \mathfrak{S}_2$ then we have the inequality

$$|\langle D_{+-}\Psi^*\Psi f, f\rangle| \leq 2^{1/2}\|K\|_2\|f\|\sqrt{(\mathbb{D}_0 f, f)}. \tag{55}$$

This implies the weaker inequality

$$|\langle D_{+-}\Psi^*\Psi f, f\rangle| \leq \|E(-i\nabla)^{-\frac{1}{2}}P_-^0 V P_+^0\|_2(\mathbb{D}_0 f, f). \tag{56}$$

Due to the fact that (56) is weaker than (55), one might think that our assumption that $K \in \mathfrak{H}$ is merely technical to obtain the relative boundedness claim in Theorem 2. However, this is not the case, the Hilbert-Schmidt condition on K in the hypothesis of Theorems 1 and 2 is in fact also necessary:

Theorem 3. *Assume V as in Theorem 1 and Theorem 2 except for the Hilbert-Schmidt condition on K but keeping the property that K has a kernel given through* (39)*. Assume in addition that the off-diagonal elements of the second quantized Dirac operator are relatively form bounded with respect to the kinetic energy. Then $K \in \mathfrak{S}_2$.*

Proof. The hypothesis implies that there is a constant c such that for all $f \in \mathfrak{F}$

$$|\langle D_{+-}\Psi^*\Psi f, f\rangle| \leq c[(\mathbb{D}_0 f, f) + (f, f)]. \tag{57}$$

Pick $f := \lambda\Omega + \int dp\, dp'\varphi(p,p')a^*(p)b^*(p')\Omega$ where Ω is the vacuum state as defined in Section 2.3.1 and $p = (\mathbf{p}, \sigma)$ and $p' = (\mathbf{p}', \tau)$ and where $\varphi \in \mathcal{S}(\mathbb{R}^3 \times \{1, \ldots, 4\} \times \mathbb{R}^3 \times \{1, \ldots, 4\})$. Using this choice we find that there is exists a constant C such that

(58)
$$\left|\lambda \int V_{+-}(p,p')\bar{\varphi}(p,p')dpdp'\right| \leq C\left(|\lambda|^2 + \int dp\, dp'(E(\mathbf{p}) + E(\mathbf{p}'))|\varphi(p,p')|^2\right),$$

for any $\lambda \in \mathbb{C}$ and any φ. (Note that $\int dp$ and $\int dp'$ means integration and summation.) Let us define $g(p,p') = (E(\mathbf{p}) + E(\mathbf{p}'))^{1/2}\ \varphi(p,p')$. We can rewrite the previous inequality in the form

$$\left|\lambda \int V_{+-}(p,p')\frac{1}{(E(\mathbf{p}) + E(\mathbf{p}'))^{1/2}}\bar{g}(p,p')dpdp'\right| \leq C\ \left(|\lambda|^2 + (g,g)\right), \tag{59}$$

for all $\lambda \in \mathbb{C}$ and all $g \in \mathcal{S}(\mathbb{R}^3 \times \{1, \ldots, 4\} \times \mathbb{R}^3 \times \{1, \ldots, 4\})$.

Taking $\lambda = \sqrt{(g,g)}$, we obtain

$$|\int K(p,p')\bar{g}(p,p')dpdp'| \leq 2C\|g\|.$$

But, if this is true for any g, this will imply

$$\int |K(p,p')|^2 dp\, dp' < \infty \tag{60}$$

which is the wanted Hilbert-Schmidt condition. □

4. Relation to other Quantization Schemes and Discussion of Allowed potentials

4.1. **Regularity of External Fields.** We wish to discuss the relation of the direct approach to second quantize a one-particle Dirac operator as presented in Section 3 with the quantization and renormalization in the Furry picture. In particular, it is interesting to know, if our class of potentials are regular external potentials as defined by Klaus and Scharf [7, 8]. For convenience we remind the reader again (see also the Introduction) of the definition of regular external fields.

Definition 1. *Given a potential*[2] $\varphi = -V$. *Consider the second quantization* $\mathbb{D}$ *of the perturbed Dirac operator* $D := D_0 + V$ *in the Furry picture, i.e., the field algebra is defined with respect to the subspaces* $\mathfrak{H}_\pm$ *of the perturbed operator* D *itself. An external potential* φ *– and if local the corresponding electric field* $\nabla\varphi$ *– is called regular, if the vacuum state of* $\mathbb{D}$ *is an element of the Fock space of the free Dirac operator.*

It is often easier to state and to investigate regularity through an equivalent characterization found by Klaus and Scharf [7].

Theorem 4. *A potential* V *is regular, if and only if*

$$Q := P_+^0 - P_+ \in \mathfrak{S}_2(\mathfrak{H}). \tag{61}$$

To show this Hilbert-Schmidt property for the potentials allowed in Theorem 3 we follow partly Nenciu and Scharf [11]. The starting point is the following formula for the projection of a self-adjoint operator H for which 0 is in the resolvent set

$$\chi_{(0,\infty)}(H) = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\eta + H)^{-1}d\eta. \tag{62}$$

This formula can be found in Kato [5] (VI,5 (Lemma 5.6)) and is meant as the Cauchy principal value

$$P_+ = \frac{1}{2} + \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{+r} R(i\eta)d\eta, \tag{63}$$

in the strong topology. Note that the convergence of the integral follows directly from the formula

$$P_+ = \frac{1}{2} + \frac{1}{2\pi}\lim_{r\to\infty}\int_0^{+r} H[H^2+\eta^2]^{-1}d\eta = \frac{1}{2} + \frac{1}{2\pi}\int_0^{\infty} H[H^2+\eta^2]^{-1}d\eta. \tag{64}$$

We will apply (62) for the operators D_0 and D and will denote the corresponding resolvents by $R_0(z)$ and $R(z)$. We now want to compare, under assumption that 0 is not in the spectrum of D, the projectors P_+ and P_+^0. The starting point is the following formula

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_0(i\eta)VR(i\eta)d\eta. \tag{65}$$

This formula transforms ([11]), Lemma 1) into

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_0(i\eta)[P_+^0VP_- + P_-^0VP_+]R(i\eta)d\eta. \tag{66}$$

This yields ([11], Lemma 2)

$$\begin{aligned} Q =& \frac{1}{2\pi}\int_{-\infty}^{\infty} R_0(i\eta)[P_+^0VP_-^0 + P_-^0VP_+^0]R(i\eta)d\eta \\ &+ \frac{1}{2\pi}(P_+^0 - P_-^0)\int_{-\infty}^{\infty} R_0(i\eta)VQR(i\eta)d\eta. \end{aligned} \tag{67}$$

We want to prove that the first term of the right hand side of (67) is Hilbert-Schmidt under the condition that $K \in \mathfrak{S}_2(\mathfrak{H})$ with K from (39). To this end we proceed as after (44) and assume that we can decompose $P_-^0VP_+^0$ as the sum of two operators

$$P_-^0VP_+^0 = B_1 + B_2,$$

where B_1 has the property that

$$|D_0|^{-\delta_1}B_1 \in \mathfrak{S}_2 \tag{68}$$

[2]We assume that $\mathfrak{A}$ vanishes, since there are no regular magnetic fields (Klaus and Scharf [7]).

with $\delta_1 = \frac{1}{2}$ and where B_2 fulfills

$$B_2|D_0|^{-\delta_1} \in \mathfrak{S}_2. \tag{69}$$

The analogous decomposition then holds for the adjoint operator.

Thus, we want to prove that (68) and (69) already imply

$$\int_{-\infty}^{+\infty} R_0(i\eta)B_jR(i\eta)d\eta \in \mathfrak{S}_2$$

for $j = 1, 2$.

The claim for B_1 is easy. We skip the proof and concentrate on B_2. We have

$$\int_{-\infty}^{+\infty} R_0(i\eta)B_2R(i\eta)d\eta = \int_{-\infty}^{+\infty} R_0(i\eta)\left(B_2\,|D_0|^{-\delta_1}\right)(|D_0|^{\delta_1}R(i\eta))d\eta.$$

The desired Hilbert-Schmidt property follows from

$$\||D_0|^{\delta_1}R(i\eta)\| \leq \text{const}\,\langle\eta\rangle^{\delta_1-1}. \tag{70}$$

To prove (70) we write down the resolvent equation

$$R(i\eta) = R_0(i\eta) - R_0(i\eta)VR(i\eta).$$

It is clear that

$$\||D_0|^{\delta_1}R_0(i\eta)\| \leq \text{const} < \eta >^{\delta_1-1}.$$

Thus it remains to prove that $VR(i\eta)$ is uniformly bounded in η in the operator norm. To this end we observe the following two facts: $R(i\eta) = R(0)DR(i\eta)$ and $DR(i\eta)$ is uniformly bounded by the spectral theorem. This yields

$$\||D_0|^{\delta_1}R_0(i\eta)\| \leq \text{const} < \eta >^{-1+\delta_1},$$

The claim follows now, since $VR(0)$ is bounded which is equivalent to $D_0R(0)$ being bounded. The latter is clear, since we know that the domain of D is $H^1(\mathbb{R}^3, \mathbb{C}^4)$ which follows, if we assume that there exists a $\delta_2 \in (0,1)$ such that $|D_0|^{-\delta_2}V \in \mathfrak{S}_\infty$.

All in all, we have proved

Theorem 5. *Assume*

- **(H1):** *Zero is in the resolvent set of D.*
- **(H2):** *There exists a $\delta_1 \in (0,1)$ such that $P_-^0VP_+^0$ can be decomposed as the sum of two operators B_1 and B_2 with $|D_0|^{-\delta_1}B_1 \in \mathfrak{S}_2$ and $B_2|D_0|^{-\delta_1} \in \mathfrak{S}_2$.*
- **(H3):** *There exists a $\delta_2 \in (0,1)$ such that $|D_0|^{-\delta_2}V \in \mathfrak{S}_\infty$.*

Then V is regular.

We remark the following:

- That the Coulomb potential is excluded follows because of two reasons: Firstly from the claim, since Klaus [6] showed that the Coulomb potential is not regular. Secondly, directly from hypothesis (H3).
- In the case when $\delta_1 = \frac{1}{2}$ the condition (H2) is implied by the symmetric condition of Section 3.
- We recall that there is a natural conjecture for the regularity of the external fields proposed by Nenciu and Scharf. They conjecture that

$$\int d^3\mathbf{p} \int d^3\mathbf{q} \frac{|\hat{V}(\mathbf{p}-\mathbf{q})|^2}{(E(\mathbf{p})+E(\mathbf{q}))^2} \frac{E(\mathbf{p})E(\mathbf{q}) - c^2\mathbf{p}\cdot\mathbf{q} - m^2c^4}{E(\mathbf{p})+E(\mathbf{q})} < \infty$$

  is necessary and sufficient. The condition (H2) is implied by

$$\int d^3\mathbf{p} \int d^3\mathbf{q} \frac{|\hat{V}(\mathbf{p}-\mathbf{q})|^2}{(E(\mathbf{p})+E(\mathbf{q}))^{2\delta_1}} \frac{E(\mathbf{p})E(\mathbf{q}) - c^2\mathbf{p}\cdot\mathbf{q} - m^2c^4}{E(\mathbf{p})E(\mathbf{q})} < \infty.$$

As shown in [11], the last inequality is sufficient for regularity for any $\delta_1 < 1$ and necessary for any $\delta_1 > 1$. Furthermore this inequality is for $\delta_1 \in (\frac{1}{2}, 2)$ equivalent to the more explicit inequality

$$A_\alpha : \qquad \int d^3p \frac{|\mathbf{p}|^2}{E(\mathbf{p})^{1-\alpha}} |\hat{V}(p)|^2 < \infty$$

with $\alpha = 2(1-\delta_1)$ (see (1.6) in [11] and Subsection 4.2 of this paper).

4.2. **Non-Locality of Potentials .** As mentioned already in the introduction, our hypothesis of Theorem 3 is a further weakening of the hypothesis of Carey and Ruijsenaars and also of Fredenhagen. Of course the question arises, if this allows for Coulomb potentials or for other local potentials, since these are used most frequently in physics. Our answer will be dimension dependent. We begin with a negative result in dimension three. The two dimensional case will be discussed in the following subsection.

To focus on the reason for this fact, we first remind of a positive fact: Let us suppose, that the external potential is purely negative. From Evans et al. [2] we know that the Coulomb potentials of the form $V = -c|\mathbf{x}|^{-1}$ satisfy the hypothesis on P_+VP_+ of Theorem 1 for $c < 2/(\pi/2 + 2/\pi)$. The same holds for all potentials that can be bounded by this potential plus a constant. In other words, the claimed negative answer cannot stem from the second quantization of the diagonal terms, at least not, if the c is small enough.

However, the Coulomb potential is immediately excluded by the conditions of Theorem 2 as mentioned in Subsection 4.1. In fact a much more drastic result is true.

Theorem 6. *Assume $V \in \mathcal{S}'(\mathbb{R}^3)$, its Fourier transform a measurable function, and the operator K defined in* (39) *in $\mathfrak{S}_2$, then $V = 0$.*

Proof. As shown by Nenciu and Scharf (see (2.5) and the argument before (2.32) in [11]), it suffices to show that the integral

$$\int d^3\mathbf{k_1} B_\epsilon(k_1) |\hat{V}(\mathbf{k_1})|^2, \tag{71}$$

is finite for $\epsilon = -1$. Here

(72)

$$B_\epsilon(k_1) = 2\pi \int_0^{+\infty} dk_2 \int_{-1}^{+1} dz \frac{k_2^2}{\left(\frac{1}{2}\sqrt{a+bz} + \frac{1}{2}\sqrt{a-bz}\right)^{2+\epsilon}} \left(1 - \frac{k_2^2 - k_1^2 + 4m^2}{\sqrt{a^2 - b^2z^2}}\right)$$

with $a = k_2^2 + k_1^2 + 4m^2$ and $b = 2k_1k_2$. We remark that the integral is well defined – with possibly infinite value – since the integrand is positive.

When $\epsilon > -1$, then the function $B_\epsilon(k_1)$ is positive continuous and

$$\lim_{k_1 \to 0^+} \frac{1}{k_1^2} B_\epsilon(k_1) = \text{const},$$

and

$$\lim_{k_1 \to +\infty} \frac{1}{k_1^{1-\epsilon}} B_\epsilon(k_1) = \text{const}.$$

This was used in [11] for ϵ close to zero. We are interested in the case that $\epsilon = -1$. Our desired result follows now from the following lemma. □

Lemma 1. *For any $k_1 > 0$, we have $B_{-1}(k_1) = +\infty$.*

Proof. We first observe the inequality

$$\frac{1}{2}\left(\sqrt{a+bz} + \sqrt{a-bz}\right) \le \sqrt{a}$$

by concavity of $t \mapsto \sqrt{t}$ on $\mathbb{R}^+$. Consequently we get the lower bound

$$\frac{k_2^2}{\left(\frac{1}{2}\sqrt{a+bz}+\frac{1}{2}\sqrt{a-bz}\right)^{2+\epsilon}} \geq \frac{k_2^2}{(\sqrt{a})^{2+\epsilon}}. \tag{73}$$

On the other hand, we have

$$\begin{aligned} \left(1-\frac{k_2^2-k_1^2+4m^2}{\sqrt{a^2-b^2z^2}}\right) &\geq \frac{1}{2}\left(1-\frac{(k_2^2-k_1^2+4m^2)^2}{a^2-b^2z^2}\right) \\ \geq\ & \frac{1}{2}\left(\frac{a^2-b^2z^2-(k_2^2-k_1^2+4m^2)^2}{a^2}\right) \\ \geq\ & \frac{1}{2}\left(\frac{b^2(1-z^2)+a^2-b^2-(k_2^2-k_1^2+4m^2)^2}{a^2}\right) \geq 2(1-z^2)k_1^2\cdot k_2^2\cdot a^{-2}. \end{aligned}$$

Thus, finally

$$B_\epsilon(k_1) \geq 4\pi k_1^2 \left(\int_{-1}^{+1}(1-z^2)dz\right)\int_0^{+\infty}\frac{k_2^4}{\sqrt{a}^{6+\epsilon}}dk_2. \tag{74}$$

Since $\sqrt{a} \sim k_2$ at ∞, we get that the integral becomes infinite when $\epsilon = -1$. □

4.3. **Local Potentials in Dimension Two.** We would like to end with a positive result. We observe that – although we developed all of our constructions for three dimensions – everything except for the no-go result for local potentials goes through also for two space dimensions. The latter, however, is dimension dependent: The quantity corresponding to $B_\epsilon(k_1)$ becomes finite for $\epsilon = -1$. The condition for regularity of a potential V which Nenciu and Scharf derive becomes the analogous condition

$$\int_{\mathbb{R}^2} d\mathbf{p}\frac{\mathbf{p}^2}{1+|\mathbf{p}|}|\hat{V}(\mathbf{p})|^2 < \infty \tag{75}$$

whereas the Hilbert-Schmidt condition for K becomes now for local potentials – as opposed to the three dimensional case – equivalent to

$$\int_{\mathbb{R}^2} d\mathbf{p}\mathbf{p}^2|\hat{V}(\mathbf{p})|^2 = \int_{\mathbb{R}^2} d\mathbf{x}|\nabla V(\mathbf{x})|^2 < \infty. \tag{76}$$

Potentials of the form $y(\mathbf{x})/|\mathbf{x}|^\alpha$ with y decreasing rapidly enough at infinity and $y(\mathbf{0}) = 1$ fulfill these conditions for $\alpha < 1/2$ and $\alpha \geq 0$.

We conclude with two remarks:

- In contrast to our construction, the hypothesis $D_{+-} \in \mathfrak{S}_2$ of Carey and Ruijsenaars excludes local potentials also in two dimensions.
- The difference between the two and three dimensional case reflects the general folklore, that quantum field theories in lower dimensions can be studied more easily perturbatively then in three dimensions.

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