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On a Matched Pair of Lie Groups for the κ-Poincaré in 2-Dimensions

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Abstract

We present the construction of the κ-Poincaré in two dimensions from matched pairs of Lie groups: $SO(1,1)$ and a κ-deformed group of translations.

1 Introduction

Quantum deformations of Lorentz and Poincaré symmetries are now an interesting field of studies with possible applications in fundamental theories [1]. The κ-deformation [2] of the Poincaré algebra is one of the most attractive and promising for physical models, mainly due to its simple structure. A significant progress in the studies of the κ-Poincaré has been made with the discovery of its bicrossproduct structure [3], with the classical Lorentz algebra and the deformed algebra of translations being its two components. It has allowed to introduce the κ-Minkowski space and to investigate its properties [4]. Moreover, it has offered us a new way of looking at the κ-Poincaré, since the bicrossproduct Hopf algebras arise in an algebraic approach to quantum gravity [5].

The bicrossproduct construction can also be seen as originating from the group factorization $P = GH$ of a group P into its two subgroups G and H. The bicrossproduct Hopf algebra $\mathbb{C}(H)\blacktriangleright\!\!\triangleleft\ \mathbb{C}G$ could be interpreted as a quantum algebra of observables of a particle moving along the orbits of G on H. For a general theory of these objects and examples we refer the reader to [6, 7, 8].)

One of the remaining interesting problems in the κ-deformation model is to look at the underlying group structure and the matched pair of Lie groups.

We shall look into this problem in our paper, keeping in mind the above interpretation and trying to establish relations with results obtained elsewhere.

First, we introduce the basic notation and definitions used in the paper.

We say that G and H form a *matched pair of Lie groups* if there exist a right action ($\triangleright$) of G on H and a left action ($\triangleleft$) of H on G satisfying the following compatibility conditions:

$$
\begin{aligned}
e \triangleright h = h, \qquad g \triangleleft e = g, \qquad g_1 \triangleright (g_2 \triangleright h) &= (g_1 g_2) \triangleright h, && (1)\\
g \triangleright e = e, \qquad e \triangleleft h = e, \qquad (g \triangleleft h_2) \triangleleft h_1 &= g \triangleleft (h_2 h_1), && (2)\\
g \triangleright (h_1 h_2) &= (g \triangleright h_1)\left((g \triangleleft h_1) \triangleright h_2\right), && (3)\\
(g_1 g_2) \triangleleft h &= (g_1 \triangleleft h)\left(g_2 \triangleleft (g_2 \triangleright h)\right), && (4)
\end{aligned}
$$

for every $g, g_1, g_2 \in G$, $h, h_1, h_2 \in H$; e denotes a neutral element of each group.

Having such a pair, one introduces the following group structure on the

Cartesian product $G \times H$:

$$\begin{aligned} \{g_1, h_1\} \circ \{g_2, h_2\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)h_2\}, \\ \{g, h\}^{-1} &= \left\{h^{-1} \triangleright g^{-1}, h^{-1} \triangleleft g^{-1}\right\}. \end{aligned} \tag{5}$$

Of course, for a group X, which factorizes into two subgroups G and H, (i.e. the map $G \times H \ni (g, h) \to g \cdot h \in P$ is a bijection) we recover a matched pair G and H, with left and right actions derived from (5).

Finally, let us mention that for every matched pair of Lie groups, their Lie algebras form a matched pair, with the right and left action derived from the relations (1-4):

$$\xi \triangleright [\eta_1, \eta_2] = [\xi \triangleright \eta_1, \eta_2] + [\eta_1, \xi \triangleright \eta_2] + (\xi \triangleleft \eta_1) \triangleright \eta_2 - (\xi \triangleleft \eta_2) \triangleright \eta_1 \tag{6}$$

$$[\xi_1, \xi_2] \triangleleft \eta = [\xi_1 \triangleleft \eta, \xi_2] + [\xi_1, \xi_2 \triangleleft \eta] + \xi_1 \triangleleft (\xi_2 \triangleright \eta) - \xi_2 \triangleleft (\xi_1 \triangleright \eta) \tag{7}$$

2 The κ-deformed group of translations.

Out of the full Hopf algebra of the κ-Poincaré we may single out the second component of its bicrossproduct structure, the κ-deformed Hopf algebra of translations $\mathcal{P}_\kappa$, with the following algebra and coproduct structure:

$$[P_\mu, P_\nu] = 0, \tag{8}$$

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \tag{9}$$

$$\Delta P_1 = P_1 \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes P_1. \tag{10}$$

Since this algebra is commutative, it corresponds to an algebra of functions on a group. The product in this group could be found directly from the coproduct structure (10). However, we shall choose here another way, which uses its structure as a matched pair of Lie groups in an explicit way. First, let us observe that the universal enveloping algebra of this group (dual to the algebra $\mathcal{P}_\kappa$) has two generators X, T with the commutation relation:

$$[X, T] = \frac{1}{\kappa} X. \tag{11}$$

Now, the Lie algebra, generated by X and T has a structure of a matched pair of two one-dimensional Lie algebras provided we define the corresponding left and right actions:

$$T \triangleright X = -\frac{1}{\kappa} X, \qquad T \triangleleft X = 0. \tag{12}$$

The corresponding pair of Lie groups can be identified with the additive group of real numbers with the matching left and right actions, for $t \in \mathbb{R}$ and $x \in \mathbb{R}$:

$$t \triangleright x = x e^{-\frac{t}{\kappa}}, \qquad t \triangleleft x = t. \tag{13}$$

The general theory of matched pairs of two one-dimensional additive groups has been studied by Majid [6]. All possible pairs are parametrized by two real numbers A and B, with the actions, for real s and u:

$$s \triangleleft u = \frac{1}{B} \ln(1 + (e^{Bs} - 1)e^{-Au}), \quad s \triangleright u = \frac{1}{A} \ln(1 + e^{-Bs}(e^{Au} - 1)). \tag{14}$$

The above derived κ-deformation corresponds to the special case of $A = 0$ and $B = \frac{1}{\kappa}$.

Using the construction (5), one recovers the group of deformed translations $\mathbb{R}^2_\kappa$, which is topologically equivalent to $\mathbb{R}^2$ but it is equipped with a nonabelian group structure:

$$\begin{bmatrix} t \\ x \end{bmatrix} \circ \begin{bmatrix} s \\ y \end{bmatrix} = \begin{bmatrix} t+s \\ x + y e^{\frac{t}{\kappa}} \end{bmatrix}. \tag{15}$$

Examples of orbits are shown on Fig.1, the lines show example orbits of horizontal and vertical translations.

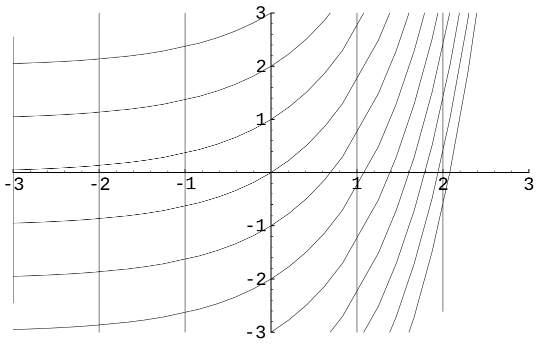


Figure 1.

3 Matched Pair P_κ and $SO(1,1)$

We identify the group $SO(1,1)$ with the additive group of real numbers and we denote its generator (boost) by N. The commutation relations between N and the X and T are

$$[N,X] = -T, \qquad [N,T] = -X - \frac{1}{\kappa}N. \tag{16}$$

The corresponding left and right actions, which are derived from the bicrossproduct structure are classical for the action of N on the κ-Minkowski:

$$N \triangleright X = -T, \qquad N \triangleright T = -X, \tag{17}$$

whereas the deformation appears in the back-action (right action) of the κ-Minkowski:

$$N \triangleleft X = 0, \qquad N \triangleleft T = -\frac{1}{\kappa}N. \tag{18}$$

In order to obtain the corresponding pair of Lie groups we build a faithful linear representation, exponentiate its matrices and find the expressions for the corresponding actions using the matrix multiplication and the rule (5).

Before we present the result, let us mention that one can use in this case either the adjoint representation:

$$X = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\kappa} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -\frac{1}{\kappa} & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & \frac{1}{\kappa} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -\frac{1}{\kappa} \\ 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

or a two-dimensional representation:

$$X = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \frac{1}{2\kappa}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & \frac{\kappa}{2} \\ \frac{1}{2\kappa} & 0 \end{pmatrix}. \tag{20}$$

Although all the relations (16-18) have a well-defined limit for $\kappa \to \infty$, the above two-dimensional representation does not.

In either case the exponentiation gives the following result for the matched pair of $\mathbb{R}$ and $\mathbb{R}^2_\kappa$:

$$\phi \triangleright \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \kappa \ln\left((\cosh\phi - 1)(\sinh\frac{t}{\kappa} + \frac{1}{2\kappa^2}x^2) - \frac{1}{\kappa}x\sinh\phi + e^{\frac{t}{\kappa}}\right) \\ x\cosh\phi - \kappa\sinh\phi\sinh\frac{t}{\kappa} - \frac{1}{2\kappa}x^2\sinh\phi e^{-\frac{t}{\kappa}} \end{bmatrix} \tag{21}$$

$$\phi \triangleleft \begin{bmatrix} t \\ x \end{bmatrix} = \ln\left(1 + \frac{(\cosh\phi - 1)e^{-\frac{t}{\kappa}}(1-\frac{1}{\kappa}x) + \sinh\phi}{(\cosh\phi-1)(\sinh\frac{t}{\kappa} + \frac{1}{2\kappa^2}x^2) - \frac{1}{\kappa}x\sinh\phi + e^{\frac{t}{\kappa}}}\right). \tag{22}$$

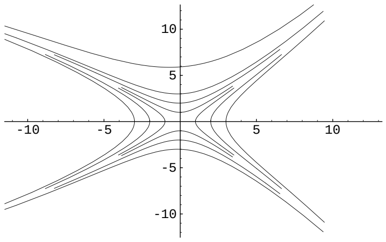
Both the action and back-action are well defined in the neighborhood of unity for both groups, however, they are not defined globally. For instance, the action of $\mathbb{R}$ on $\mathbb{R}^2_\kappa$ becomes ill-defined for ϕ big enough if $\sinh\frac{t}{\kappa} - +\frac{1}{2\kappa^2}x^2 - x$ is negative.

It is easy to verify that in the limit $\kappa \to \infty$ we recover the usual action of $SO(1,1)$ on $\mathbb{R}^2$ (and the back-action as well):

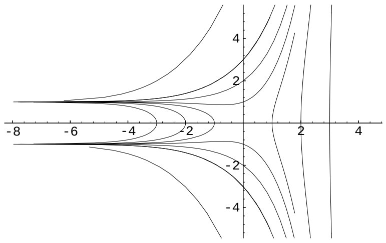
$$\phi \triangleright \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} t\cosh\phi - x\sinh\phi \\ x\cosh\phi - t\sinh\phi \end{bmatrix}, \tag{23}$$

$$\phi \triangleleft \begin{bmatrix} t \\ x \end{bmatrix} = \phi. \tag{24}$$

Example orbits of the group action on the plane are presented below for two values of κ.



$\kappa = 25$



$\kappa = 1$

4 Conclusions

As we have indicated earlier, the κ-deformed group of translations is a special case of a matched pair of two one-dimensional additive groups:

$$
\begin{aligned}
[X, T] &= \frac{A}{B}(1 - e^{-BX}), && (25)\\
\Delta X &= 1 \otimes X + X \otimes 1, && (26)\\
\Delta T &= T \otimes e^{-BT} + 1 \otimes T && (27)
\end{aligned}
$$

where the classical limit is $A = B = 0$ and the κ-deformation is $A = 0$, $B = \frac{1}{\kappa}$.

Assuming the interpretation of the bicrossproduct construction of the deformed Minkowski space (or the momentum space in the dual picture) as an $A \to 0$ limit of a geometrically curved phase space, we see that the two-dimensional κ Poincaré might be seen as a general symmetry structure of the phase-space.

So far we have explored only one limit, with $h = \frac{A}{B} = 0$. An interesting problem would be to investigate the other limit, with h fixed and $B \to \infty$, which would correspond to a flat phase-space of a quantum mechanical system. However, the cross-algebra of the quantum-like limit is degenerate because the obtained algebra has no Hopf algebra structure (there exists a coproduct but no counit and no antipode). Therefore, one should rather build a symmetry structure on a two-parameter family of deformations arising from the bicrossproduct of two real lines and take the limit afterwards. The symmetry structure would be a two-parameter deformation of the Poincaré, with the deformed translation part being neither commutative nor cocommutative.

References

[1] P.Podles, S.L.Woronowicz, On the classification of quantum Poincaré groups. *Commun.Math.Phys.* 178:61-82, 1996

[2] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy. q-Deformation of Poincaré algebra. *Phys. Lett.* B268:331-338, 1991.

J. Lukierski, A. Nowicki and H. Ruegg. New quantum Poincaré algebra and κ-deformed field theory. *Phys. Lett.* B293:344-352, 1992.

A. Nowicki, E. Sorace and M. Tarlini. The quantum deformed Dirac equation from the κ-Poincaré algebra. *Phys. Lett.* B302:419-422, 1993.

[3] S.Majid,H.Ruegg. Bicrossproduct structure of κ-Poincaré and noncommutative geometry. *Phys.Lett.*B 334, 348, (1994)

[4] A.Sitarz Noncommutative differential calculus on the κ-Minkowski *Phys.Lett.*B 349, 42-48, (1995)

[5] S.Majid,
Hopf Algebras for physics at the Planck scale, Class.Quant.Grav. 5, 1587-1606, (1988)

[6] S.Majid, Foundations of Quantum Group Theory, *Cambridge University Press*, Cambridge 1995,

[7] S.Majid, Cross Product, quantization, nonabelian cohomology and twisting of Hopf algebras, in: Proceedings *Generalized Symmetries*, Clausthal, (1993), World Scientific

[8] S.Majid, Matched Pairs of Lie groups associated to solutions of yang-Baxter Equations, Pacific Journal of Mathematics, 141(2), 311, (1990)