M P E J

MATHEMATICAL PHYSICS ELECTRONIC JOURNAL

ISSN 1086-6655
Volume 2, 1996

Paper 3
Received: April 22, 1996, Accepted: May 23, 1996
Editor: G. Gallavotti

Renormalization group approach to zero temperature Bose condensation

G. Benfatto[1]

Dipartimento di Matematica, Università di Roma "Tor Vergata"
00133 Roma, Italy

Abstract - We study the problem of Bose condensation at zero temperature and weak coupling for a three dimensional system of bosons, interacting with a repulsive short range potential, in the Bogoliubov approximation. We prove that the properties of the model can be explained in terms of an anomalous asymptotically free renormalization group flow and we show that the two-point correlation function has the typical superfluid behaviour at long wavelengths, as generally expected. The proof is, for the moment, only at the level of perturbation theory in the running coupling constants. We also obtain an expression for the sound speed, whose leading term (when the coupling goes to zero) coincides with the sound speed in the exactly soluble Bogoliubov model.

[1]Supported by Ministero della Ricerca Scientifica, Italy

1 Introduction

It is commonly believed, following the pioneering paper by Bogoliubov [1], that a three dimensional system of bosons at zero temperature, interacting with a weak repulsive short range potential, exhibits condensation and superfluid behaviour. In the perturbative theoretical analysis, the two phenomena are related to the asymptotic behaviour of the two-point correlation function $S(x-y)$, where $x-y$ is the space-time (imaginary time) separation. There is condensation if $S(x-y) \to \rho > 0$ as $x-y \to \infty$, ρ being the condensate density, while the superfluidity is related to an *anomalous* behaviour of $S(x-y)-\rho$, whose Fourier transform is expected to have a singularity of the type $(k_0^2+c^2\mathbf{k}^2)^{-1}$, to compare with the singularity $(-ik_0+\mathbf{k}^2/2m)^{-1}$ of the free Bose gas (see, for example, [2]). In fact this anomaly explains, according to Landau's criterion [3], the superfluid properties of the system, whose spectrum is expressed, for small momenta, in terms of collective excitations with speed c.

The renormalization group approach allows us to solve in a relatively simple (but not standard) and non ambiguous way the difficult infrared divergences problems, which plague the perturbation theory of the model, by reducing them to the study of the flow equations for a finite number of quantities (the *running couplings* and the *renormalization constants*, see below), which describe the relevant properties of the interaction on different scales. Our main result is that one can control this flow at level of *perturbation theory in the running couplings*, so obtaining a strong justification of the generally accepted picture of Bose condensation and superfluid behaviour. Moreover, it is reasonable to guess (at least for people working in rigorous renormalization group theory) that our results could be improved, by solving in this case the so called *large fields problem*, so that one can get a rigorous construction of the model, valid for small coupling but not perturbative.

There is of course a huge physical literature on the perturbative theory of Bose condensation at zero or small temperature. As far as we know, up to 1994, the more convincing results about the superfluid behaviour at zero temperature were contained in [4] (see also references therein), where the authors used arguments similar to the ones that we will explain in this paper. However, at our knowledge, there was no previous treatment of the problem explicitly based on renormalization group arguments. The results contained in this paper were presented at the Workshop in Constructive Physics, held at the École Polytechnique, Palaiseau, in july 1994 [8], and were discussed also in [9].

Recently, there was a renewed interest on the subject of Bose condensation, motivated by new experimental results [12], and a new paper appeared [10], explicitly based on renormalization group ideas, where a more heuristic approach to the problem was used. In [10] the authors consider also the two-dimensional case, which will not be treated here, because the corresponding renormalization group flow can not be rigorously studied, even at level of perturbation theory, at least by using only the ideas of this paper; in fact in the two dimensional case, there is a relevant non gaussian contribution to the effective potential. In the present paper we present a simplified version of the derivation of the main results discussed in [8], [9] (which agree with those of [10]).

Of course, in the three dimensional case the main results of [10], which essentially follow from dimensional arguments and symmetry considerations, are in full agreement with the results of this paper.

2 The model

The theoretical analysis is generally based on the functional representation of the model. Let H be the Hamiltonian describing a system of N bosons in $\mathbf{R}^3$, enclosed in a periodic box of side size L, interacting with a pair potential $\lambda v(\mathbf{x}-\mathbf{y})$ which is supposed to be integrable, with short range p_0^{-1} and repulsive, in the sense that $\hat v(0) = \int d\mathbf{x} v(\mathbf{x}) > 0$ and $\lambda \ge 0$. H contains also a chemical potential term $-\mu N$.

Let $\varphi^{\pm}_{\mathbf{x}}$ be the creation and the annihilation operators for the bosons and $\varphi^\sigma_x = e^{Ht}\varphi^\sigma_{\mathbf{x}} e^{-Ht}$, $\sigma = \pm$, $x = (t, \mathbf{x})$.

Define, for $\beta > t_i \ge 0$, $i = 1, \ldots, n$, with $t_i \ne t_j$ for $i \ne j$,

$$S^{\sigma_1 \cdots \sigma_n}_{x_1,\ldots,x_n} = \lim_{\substack{\beta\to\infty \\ L\to\infty}} \frac{\mathrm{Tr}\left[e^{-\beta H}\varphi^{\sigma_{\pi(1)}}_{x_{\pi(1)}}\cdots\varphi^{\sigma_{\pi(n)}}_{x_{\pi(n)}}\right]}{\mathrm{Tr}\ e^{-\beta H}}\,, \tag{1}$$

where $\sigma_i = \pm 1$ and π is the permutation of $(1, \ldots, n)$, such that $t_{\pi(1)} > t_{\pi(2)} > \ldots > t_{\pi(n)}$. In particular define $S(x) \equiv S^{-+}_{x,0}$.

The functions (1), which we shall call the *Schwinger functions*, describe the properties of the *ground state* of the above bosons system (essentially *by definition*) in the grand canonical ensemble with chemical potential μ.

The case $\lambda = 0$ is trivial and one finds that, if $\mu < 0$ is taken as a function of β, L, which goes to 0 as $L, \beta \to \infty$, in such a way that the number of particles in the *condensate* (the state $\mathbf{k} = 0$) is fixed (that is, $e^{\beta\mu}(1-e^{\beta\mu})^{-1} = L^d\rho$), then

$$S(x) = S_\rho(x) \equiv \rho + \frac{1}{(2\pi)^4}\int dk\, \frac{e^{-ikx}}{-ik_0 + \frac{\mathbf{k}^2}{2m}}\,, \tag{2}$$

where $k = (k_0, \mathbf{k})$.

The other Schwinger functions can be calculated by the Wick rule; they will describe the ground state of a system of non interacting bosons with density ρ, a *Bose condensed state.*

Let us now consider the case $\lambda > 0$. As it is well known (see, for example, [2]) the Schwinger functions can be expressed as functional integrals in the following way:

$$S^{\sigma_1\ldots\sigma_n}_{x_1\ldots x_n} = \int \varphi^{\sigma_1}_{x_1}\cdots\varphi^{\sigma_n}_{x_n}\frac{e^{-V(\varphi)}P(d\varphi)}{\int e^{-V(\varphi)}P(d\varphi)}\,, \tag{3}$$

where $P(d\varphi)$ is a complex gaussian measure, such that the fields φ^-_x, $\varphi^+_x = (\varphi^-_x)^*$ have covariance:

$$\begin{aligned} \int \varphi^-_x\varphi^+_y P(d\varphi) &= S(x-y)\,, \\ \int \varphi^-_x\varphi^-_y P(d\varphi) &= \int \varphi^+_x\varphi^+_y P(d\varphi) = 0\,, \end{aligned} \tag{4}$$

and, if $x = (x_0, \mathbf{x})$, $y = (y_0, \mathbf{y})$ and $\nu = -\mu$,

$$\begin{aligned} V(\varphi) &= \lambda \int v(\mathbf{x}-\mathbf{y})\delta(x_0 - y_0)\varphi_x^+\varphi_x^-\varphi_y^+\varphi_y^- \, dx\, dy \\ &+ \nu \int \varphi_x^+\varphi_x^- \, dx \ . \end{aligned} \tag{5}$$

ν has the role of a control parameter that should be fixed so that the limiting theory is meaningful as a perturbation of the free theory with propagator (2). If this program is successful, it is natural to say that there is Bose condensation at $T = 0$ with given density ρ and chemical potential $-\nu$.

The form of the covariance (2) shows that the fields $\varphi_x^\pm$ can be represented as

$$\varphi_x^\pm = \xi^\pm + \psi_x^\pm \ , \tag{6}$$

$\xi^\pm$ being variables independent from $\psi_x^\pm$ and with covariance $\langle \xi^- \xi^+ \rangle = \rho$, while the fields $\psi_x^\pm$ have covariance $\langle \psi_x^- \psi_y^+ \rangle = S_0(x-y)$, $\langle \psi_x^- \psi_y^- \rangle = \langle \psi_x^+ \psi_y^+ \rangle = 0$.

The integration with respect to $\xi^\pm$ can be thought as a Gaussian integral by writing $\xi^\pm = \xi_1 \pm i\xi_2$ and $P(d\xi) = (\pi\rho)^{-1}\exp[-\rho^{-1}(\xi_1^2 + \xi_2^2)]d\xi_1 d\xi_2$. Hence, if we define $\Lambda = [-\frac{1}{2}\beta, \frac{1}{2}\beta] \times [-\frac{1}{2}L, \frac{1}{2}L]^3$ and:

$$W(\xi) = -\lim_{|\Lambda|\to\infty} \frac{1}{\Lambda} \log \int e^{-V(\xi+\psi)} P(d\psi) \ , \tag{7}$$

we see that the computation of $\langle \xi^+ \xi^- \rangle$ in presence of interaction will lead to the integral:

$$\rho = \lim_{|\Lambda|\to\infty} \int \frac{d\xi_1 d\xi_2}{2\pi\rho} (\xi_1^2 + \xi_2^2) e^{-\frac{\xi_1^2+\xi_2^2}{\rho} - [W(\xi) - W_0]|\Lambda|} \ , \tag{8}$$

where W_0 is a normalization constant and the equality to ρ of the above integral is just the requirement that the condensate density should be ρ. Therefore equality (8) can hold if and only if the function $W(\xi)$, which is a function of the product $\xi^+\xi^-$, by symmetry considerations, reaches its minimum at $\xi^+\xi^- = \rho$. And in this case $\xi^+\xi^-$ will be a sure random variable, provided the minimum is not degenerate.

This implies that, in order to get a condensate with density ρ, one has to choose ν so that the free energy (7) has a minimum in $\xi^+ = \xi^- = \sqrt{\rho} > 0$.

These considerations are a heuristic justification of the so called *Bogoliubov approximation*, very usually used in the literature, consisting in replacing the fields $\xi^\pm$ by a real positive constant external field and by choosing its value so that the ground state energy is minimum. In agreement with this approximation, we shall assume that, in order to study Bose condensation at $T = 0$, one has to consider the measure

$$\mathcal{N}^{-1} e^{-V_\rho(\psi)} P(d\psi) \ , \tag{9}$$

where $\mathcal{N}$ is a normalization constant, $V_\rho(\psi)$ is obtained from (5) trough the substitution (6), $\xi^+ = \xi^- = \sqrt{\rho}$ and ν has to be chosen so that the free energy is minimum for the fixed value of $\xi^\pm$.

The Schwinger functions of the field ψ are defined by an expression similar to (3) and we shall use the symbol $\tilde{S}$ to denote them. Their perturbation expansion is obtained in the usual way in terms of the propagator $S_0(x-y)$. Note that the measure (9) does not preserve the free measure property $\tilde{S}_{--}(x-y) = \tilde{S}_{++}(x-y) = 0$.

An old perturbative argument [5] allows one to show (see also [2]) that the free energy is minimum if the following formal equation is satisfied:

$$\Sigma_{-+}(0) = \Sigma_{++}(0) \; , \tag{10}$$

where $\Sigma_{\sigma_1\sigma_2}(k)$ is the Fourier transform of the sum of all one particle irreducible graphs (connected graphs which can not become disconnected by cutting one leg) with two external lines $\psi_x^{\sigma_1}$, $\psi_y^{\sigma_2}$.

The *renormalization condition* (10) has here the same role of the condition which determines the critical temperature in Statistical Mechanics; therefore it is natural to use it, instead of the free energy minimum condition, as the condition fixing the chemical potential, in a renormalization group analysis of the measure (9).

The problem we want to study is an infrared problem; therefore we shall consider a simplified model by substituting $S_0(x)$ with

$$g_{\leq 0}(x) = \frac{1}{(2\pi)^4} \int dk \, t_0(k) \frac{e^{-ikx}}{-ik_0 + \frac{\mathbf{k}^2}{2m}} \; , \tag{11}$$

where $t_0(k)$ is a smooth function, which imposes an ultraviolet cutoff on scale p_0 (the scale of the potential). The choice of $t_0(k)$ is not relevant, so that in the following, for simplicity, we shall suppose that $t_0(k)$ is the characteristic function of the set $\{k_0^2 + \frac{\mathbf{k}^2}{2m}\frac{p_0^2}{2m} \leq (\frac{p_0^2}{2m})^2\}$.

Note that the assumed presence of the ultraviolet cutoff on the scale of the interaction potential is reasonable only if $\rho p_0^{-3} \ll 1$, that is, only if there is in mean much less than one particle in a cube with side equal to the range p_0^{-1} of the potential.

Hence we have to study the measure:

$$\mathcal{N}^{-1} e^{-V_\rho(\psi)} P^{(\leq 0)}(d\psi) \; , \tag{12}$$

where $P^{(\leq 0)}(d\psi)$ is the measure with covariance (11). We can do that by a multiscale analysis, in the form presented in [6] and applied to a "similar" infrared problem, the one dimensional Fermi gas, in [9]. We shall now give a rough description of this analysis. A more detailed discussion will appear in the next future [11].

3 The renormalization group flow

The infrared behaviour of the model should not change, if we localize the potential V_ρ, so obtaining:

$$\begin{aligned} V_\rho(\psi) \simeq{}& \lambda\hat{v}(0) \int (\psi_x^+ \psi_x^-)^2 \, dx + \\ &+ 2\lambda\hat{v}(0)\sqrt{\rho} \int \psi_x^+\psi_x^- \, (\psi_x^+ + \psi_x^-) \, dx + \\ &+ (2\lambda\hat{v}(0)\rho + \nu) \int \psi_x^+\psi_x^- \, dx + \lambda\hat{v}(0)\rho \int (\psi_x^+ + \psi_x^-)^2 \, dx \; . \end{aligned} \tag{13}$$

We now observe that, if the local terms of order greater than two were not present (so obtaining the exactly soluble *Bogoliubov model*), then the condition (10) could be imposed, by choosing $\nu = -2\lambda\rho\hat{v}(0)$, that is by putting equal to zero the coefficient of $\psi^+\psi^-$. This observation implies that the "natural" way to study the measure (12) is to change the free measure, by absorbing in it the local terms of V_ρ, quadratic in the field, which remain after imposing the condition $\nu^0 \equiv \nu + 2\lambda\rho\hat{v}(0) = 0$. We shall denote $P_B(d\psi)$ the new measure and we shall call it the *renormalized free measure*.

As it is well known [1], the renormalized free measure has indeed the superfluid behaviour for $k \to 0$, with $c^2 = c_B^2 \equiv 2\lambda\rho\hat{v}(0)$. We can show that this is true also for the measure (12), that we shall write in the form:

$$\mathcal{N}^{-1} e^{-\tilde{V}_0(\psi)} P_B(d\psi) \,. \tag{14}$$

It is convenient, before proceeding, to change the basic fields by defining:

$$\chi_x^\pm = \frac{1}{\sqrt{2\rho}}(\psi^+ \pm \psi^-)\,, \quad \psi^\pm = \sqrt{\frac{\rho}{2}}(\chi^+ \pm \chi^-)\,, \tag{15}$$

so that $\tilde{V}_0$ becomes a function of χ. The covariance of the fields $\chi^\pm$ in the distribution P_B is given by:

$$\tilde{C}^{\sigma_1\sigma_2}(x) = \frac{1}{(2\pi)^4}\int e^{-ikx} t_0(k) \tilde{G}_0^{-1}(k)_{\sigma_1\sigma_2}\,, \tag{16}$$

where, if $\varepsilon = \lambda\hat{v}(0)\rho 2mp_0^{-2}$, the matrix $\tilde{G}_0(k)$ is defined by

$$\tilde{G}_0(k) = \rho \begin{pmatrix} \frac{\mathbf{k}^2}{2m} + 4\varepsilon\frac{p_0^2}{2m}t_0(k) & ik_0 \\ -ik_0 & -\frac{\mathbf{k}^2}{2m} \end{pmatrix}. \tag{17}$$

In order to study the measure (14), we make the scale decomposition $\tilde{C}^{\sigma_1\sigma_2}(x) = \sum_{h=-\infty}^{0} \tilde{C}_h^{\sigma_1\sigma_2}(x)$, where $\tilde{C}_h^{\sigma_1\sigma_2}(x)$ is obtained from (16) by substituting $t_0(k)$ with $\tilde{T}_h(k) = t_0(\gamma^{-h}k) - t_0(\gamma^{-h+1}k)$, and the *scaling parameter* γ is a any number greater than 1. We have the scaling relations:

$$\tilde{C}_h^{\sigma_1\sigma_2}(x) = \gamma^{(3+\frac{\sigma_1+\sigma_2}{2})h}\tilde{g}_h^{\sigma_1\sigma_2}(\gamma^h x)\,, \tag{18}$$

with

$$\tilde{g}_h^{\sigma_1\sigma_2}(x) = \frac{1}{(2\pi)^4}\int dk e^{-ikx}\tilde{T}_0(k)\tilde{G}_h^{-1}(k)_{\sigma_1\sigma_2} \tag{19}$$

and

$$\tilde{G}_h(k) = \rho \begin{pmatrix} \frac{\gamma^{2h}\mathbf{k}^2}{2m} + 4\varepsilon\frac{p_0^2}{2m}t_0(\gamma^h k) & ik_0 \\ -ik_0 & -\frac{\mathbf{k}^2}{2m} \end{pmatrix}. \tag{20}$$

Note that $\tilde{g}_h^{\sigma_1\sigma_2}(x)$ are essentially independent of h, for $h \to -\infty$.

Let us now define the *effective potentials* $\tilde{V}_h(\chi)$ in the usual way [6], by iterating the relation:

$$e^{-\tilde{V}_h(\chi)} = \int \tilde{P}_B^{(h+1)}(d\chi^{(h+1)}) e^{-\tilde{V}_{h+1}(\chi+\chi^{(h+1)})}\,, \tag{21}$$

where $\tilde{P}_B^{(h)}(d\chi)$ denotes the measure with covariance $\tilde{C}_h$.

The scaling relations (18) imply that we can define dimensionless fields $\bar{\chi}^\pm$ by the relations: $\chi_x^- = \gamma^h \bar{\chi}^-_{\gamma^h x}$, $\chi_x^+ = \gamma^{2h}\bar{\chi}^+_{\gamma^h x}$ and this allows us to analyze the relevant local part $\mathcal{L}\tilde{V}_h(\chi)$ of the effective potentials $\tilde{V}_h(\chi)$. By a trivial dimensional analysis and by taking into account the symmetries of the model, one can show that $\mathcal{L}\tilde{V}_h(\chi)$ has to be of the form:

$$\begin{aligned}\mathcal{L}\tilde{V}^{(h)}(\chi) &= \frac{p_0^2\rho}{2m}(\tilde{\lambda}_h F_{40} + \tilde{\mu}_h F_{21} + \gamma^{2h}\tilde{\nu}_h(F_{02}-F_{20}) \\ &+ 2\tilde{z}_h F_{02} + 2\tilde{\zeta}_h D_{tt} + 2\tilde{\alpha}_h D_{ss} + 2\tilde{d}_h D_t)\,, \end{aligned} \tag{22}$$

where $F_{m_1m_2} = \int dx\, \chi_x^{-m_1}\chi_x^{+m_2}$, $D_{tt} = -(2mp_0^{-2})^2\int(\partial_{x_0}\chi_x^-)^2\,dx$, $D_{ss} = -p_0^{-2}\int(\partial_{\mathbf{x}}\chi_x^-)^2\,dx$, $D_t = -2mp_0^{-2}\int\chi_x^+\partial_{x_0}\chi_x^-\,dx$.

As usual, we can hope to have a perturbative control of the model only if we can show that all the running constants appearing in (22) stay small for $h\to-\infty$, for a suitable choice of $\tilde{\nu}_0$ (that is, of the chemical potential), such that $|\tilde{\nu}_h| \le \bar{\nu}$, for $h\to-\infty$. This condition is equivalent to the renormalization condition (10), if we can also show that $\tilde{\lambda}_h$ and $\tilde{\mu}_h$ go to 0 as $h\to-\infty$ (*asymptotic freedom*). In fact, in this case, it implies that, for $h\to-\infty$, the effective potential is of the second order in the fields $\psi^\pm$ and that they appear only in the combination $(\psi_x^+ + \psi_x^-)^2 = 2\rho F_{02}$, up to terms containing field derivatives, which do not influence (10). A simple calculation allows one to prove that this structure of the effective potential implies the renormalization condition.

For $h=0$ we have: $\tilde{\lambda}_0 = \frac{\varepsilon}{4}$, $\tilde{\mu}_0 = -\varepsilon\sqrt{2}$, $\tilde{\nu}_0 = \frac{\nu^0}{2}\frac{2m}{p_0^2}$, and $\tilde{z}_0 = \tilde{\zeta}_0 = \tilde{\alpha}_0 = \tilde{d}_0 = 0$. However $\tilde{z}_h$, $\tilde{\zeta}_h$, $\tilde{d}_h$ and $\tilde{\alpha}_h$ will be different from zero for $h<0$ and they could grow as $h\to-\infty$. This problem can be solved by the same strategy used in passing from the representation (12) of the measure to the representation (14). We define iteratively a new family of effective potentials $V_h(\chi)$ in the following way.

Given $V_0(\chi) = \tilde{V}_0(\chi)$, we define $\tilde{V}_{-1}(\chi)$ as before; we then define $V_{-1}(\chi)$ by absorbing the terms proportional to $\tilde{z}_{-1}$, $\tilde{\zeta}_{-1}$, $\tilde{d}_{-1}$ and $\tilde{\alpha}_{-1}$ in the measure $\tilde{P}_B^{(\le -1)}(d\chi)$ (the measure with covariance $\sum_{h=-\infty}^{-1}\tilde{C}_h^{\sigma_1\sigma_2}$), so that:

$$\int \tilde{P}_B^{(\le -1)}(d\chi)e^{-\tilde{V}_{-1}(\chi)} = \frac{1}{\mathcal{N}}\int P_B^{(\le -1)}(d\chi)e^{-V_{-1}(\chi)}\,. \tag{23}$$

We can iterate this procedure, so defining a family of measures $P_B^{(\le h)}(d\chi)$ and a family of effective potentials $V_h(\chi)$, such that:

$$\mathcal{L}V_h(\chi) = \frac{p_0^2\rho}{2m}\left[\lambda_h F_{40} + \mu_h F_{21} + \gamma^{2h}\nu_h(F_{02}-F_{20})\right]\,, \tag{24}$$

and the covariance $C_{\le h}^{\sigma_1\sigma_2}$ of $P_B^{(\le h)}(d\chi)$ is of the form:

$$C_{\le h}^{\sigma_1\sigma_2}(x) = \frac{1}{(2\pi)^4}\int dk e^{-ikx}\, t_0(\gamma^{-h}k)\, G_{\le h}^{-1}(k)_{\sigma_1\sigma_2}\,, \tag{25}$$

$$G_{\leq h}(k) = \rho \begin{pmatrix} \frac{\mathbf{k}^2}{2m} + \frac{2p_0^2 Z_h}{m} & ik_0 E_h \\ -ik_0 E_h & -\frac{\mathbf{k}^2}{2m} - \frac{8mB_h k_0^2}{p_0^2} - \frac{2A_h \mathbf{k}^2}{m} \end{pmatrix} . \tag{26}$$

We have studied perturbatively the flow of the *running couplings* λ_h, μ_h and ν_h and of the *renormalization constants* Z_h, A_h, B_h and E_h (the *beta function* of our problem), by keeping only the leading terms in the expansion of r_{h-1} in terms of $\{r_{h'}, h' \geq h\}$, if $r_h \equiv \{\lambda_h, \mu_h, \nu_h, Z_h, A_h, B_h, E_h\}$; the details of this calculations can be found in [8]. It turns out that the leading terms are associated with the one-loop Feynman graphs, calculated with the single scale propagator which follows from the previous definitions (note that it is a function of the renormalization constants). Moreover it is possible to prove that, if ε is sufficiently small, there is no important change in the values of r_h up to $h \simeq h_0$, if $\varepsilon \simeq \gamma^{2h_0}$, that is, in the momentum region where the superfluid behaviour is not yet dominating.

Finally an important role is played by the exact identity, following from gauge invariance of the model [9],[8],[10]:

$$Z_h(0) = -\mu_h/\sqrt{2} \tag{27}$$

valid for any $h \leq 0$. There is another exact identity, that is:

$$\lambda_h = \frac{Z_h}{4} \tag{28}$$

which however has a minor role (and in fact was not used in [8]).

A detailed calculation of the beta function in the region $h \ll h_0$ allows one to prove that, for $h \to -\infty$, A_h and B_h converge to some finite positive constants $A_{-\infty}$ and $B_{-\infty}$, while $(E_{h-1} - E_h)/E_h = (Z_{h-1} - Z_h)/Z_h$ and:

$$Z_{h-1} = Z_h - c_0 Z_h^2 \tag{29}$$

where c_0 is a suitable positive constant, depending on $A_{-\infty}$ and $B_{-\infty}$.

The discussion of the above equations is elementary and, starting from initial datum $Z_0 = \varepsilon$ (or any other close to it), the result is that, if ν_0 is chosen so that ν_h is bounded uniformly in h, then, asymptotically: $Z_h = 4\lambda_h = -\mu_h/\sqrt{2} = \bar{c}\,|h|^{-1}$, $\nu_h = O(\lambda_h)$, if $\bar{c}$ is a suitable constant (ε independent).

At this point it is a standard task to check that all the neglected terms in the beta function are at least of order $1/|h|^2$. Hence they can not change in a substantial way the asymptotic properties of the flow (up to convergence problems). By using the techniques of Ref. [6], one could also prove that the sum of all terms of order n, in the expansion of r_{h-1} in terms of $\{r_{h'}, h' \geq h\}$, is bounded by $C^n n!$ for a suitable constant C, independent of h.

The main consequence of the previous discussion is that, for $k \to 0$ (that is, for $h \to -\infty$), the model is gaussian (asymptotic freedom) and the pair Schwinger function of the fields $\psi^\pm$ behaves as:

$$\tilde{S}_{-+}(k) = -\tilde{S}_{--}(k) = -\tilde{S}_{++}(k) \simeq \frac{q_0}{8B_{-\infty}} \frac{1}{k_0^2 + c^2\mathbf{k}^2} , \tag{30}$$

where $q_0 = p_0^2/(2m)$ and, if $v_0 = p_0/m$, the sound speed c is given by:

$$c^2 = \frac{(1+4A_{-\infty})v_0^2}{16B_{-\infty}} = c_B^2[1+O(\epsilon^{1/2})]. \qquad (31)$$

Acknowledgments - I have to acknowledge the contribution of many other people in the different stages of this work, in particular G. Gallavotti, whose comments and suggestions had an essential role; I also profited by many discussions with J. F. Perez, during a visit to the University of Sao Paulo, as well as by the criticism of C. Castellani, C. Di Castro, M. Grilli and J. Magnen, which allowed me to find and correct some mistakes in my original calculations.

References

[1] N. N. Bogoliubov, J. Phys. (Moscow) **II**, 23 (1947).

[2] A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinski, *Methods of quantum field theory in Statistical Physics* (Dover, New York, 1963); V. N. Popov, *Functional integration in Quantum Field Theory and Statistical Mechanics* (Reidel, Dordrecht, 1983).

[3] L. D. Landau and E. M. Lifshits, *Statistical Physics* (Pergamon, Oxford, 1968).

[4] V. N. Popov and A. V. Seredniakov, Sov. Phys. JETP **50**, 193–195 (1979); Yu. A. Nepomnyashchii, A. A. Nepomnyashchii, Sov. Phys. JETP **48**, 493–501 (1978).

[5] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).

[6] G. Gallavotti, Rev. Mod. Phys. **57**, 471–569 (1985).

[7] G. Benfatto and G. Gallavotti, J. Stat. Phys. **59**, 541–664 (1990); G. Benfatto, G. Gallavotti, A. Procacci, and B. Scoppola, Commun. Math. Phys. **160**, 93–172 (1994).

[8] G. Benfatto, in *Constructive Physics*, V. Rivasseau ed., Proceedings of the International Workshop on Constructive Results in Field Theory, Statistical Mechanics and Condensed Matter Physics, Palaiseau, july 25–27 1994, Lecture Notes in Physics **446** (1995), Springer-Verlag.

[9] G. Benfatto and G. Gallavotti, *Renormalization group*, Physics Notes **1**, Princeton University Press (1995).

[10] C. Castellani, C. Di Castro, F. Pistolesi, G. C. Strinati, preprint (1996).

[11] G. Benfatto, P. Lemberger, work in progress.

[12] M. H. Anderson *et al.*, Science **269**, 198 (1995).